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The holonomy of a singular foliation [1] [2] [3]

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Abstract

We give an overview of [1], in collaboration with G. Skandalis, where we construct the holonomy groupoid and the C^*-algebras associated with *any* singular foliation (in the sense of Stefan and Sussmann).

1 Introduction

Foliations arise naturally in several situations, including Poisson geometry (every Poisson manifold is endowed with a canonical symplectic foliation). The relation of foliations with groupoids is well known: The orbits of a Lie groupoid define a foliation; on the other hand, to a *regular* foliation there corresponds its holonomy groupoid constructed by Ehressmann [8] and Winkelnkemper [18]. (For an account of this see [10].) In the regular case the crucial properties of the holonomy groupoid are

- *smoothness*, namely in the regular case the holonomy groupoid is a Lie groupoid
- *minimality*, namely every other Lie groupoid which defines the same foliation maps onto the holonomy groupoid.

The holonomy groupoid is the first step towards a number of different important results:

- Due to minimality it records the necessary information for the space of leaves of the foliation in hand. This space presents considerable topological pathology, and A. Connes showed that it can be replaced by a certain C^*-algebra constructed from the holonomy groupoid.
- Using this C^*-algebra, A. Connes and G. Skandalis in [4] developed an index theory for foliations.

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- By extending the construction of the C^*-algebra to an arbitrary Lie groupoid we get a formal deformation quantization for the Poisson structure of the dual of an integrable Lie algebroid.
- The C^*-algebra above corresponds to a family of pseudodifferential operators along the s-fibers of the holonomy groupoid, and these operators play an important role in both index theory and deformation quantization.

Attempts to generalize the holonomy groupoid in the singular case were made by several authors, mainly Pradines and Bigonnet, [2], [14]. Their construction was very well understood and its range of applicability explained by Debord in [7]. This work deals with the foliations which arise from an *almost injective* Lie algebroid, namely an algebroid whose anchor map is injective in a dense subset of the base manifold. The integrability of such algebroids is proven (independently) by both Debord [7] as well as Crainic and Fernandes [5]. The difference of the two approaches is the following: For any given Lie algebroid, the latter authors construct a certain topological groupoid (Weinstein groupoid) and give the criteria under which it enjoys a smooth structure making it a Lie groupoid. Then it is explained that in the almost injective case these criteria are satisfied. There might, however, exist other integrating Lie groupoids which may as well be smaller. On the other hand, Debord constructs a groupoid which is a priori minimal among all Lie groupoids possibly integrating the algebroid, and shows that the almost injectivity assumption endows it with a good smooth structure. So in the almost injective case, it is legitimate to call Debord's groupoid the holonomy groupoid.

On the other hand, Paterson in [13] pointed out that in order to define the correct pseudodifferential calculus all we need is a groupoid with smooth s-fibers, rather than universal smoothness. It is therefore understood that there might be an alternative approach to problems such as the ones mentioned above, rather than trying to fully integrate an algebroid. Furthermore, in [7] there are given examples of foliations which cannot arise from almost injective Lie algebroids. It is therefore necessary to treat the problem of the existence of a holonomy groupoid independently from the integrability of some algebroid.

This is exactly the approach we adopt here. Once we think along these terms, the holonomy groupoid $\mathcal{H}(\mathcal{F})$ needs merely to be the smallest groupoid which desingularizes the foliation $\mathcal{F}$. An obvious choice for such a groupoid is the equivalence relation of belonging in the same leaf. The s-fibers of this are $L \times L$, where L is a leaf of $\mathcal{F}$. Thus they are smooth although the groupoid itself is not. In this work we show that in fact there is always a better holonomy groupoid. A rough description of our results is:

Theorem 1.1. *Let $\mathcal{F}$ be a (possibly singular) Stefan-Sussmann foliation on a manifold M. Then there exists a topological groupoid $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ such that:*

- *Its orbits are the leaves of the given foliation $\mathcal{F}$.*

- *$\mathcal{H}(\mathcal{F})$ is minimal in the sense that if $G \rightrightarrows M$ is a Lie groupoid which defines the foliation $\mathcal{F}$ then there exists an open subgroupoid G_0 of G (namely its s-connected component) and a morphism of groupoids $G_0 \to \mathcal{H}(\mathcal{F})$ which is onto.*

- *If $\mathcal{F}$ is regular or almost regular, i.e. defined by an almost injective Lie algebroid, then $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ is the holonomy groupoid defined in [7].*

Our holonomy groupoid follows in some sense the view of Bigonnet-Pradines, who consider local holonomies abstractly, as local diffeomorphisms of local transversals. We first define a notion of *atlas* for $\mathcal{F}$ where the role of a chart is played by the notion of a *bi-submersion*, which we introduce here. Bi-submersions are, roughly, those objects which record locally the holonomies defined by exponentiating the vector fields of $\mathcal{F}$. Then $\mathcal{H}(\mathcal{F})$ arises as a quotient of the minimal $\mathcal{U}_0$ among such atlases. Its topology (as a quotient space) is quite bad, but in several cases it has smooth s-fibers.

