

An approach toward a finite-dimensional definition of twisted K -theory ¹

by Kiyonori Gomi

Abstract

This is an expository account of the following result: we can construct a group by means of twisted \mathbb{Z}_2 -graded vectorial bundles which is isomorphic to K -theory twisted by any degree three integral cohomology class.

1 Introduction

Topological K -theory admits a twisting by a degree three integral cohomology class. The resulting K -theory, known as *twisted K -theory* [2], has its origin in the works of Donovan-Karoubi [8] and Rosenberg [15], and has applications to D-brane charges [5, 12, 17], Verlinde algebras [9] and so on.

As is well-known, ordinary K -theory has definitions by means of:

- (1) vector bundles;
- (2) the C^* -algebra of continuous functions; and
- (3) the space of Fredholm operators.

Twisted K -theory is usually defined by twisting the definitions (2) or (3). For a definition parallel to (1), there are the notions of *twisted vector bundles* ([13, 14, 16], see also [5, 8, 12]), and of *bundle gerbe K -modules* [4]. However, the definitions by means of these geometric objects are only valid for twisted K -theory whose “twisting”, a third integral cohomology class, is of finite order: otherwise, there is no non-trivial such geometric object in finite-dimensions.

Toward a finite-dimensional definition of twisted K -theory valid for degree three integral cohomology classes of infinite order, we explain in this article Furuta’s notion of generalized vector bundles [10], which we call *vectorial bundles*. We also explain a notion of finite dimensional approximation of Fredholm operators, which provides a linear version of the finite dimensional approximation of the monopole equations [11]. We can use these notions to construct a group and an isomorphism from K -theory twisted by any degree three integral cohomology class. The proof of the result is only outlined. The detailed treatment will be provided elsewhere.

¹Received: December 16, 2006

A possible application of the result above is to generalize the notions of *2-vector bundles* [3, 6]. A 2-vector bundle of rank 1 due to Brylinski [6] is a stack which reproduces the category of twisted vector bundles. Replacing the category of vector bundles by that of vectorial bundles, we can directly generalize the 2-vector bundles in [6]. Similarly, we can also generalize the 2-vector bundles due to Baas, Dundas and Rognes [3], which they studied in an approach to geometric realization of elliptic cohomology. It seems interesting to apply their study of 2-vector bundles to the generalization of 2-vector bundles made of vectorial bundles.

Acknowledgments. I am grateful to M. Furuta for helpful discussions concerning this work. I am also indebted to T. Moriyama and A. Henriques for useful suggestions. I thank the organizers of School on Poisson geometry and related topics for the invitation to give a talk.

2 Twisted K -theory and twisted vector bundles

In this section, we review the definition of twisted K -theory by means of the space of Fredholm operators, following [2]. We also review the notion of twisted vector bundles [4, 5, 8, 12, 13, 14, 16].

Unless otherwise mentioned, X is a compact manifold through this article.

2.1 Twisted K -theory

The twisted K -theory we consider in this article is associated to a degree three integral cohomology class. To give the precise definition, we represent the class by a projective unitary bundle. Let \mathcal{H} be a separable Hilbert space of infinite dimension, and $PU(\mathcal{H})$ the projective unitary group $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$.

Definition 2.1. For a