The importance of this approach is that it proposes a different point of view for the issue of the holonomy groupoid, namely that it is inextricably linked with a certain atlas. In fact, we show that smoothness may always be modified so that the quotient map $\mathcal{U} \to \mathcal{H}(\mathcal{F})$ becomes a submersion along the s-fibers. This way one may speak about a *holonomy pair* $(\mathcal{U}, \mathcal{H}(\mathcal{F}))$, rather than just the holonomy groupoid. Such pairs may arise in various ways for a foliation $\mathcal{F}$, e.g. by considering different atlases. When the minimal atlas $\mathcal{U}_0$ does not quotient to a longitudinally smooth (holonomy) groupoid, we may consider the groupoid $R \rightrightarrows M$ defined by the equivalence relation, and $(\mathcal{U}_0, R)$ becomes the appropriate holonomy pair. So the notation $\mathcal{H}(\mathcal{F})$ may mean either our quotient when it has smooth s-fibers, or R when the quotient doesn't have smooth s-fibers.

There is, however, a more important reason for adopting this approach. In [1] we explain how the quotient map $\mathcal{U} \to \mathcal{H}(\mathcal{F})$ allows us to construct the C^*-algebra of the foliation $\mathcal{F}$: The usual construction cannot be applied for $\mathcal{H}(\mathcal{F})$, since its topology is pathological, and the functions defined on it are highly non-continuous. Thinking, however, in terms of the holonomy pair $(\mathcal{U}, \mathcal{H}(\mathcal{F}))$, we can work with functions defined on the bi-submersions of $\mathcal{U}$. Since the quotient map is a submersion, we can then integrate such functions along the fibers of the quotient map to obtain functions on $\mathcal{H}(\mathcal{F})$. Then, pullbacks of bi-submersions translate to involution and convolution on the space of such functions, which is completed appropriately to the full and reduced C^*-algebras of the foliation. This approach pushes a little further the construction of the C^*-algebra given by A. Connes for non-Hausdorff groupoids. In the regular case the resulting C^*-algebra(s) are exactly the ones we get with the usual construction.

The construction of the C^*-algebra is beyond the scopes of this sequel. Let us merely state at this point that bi-submersions also make possible the definition of the appropriate pseudodifferential calculus realizing the above (reduced) C^*-

algebra. In a forthcoming paper we give this construction, together with the extension of the longitudinal index map given in [4] to any singular foliation.

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2 What is a foliation?

Before describing the method leading to the holonomy groupoid though, some clarifications are necessary. The term "foliation" on a manifold M may be understood in either of the following ways:

- A partition of M to disjoint submanifolds (leaves), possibly of different dimension (hence the singularities), or
- A distribution F on the tangent bundle TM which is locally finitely generated by (globally defined) vector fields and involutive (satisfying the conditions given by Stefan [16] and Sussmann [17]).

If a foliation is *regular*, then the two notions coincide, namely the leaves determine the vector fields which define the distribution. Another way to see this is that in this case F is a (constant rank) vector subbundle of TM, so locally its module of sections does not depend on the choice of vector fields which generate it.

In the *singular* case though, this is no longer true. One can get the same leaves from different choices of vector fields. For example, consider the partition of $\mathbb{R}$ to three leaves, $L_1 = \mathbb{R}^*_-$, $L_2 = \{0\}$ and $L_3 = \mathbb{R}^*_+$. These may be considered integral submanifolds to any of the submodules $\mathcal{F}_n = \langle x^n \frac{d}{dt} \rangle$ of $\mathcal{X}(M)$ for a positive integer n. Although $\mathcal{F}_{n+1}$ lies inside $\mathcal{F}_n$ the converse does not hold. In this example we have a preferred choice of module, say $\mathcal{F}_1$, but in several other cases no such choice is possible. For instance, take the foliation on $\mathbb{R}$ whose leaves are $\mathbb{R}_+$ and $\{x\}$ for any $x \leq 0$. Then we can take $\mathcal{F} = \langle f \frac{\partial}{\partial x} \rangle$ for any function f which vanishes on every non-positive real. Note that we cannot consider the module of all vector fields which vanish on $\mathbb{R}_-$, as it is not locally finitely generated.

So in the singular case one needs to determine a priori the module of vector fields which gives the distribution. We therefore need to postulate the following definition:

Definition 2.1. Let M be a smooth manifold. A (Stefan-Sussmann) *foliation* on M is a locally finitely generated submodule of the $C^\infty(M)$-module of compactly supported vector fields $\mathcal{X}_c(M)$, stable under Lie brackets.

In what follows we assume the choice of such a submodule $\mathcal{F}$. A different choice of submodule leads to a different holonomy groupoid.

2.1 Pseudogroups of diffeomorphisms

There is a deeper approach to what a foliation really is. A module $\mathcal{F}$ on a manifold M as above gives rise to two pseudogroups of local diffeomorphisms:

- The pseudogroup $\mathrm{Aut}(M, \mathcal{F})$ of local diffeomorphisms g of M which preserve the foliation, namely such that $g_*\mathcal{F} = \mathcal{F}$.
- The pseudogroup $\exp \mathcal{F}$ generated by $\exp X$ with $X \in \mathcal{F}$.

The next result is the key ingredient of the Frobenius theorem:

Proposition 2.2. *The pseudogroup* $\exp \mathcal{F}$ *is a normal sub-pseudogroup of* $\mathrm{Aut}(M, \mathcal{F})$.

Proof. Let $X \in \mathcal{F}$; we have to show that $\exp X \in \mathrm{Aut}(M, \mathcal{F})$. Replacing M by a neighborhood of the support of X, we may assume that $\mathcal{F}$ is finitely generated. Take $Y_1, \ldots, Y_n$ to be global sections of $\mathcal{F}$ generating $\mathcal{F}$. Since $[X, Y_i] \in \mathcal{F}$, there exist functions $\alpha_{i,j} \in C_c^\infty(M)$ such that $[X, Y_i] = \sum_j \alpha_{j,i} Y_j$.

Denote by $L : C^\infty(M)^n \to C^\infty(M)^n$ the linear mapping given by $L(f_1, \ldots, f_n) = (g_1, \ldots, g_n)$, where $g_i = X(f_i) + \sum_j \alpha_{i,j} f_j$.

Let $S : C^\infty(M)^n \to \mathcal{F}$ be the map $(f_1, \ldots, f_n) \mapsto \sum f_i Y_i$; since $L_X \circ S = S \circ L$, we find $\exp X \circ S = S \circ \exp L$. Therefore, $\exp X(\mathcal{F})$, which is the image of $\exp X \circ S$, is contained in the image of S, *i.e.* it is contained in $\mathcal{F}$.

Furthermore, if $g \in \mathrm{Aut}(M, \mathcal{F})$, we find $g \circ \exp X \circ g^{-1} = \exp(g_* X) \in \exp \mathcal{F}$. □

More that that, the leaves of $\mathcal{F}$ are just the orbits of the action of $\exp \mathcal{F}$ on M. There may be other pseudogroups of local diffeomorphisms preserving $\mathcal{F}$. For instance, suppose the foliation $\mathcal{F}$ is defined by a Lie groupoid $G \rightrightarrows M$; this means that $\mathcal{F}$ is the image by the anchor map of the sections of the Lie algebroid AG. (In this case a leaf at $x \in M$ is $t(s^{-1}(x))$.) Any Lie groupoid has local bisections (see [9, 1.4.9]). Let us give a slightly different (but equivalent) definition here:

Definition 2.3. Let $G \rightrightarrows M$ be a Lie groupoid.

1. A *bisection* is a locally closed submanifold V of G such that the restrictions of both s and t to V are diffeomorphisms from V onto open subsets of M.
2. The *local diffeomorphism* associated to a bisection V is $\varphi_V = t \circ s^{-1}$ of M, where $s_V : V \to s(V)$ and $t_V : V \to t(V)$ are the restrictions of s and t to V.

So if $\mathcal{F}$ is defined by a Lie groupoid $G \rightrightarrows M$, then the local diffeomorphisms defined from all the bisections form a pseudogroup which sits inside $\mathrm{Aut}(M, \mathcal{F})$. In the regular case the holonomy groupoid is the one whose bisections define the pseudogroup $\exp \mathcal{F}$. This is a key observation, pointing us to the correct direction for the singular case: The holonomy groupoid should record this particular pseudogroup of local diffeomorphisms.

From this point of view it is clearer why the construction of the holonomy groupoid is usually treated as a local problem. For instance, Debord defines an atlas of quasi-graphoids for the foliation, and the holonomy groupoid turns out to be a quotient of this atlas. Our approach is along similar lines.

3 Atlases for foliations

Let us start with the usual notion of atlas for a compact and connected n-dimensional manifold M. An atlas is understood as a collection of charts, namely bijections $(x_1, \ldots, x_n) : U \to B^n(\mathbb{R})$, where $B^n(\mathbb{R})$ is an open ball in $\mathbb{R}^n$ and U a subset of M, which we declare open. The manifold M may be regarded as a foliation with one leaf, or, equivalently, with $\mathcal{F} = \mathcal{X}(M)$. Locally $\mathcal{F}$ is generated by the vector fields $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ induced by the coordinate functions.

These vector fields tell us how to "move" along the (unique, in this case) leaf by following their integral curves. To see this, consider a chart $(U, (x_1, \ldots, x_n))$ at a point $m \in M$. Then there is an open neighborhood Ω of $U \times \mathbb{R}^n$ at $(m, 0)$ such that the maps

$$s, t : \Omega \to M, \quad s = pr_1, \quad t(x, y_1, \ldots, y_n) = \exp\left(\sum t_i \frac{\partial}{\partial x^i}\right)(x)$$

are defined in Ω and are submersions. Now $\Omega \rightrightarrows M$ is not a groupoid, nevertheless its "orbits" $t(s^{-1}(m))$ still tell us how to move close to m following the flows of the vector fields which define the foliation. So the usual notion of an atlas for a smooth manifold can be reformulated to provide the necessary information concerning how we can move along a leaf.

A different approach would be to consider bisections (the definition is given in 2.3). Considering all the bisections of $\Omega \rightrightarrows M$ above, it is straightforward that the pseudogroup of local diffeomorphisms they induce is exactly $\exp \mathcal{X}(M)$.

This picture is similar to what is going on when a Lie groupoid G exists for a foliation $\mathcal{F}$. In fact, choosing locally a base of sections for the Lie algebroid AG and using the exponential map we can show that locally a Lie groupoid G is diffeomorphic to an Ω as above. Furthermore, if $\mathcal{F}$ is regular we can choose locally a base of vector fields which generate $\mathcal{F}$ and apply the previous construction. But with a general singular foliation $\mathcal{F}$, things are more complicated because there are several leaves of varying dimensions.

3.1 Bi-submersions

Let us make a fresh start now and see how we can generalize the notion of an atlas in the above sense to fit foliations. Take a point $m \in M$. For any open $U \subset M$ at

m, the linear subspace $F_m \subseteq T_mM$ is defined by the values of the vector fields in the $C^\infty(U)$-module $\mathcal{F}(U)$. This module is finitely generated, but there is no base; if there was a choice of linearly independent generators, say $X_1, \ldots, X_n$, then we might use them to imitate the previous construction. We ensure the existence of such generators in the following way:

Let $I_m = \{f \in C^\infty(M) \mid f(m) = 0\}$ and denote $\mathcal{F}_m = \mathcal{F}/I_m\mathcal{F}$. This is a finite dimensional vector space, and the evaluation map induces an exact sequence of vector spaces

$$0 \to \mathfrak{g}_m \to \mathcal{F}_m \overset{ev_m}{\to} F_m \to 0$$

It follows that the kernel $\mathfrak{g}_m$ of this extension has a bracket which makes it a Lie algebra. Actually $\mathfrak{g}_m$ records the *isotropy* of the foliation at m.

Example 3.1. Consider the partition of $\mathbb{R}^2$ into two leaves: $\{0\}$ and $\mathbb{R} - \{0\}$. It is given by the action of either of the Lie groups $GL(2,\mathbb{R}), SL(2,\mathbb{R})$ or $\mathbb{C}^*$. The module $\mathcal{F}$ is different for each of these actions. The corresponding $\mathcal{F}_x$ are equal to $T_x\mathbb{R}^2$ at each non-zero $x \in \mathbb{R}^2$ and at zero they are exactly the Lie algebra of the group.

The $\mathcal{F}_x$s allow us to choose local generators for $\mathcal{F}$ as the next proposition shows:

Proposition 3.2. *If the images of $X_1, \ldots, X_n \in \mathcal{F}$ form a base of $\mathcal{F}_x$, then there exists a neighborhood U of x in M such that $\mathcal{F}$ restricted to U is generated by $X_1, \ldots, X_n$.*

Proof. As in the proof of 2.2, we assume that $\mathcal{F}$ is finitely generated as a module and consider global sections $Y_1, \ldots, Y_n$ generating $\mathcal{F}$. Since the images of $X_1, \ldots, X_k$ form a basis of $\mathcal{F}_x$, there exist $a_{\ell,i} \in \mathbb{C}$ for $1 \leq i \leq N$ and $1 \leq \ell \leq k$ such that $Y_i - \sum_{\ell=1}^k a_{\ell,i}X_\ell \in I_x\mathcal{F}$. It follows that there exist functions $\alpha_{i,j} \in C^\infty(M)$ for $1 \leq i,j \leq N$ such that $\alpha_{i,j}(x) = 0$ and for all i we have $Y_i - \sum_{\ell=1}^k a_{\ell,i}X_\ell = \sum_{i=1}^n \alpha_{j,i}Y_j$ in a neighborhood of x. This can be written as $\sum_{j=1}^N \beta_{j,i}Y_j = \sum_{\ell=1}^k a_{\ell,i}X_\ell$ for all $1 \leq i \leq N$, where $\beta_{i,j} = -\alpha_{i,j}$ if $i \neq j$ and $\beta_{i,i} = 1 - \alpha_{i,i}$

For $y \in M$, let B_y denote the matrix with entries $\beta_{i,j}(y)$. Since B_x is the identity matrix, for y in a neighborhood U of x, the matrix $B(y)$ is invertible. Write $(B(y))^{-1} = (\gamma_{i,j}(y))$. We find on U, $Y_i = \sum_{\ell=1}^k c_{\ell,i}X_\ell$, where $c_{\ell,i} = \sum a_{\ell,j}\gamma_{j,i}$. □

Proposition 3.3. *Let $X_1, \ldots, X_n$ be vector fields whose images form a base of $\mathcal{F}_m$. Then there exists an open neighborhood Ω of $(m,0)$ in $M \times \mathbb{R}^n$ such that*

1. *The maps $s,t : \Omega \to M$ defined by $s = pr_1$ and $t(x,y) = exp(\sum y_iX_i)(x)$ are submersions*

2. *$s^{-1}(\mathcal{F}) = t^{-1}(\mathcal{F})$ and $s^{-1}(\mathcal{F}) = C_c^\infty(\Omega; \ker ds) + C_c^\infty(\Omega; \ker dt)$.*

(Here $s^{-1}(\mathcal{F})$ stands for the vector fields of Ω which map to $\mathcal{F}$ by ds.)

Proof. Let $y, z \in \mathbb{R}^n$; set $Y = \sum y_i X_i$ and $Z = \sum z_i X_i$. For $\alpha \in \mathbb{R}$ define $\psi_\alpha = t(\cdot, \alpha y)$. The formula for the derivative of $X \mapsto \exp X$ yields

$$(dt)_{(x,y)}(0, z) = \int_0^1 (\psi_{1-\alpha})_*(Z_{\psi_\alpha(x)})\, d\alpha.$$

Let us show that $C_c^\infty(M \times \mathbb{R}^n; \ker ds) \subset t^{-1}\mathcal{F}$: Consider the vector field $\hat{Z}$ on $M \times \mathbb{R}^n$ defined by $\hat{Z}(x, y) = (\sum y_i X_i(x), 0_y)$. It belongs to $s^{-1}(\mathcal{F})$. The local diffeomorphism $\varphi = \exp(\hat{Z})$ fixes $s^{-1}(\mathcal{F})$, namely $\varphi \in \exp(s^{-1}\mathcal{F})$. Define $\psi = \alpha \circ \varphi$, where $\alpha : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is $\alpha(x, y) = (x, -y)$. Then $\psi \in \exp(s^{-1}\mathcal{F})$ and $\psi^2 = id$, $s \circ \psi = t$. The result follows.

Now, since $\mathcal{F}$ is spanned by the X_i's in a neighborhood U of x in M, there exists a smooth function $h = (h_{i,j})$ defined in a neighborhood W_0 of $(x, 0)$ in $U \times \mathbb{R}^n$ with values in the space of $n \times n$ matrices such that: $(dt)_{(x,y)}(0, z) = \sum z_i h_{i,j}(x, y) X_j$, and $h_{i,j}(x, 0) = \delta_{i,j}$. Taking a smaller neighborhood W, we may assume that $h(x, y)$ is invertible. In this neighborhood we have

$$t^{-1}(\mathcal{F}) = C_c^\infty(W; \ker ds) + C_c^\infty(W; \ker dt).$$

Let $\kappa : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ be the map defined by $\kappa(y, u) = (t(x, y), -y)$. Note that κ is an involution of $M \times \mathbb{R}^k$ and $s \circ \kappa = t$ (whence $t \circ \kappa = s$). Put then $\Omega = W \cap \kappa(W)$. □

Notice that the second assertion is exactly what happens with the source and target maps of a Lie groupoid. Here we don't consider any multiplication though. The usual notion of a manifold chart may thus be generalized in the following way:

Definition 3.4. A **bi-submersion** for a foliation $\mathcal{F}$ on a smooth manifold M is a triple (U, t_U, s_U) such that U is a smooth manifold, $t_U, s_U : U \to M$ are submersions and they satisfy 2 as above. A bi-submersion Ω as above is called a **bi-submersion near the identity**.

Let (U_i, t_i, s_i) be two bi-submersions. A **morphism** is a smooth map $\phi : U_1 \to U_2$ such that $s_2 \circ \phi = s_1$ and $t_2 \circ \phi = t_1$.

3.2 Bisections

One may consider various bi-submersions for a foliation $\mathcal{F}$, each of them reflecting a different way one can move along a leaf in the vicinity of a point. The question that arises naturally is

when do two bi-submersions provide the same local diffeomorphisms for the foliation?

The obvious answer is when they locally record the same local diffeomorphisms. Perhaps the best way to understand this is by examining the (local) **bisections** of a bi-submersion. It follows from the proof of [9, 1.4.9] that bisections exist for any pair of submersions $U \rightrightarrows M$ between smooth manifolds. It is therefore legitimate to talk about the bisections of a bi-submersion in the sense of 2.3.

Definition 3.5. Let (U, t_U, s_U) be a bi-submersion of $(M, \mathcal{F})$. Let $u \in U$ and φ a local diffeomorphism of M. We say that φ *is carried by* (U, t_U, s_U) *at* u if there exists a bisection V such that $u \in V$ and whose associated local diffeomorphism coincides with ϕ in a neighborhood of u.

Thus a bi-submersion may be thought of as a manifestation of a certain collection of local diffeomorphisms of M which respect the foliation $\mathcal{F}$. For example, the bi-submersions near the identity carry the local diffeomorphisms of the form $\exp \mathcal{F}$, namely the ones generated by $\{\exp X \mid X \in \mathcal{F}\}$.

Returning to the question we posed above, two bi-submersions will locally record the same local diffeomorphisms if both of them have local bisections which induce such diffeomorphisms. The simplest way to ensure the existence of such bisections is to assume there is a local morphism between the two bi-submersions. This leads to the next definition:

Definition 3.6. Let $\mathcal{U} = (U_i, t_i, s_i)$ be a family of bi-submersions. A bi-submersion (U, t_U, s_U) is **adapted** to $\mathcal{U}$ if for all $u \in U$ there exists an open subset $U' \subseteq U$, an element $i \in I$ and a map $U' \to U_i$ which preserves s and t.

The following result is crucial:

Proposition 3.7. *Let $x \in M$. Let $X_1, \ldots, X_n \in \mathcal{F}$ be vector fields whose images form a basis of $\mathcal{F}_x$. For $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, put $\varphi_y = \exp\left(\sum y_i X_i\right) \in \exp \mathcal{F}$. Put $\mathcal{W}_0 = \mathbb{R}^n \times M$, $s_0(y, x) = x$ and $t_0(y, x) = \varphi_y(x)$.*

a) *There is a neighborhood $\mathcal{W}$ of $(0, x)$ in $\mathcal{W}_0$ such that $(\mathcal{W}, t, s)$ is a bi-submersion where s and t are the restrictions of s_0 and t_0.*

b) *Let (V, t_V, s_V) be a bi-submersion and $v \in V$. Assume that $s_V(v) = x$ and that the identity of M is carried by (V, t_V, s_V) at v. There exists an open neighborhood V' of v in V and a submersion $g : V' \to \mathcal{W}$ which is a morphism of bi-submersions and $g(v) = (0, x)$.*

Proof. a) This is proposition 3.3.

b) Replacing V by an open subset containing v, we may assume that $s_V(V) \subset s(\mathcal{W})$ and that the bundles $\ker dt_V$ and $\ker ds_V$ are trivial. Since $t_V^{-1}(\mathcal{F}) = C_c^\infty(V; \ker ds_V) + C_c^\infty(V; \ker dt_V)$, the map $dt : C_c^\infty(V; \ker ds_V) \to t_V^*(\mathcal{F})$ is onto, and there exist $Y_1, \ldots, Y_n \in C^\infty(V; \ker ds_V)$ such that $dt_V(Y_i) = X_i$. Since $X_i(x)$ form a basis of $\mathcal{F}_x$, the $Y_i(v)$ are independent. Replacing V

by an open neighborhood of v, we may assume that the Y_i's are independent everywhere on V. Let $Z_{n+1}, \ldots, Z_k$ be sections of $\ker ds$ such that $(Y_1, \ldots, Y_n, Z_{n+1}, \ldots, Z_k)$ is a basis of $\ker ds_V$. Since $dt_V(Z_i) \in t_V^*(\mathcal{F})$ which is spanned by the Y_i's, we may subtract a combination of the X_i's so to obtain a new basis $(Y_1, \ldots, Y_n, Y_{n+1}, \ldots, Y_k)$ satisfying $dt_V(Y_i) = X_i$ if $i \leq n$ and $dt_V(Y_i) = 0$ if $i > n$. For $y = (y_1, \ldots, y_k) \in \mathbb{R}^k$ small enough we denote by ψ_y the (partially defined) diffeomorphism $\exp\left(\sum y_i Y_i\right)$ of V.

Let $U_0 \subset V$ be a bisection through v representing the identity, *i.e.* such that s_V and t_V coincide on U_0. We identify U_0 with an open subset of M via this map. There exists an open neighborhood U of v in U_0 and an open ball B in $\mathbb{R}^k$ such that $h : (y, u) \mapsto \psi_y(u)$ is a diffeomorphism of $U \times B$ into an open neighborhood V' of v. Let $p : \mathbb{R}^k \to \mathbb{R}^n$ be the projection to n first coordinates. The map $p \circ h^{-1}(V') \to \mathcal{W}$ is the desired morphism. It is a submersion.

□

Proposition 3.7 really shows that bi-submersions near the identity are adapted to any other bi-submersion. More clearly, it means that:

Corollary 3.8. *Let (U, t_U, s_U) and (V, t_V, s_V) be bi-submersions and let $u \in U$, $v \in V$ be such that $s_U(u) = s_V(v)$.*

a) If the identity local diffeomorphism is carried by U at u and by V at v, there exists an open neighborhood U' of u in U and morphism $f : U' \to V$ such that $f(u) = v$.

b) If there is a local diffeomorphism carried both by U at u and by V at v, there exists an open neighborhood U' of u in U and morphism $f : U' \to V$ such that $f(u) = v$.

c) If there is a morphism of bi-submersions $g : V \to U$ such that $g(v) = u$, there exists an open neighborhood U' of u in U and morphism $f : U' \to V$ such that $f(u) = v$.

3.3 Moving far along a leaf

The picture given by local bisections describes a neighborhood of a leaf L_m of $\mathcal{F}$ around m. The question that arises naturally is

how can one move far away from m along the same leaf?

Lie groupoids give us this information because they enjoy a (partially defined) multiplication among arrows. But since we care about points on the leaf, it is not really the product of a sequence of arrows on the groupoid level that we need. It suffices to know the collection of all composable arrows. For bi-submersions we observe:

Proposition 3.9. *Let (U_1, t_1, s_1) and (U_2, t_2, s_2) be two bi-submersions for the foliation $\mathcal{F}$ on M. Then the pullback manifold $U_1 \times_{(s_1,t_2)} U_2$ is also a bi-submersion for $\mathcal{F}$. We call $U_1 \times_{(s_1,t_2)} U_2$ the* composition *of U_1 and U_2 and write $U_1 \circ U_2$.*

So if we consider a family of bi-submersions and all its compositions, we have a way to move along the leaves of a foliation. With these ingredients in hand we can now give the definition of an *atlas* for a foliation $\mathcal{F}$:

Definition 3.10. We say that $\mathcal{U} = \{(U_i, t_i, s_i)\}_{i \in I}$ is an **atlas** for the foliation $\mathcal{F}$ if:

1. $\bigcup_{i \in I} s_i(U_i) = M$
2. The inverse (U_i, s, t) of every (U_i, t, s) is adapted to $\mathcal{U}$.
3. The composition of two elements is adapted to $\mathcal{U}$.

We say that an atlas $\mathcal{U}_1$ is **adapted** to the atlas $\mathcal{U}$ if every bi-submersion in $\mathcal{U}_1$ is adapted to a bi-submersion of $\mathcal{U}$.

An obvious example of an atlas is a Lie groupoid $G \rightrightarrows M$ (for the foliation it induces on M).

By abuse of notation we write $\mathcal{U}$ for the maximal atlas associated to an atlas for $\mathcal{F}$ as above. Namely the family $\mathcal{U}$ will include all the bi-submersions which are adapted to it. An atlas, as above, gives a full description of the leaves of a foliation $\mathcal{F}$. Bearing in mind the role of bisections, one understands that the choice of an atlas corresponds to the choice of a certain family of local diffeomorphisms which preserve $\mathcal{F}$. Such a family provides information regarding the leaves of the foliation.

We denote $\mathcal{U}_0$ the atlas generated by the bi-submersions near the identity Ω as in proposition 3.3. In the context of local diffeomorphisms $\mathcal{U}_0$ can be thought to represent the family of local diffeomorphisms $\exp \mathcal{F}$. It turns out that $\mathcal{U}_0$ is adapted to any other atlas. It is called the *path holonomy atlas.*

4 The holonomy groupoid

An atlas $\mathcal{U} = \{(U_i, t_i, s_i)\}_{i \in I}$ as above gives rise to a **groupoid** $G(\mathcal{U}) \rightrightarrows M$, whose orbits are exactly the leaves of the foliation $\mathcal{F}$. To make this more precise, consider the equivalence relation in the discrete union $\coprod_{i \in I} U_i$ defined by

$$U_1 \ni u_1 \sim u_2 \in U_2 \Leftrightarrow \text{there exists locally a morphism of bi-submersions } f : U_1 \to U_2 \text{ such that } f(u_1) = u_2.$$

Then $G(\mathcal{U})$ is the quotient space of this relation, and it is obviously a groupoid over M. It has the quotient topology, which is usually ill-behaved. This way, starting from any atlas $\mathcal{U}$ for $\mathcal{F}$ we get a topological groupoid for the leaf space of $\mathcal{F}$. The following result is the key ingredient to show that the atlas $\mathcal{U}_0$ is the one that gives the holonomy groupoid.

Proposition 4.1. *Every s-connected Lie groupoid which realizes $\mathcal{F}$ is adapted to the atlas $\mathcal{U}_0$ of bi-submersions near the identity.*

To see this, first note:

Proposition 4.2. *Let $\mathcal{U}$ be an atlas for the foliation $\mathcal{F}$ and (U, t_U, s_U) a bi-submersion adapted to $\mathcal{U}$. Then there exists a map $q_U : U \to G(\mathcal{U})$ which preserves the source and target maps.*

Now consider the atlas $\mathcal{U}_0$ generated by the bi-submersions near the identity and denote $\mathcal{H}(\mathcal{F}) \rightrightarrows M$ its associated groupoid. Since any s-connected Lie groupoid $G \rightrightarrows M$ which realises $\mathcal{F}$ is adapted to $\mathcal{U}_0$ it follows that there is a map $q_G : G \to \mathcal{H}(\mathcal{F})$. This map is easily seen to be a *morphism of groupoids* onto $\mathcal{H}(\mathcal{F})$. This shows that $\mathcal{H}(\mathcal{F})$ **is the holonomy groupoid.**

The holonomy groupoid constructed by Debord is really $\mathcal{H}(\mathcal{F})$ in case the module $\mathcal{F}$ is projective. This is shown by the following proposition which is a straightforward application of the Serre-Swan theorem.

Proposition 4.3. *If the module $\mathcal{F}$ is projective then it carries a natural Lie algebroid structure and $\mathcal{H}(\mathcal{F})$ is a Lie groupoid.*

4.1 The topology of the holonomy groupoid

The topology of the holonomy groupoid, as well as all groupoids associated with other atlases, is usually quite bad, as it is a quotient topology.

Let us fix an atlas $\mathcal{U}$ and let $G_{\mathcal{U}}$ be the associated groupoid. For every bi-submersion (U, t, s) adapted to $\mathcal{U}$, let $V_U \subset U$ be the set of $u \in U$ such that $\dim T_u U = \dim M + \dim \mathcal{F}_{s(u)}$. It is an open subset of U when U is endowed with the smooth structure along the leaves of the foliation $t^{-1}(\mathcal{F}) = s^{-1}(\mathcal{F})$.

Proposition 4.4. *a) For every $x \in G$, there is a bi-submersion (U, t, s) adapted to $\mathcal{U}$ such that $x \in q_U(V_U)$.*

b) Let (U, t, s) and (U', t', s') be two bi-submersions and let $f : U \to U'$ be a morphism of bi-submersions. Let $u \in U$. If $u \in V_U$, then $(df)_u$ is injective; if $f(u) \in V_{U'}$, then $(df)_u$ is surjective.

Proof. a) Let $x \in G$. Let (W, t, s) be a bi-submersion adapted to $\mathcal{U}$ and $w \in W$ and such that $x = q_W(w)$. Let $A \subset W$ be a bi-section through w. Let $g : s(A) \to t(A)$ be the local diffeomorphism of M defined by A. By proposition 3.7 there exists a bi-submersion (U_0, t_0, s_0) and $u_0 \in U_0$ such that $\dim U_0 = \dim \mathcal{F}_{s(u)} + \dim M$, $s_0(u_0) = s(u)$ and carrying the identity through u_0. Put then $U = \{x \in U_0;\ t_0(u) \in s(A)\}$ and let s be the restriction of s_0 to U and t be the map $u \mapsto g(t_0(u))$. Obviously (U, t, s) is a bi-submersion which carries g at u_0. It follows that (U, t, s) is adapted to $\mathcal{U}$ at u_0 and $q_U(u_0) = q_W(w) = x$. It is obvious that $u_0 \in V_U$.

b) Since s and s' are submersions and $s' \circ f = s$, df_u is injective or surjective if and only if its restriction $(df_{|\ker ds})_u : \ker(ds)_u \to \ker(ds')_{f(u)}$ is. Consider the composition

$$\ker(ds)_u \xrightarrow{(df_{|\ker ds})_u} \ker(ds')_{f(u)} \xrightarrow{t'_*} \mathcal{F}_{s(u)}.$$

By definition of bi-submersions the maps t'_* and $t_* = t'_* \circ (df)_u$ are onto; if $u \in \Gamma_U$, then $t_* : \ker(ds)_u \to \mathcal{F}_{s(u)}$ is an isomorphism; if $f(u) \in \Gamma_{U'}$, then $t'_* : \ker(ds')_{f(u)} \to \mathcal{F}_{s(u)}$ is an isomorphism. The conclusion follows.

□

It follows from 4.4(b) that the restriction of f to a neighborhood of $V_U \cap f^{-1}(V_{U'})$ is étale. This restriction preserves the foliation, and is therefore étale also with respect to this structure. Now the V_U are open in U with respect to the longitudinal structure; they are manifolds. The groupoid G is obtained by gluing them through local diffeomorphisms.

Remark 4.5. Looking at $G_{\mathcal{U}}$ longitudinally, we observe that the necessary condition for it to be a manifold is the following: We need to ensure that for every $x \in M$ there exists a bi-submersion (U, t, s) in the path holonomy atlas and a $u \in U$ which has an open neighborhood $U_u \subseteq U$ with respect to the leaf topology such that the quotient map $U_u \to G_{\mathcal{U}}$ is injective. Under this condition the s (t)-fibers of $G_{\mathcal{U}}$ are smooth manifolds (of dimension equal to the dimension of the underlying leaf) and the quotient map $q_U : U \to G_{\mathcal{U}}$ is a submersion along the s (t)-fibers. However, this condition is not always satisfied, and this fact leads to the following definition:

Definition 4.6. A *holonomy pair* for a foliation $(M, \mathcal{F})$ is a pair $(\mathcal{U}, G)$ where $\mathcal{U}$ is an atlas of bi-submersions and G is a groupoid over M which is a Lie groupoid for the longitudinal smooth structure, together with a surjective groupoid morphism $\alpha : G_{\mathcal{U}} \to G$ such that the maps $\alpha \circ q_U$ are leafwise submersions for each $U \in \mathcal{U}$.

In case $(\mathcal{U}, G_U)$ is not a holonomy pair for some atlas $\mathcal{U}$ we can always replace G_U with the groupoid $R_{\mathcal{F}}$ defined naturally by the equivalence relation of "belonging in the same leaf". This groupoid is not smooth but has smooth fibers. Moreover, for every bi-submersion (U, t, s) the maps $(t_U, s_U) : U \to R_{\mathcal{F}}$ play the role of q_U.

This deals with the cases where the atlas $\mathcal{U}$ does not satisfy the condition we mentioned above. In the overall, given a foliation $\mathcal{F}$ there always exists a minimal holonomy pair $(\mathcal{U}, R_{\mathcal{F}})$. Minimal here means that the atlas $\mathcal{U}$ is adapted to any other atlas. In many cases there exists an even "better" one, namely $(U, \mathcal{H}(\mathcal{F}))$. That is, when $\mathcal{H}(\mathcal{F})$ happens to have smooth s-fibers. It is explained in [1] that the minimal holonomy pair arises when we consider the path holonomy atlas.

It is also explained in [1] that holonomy pairs are all we need in order to construct a C^*-algebra. Such data is always available.

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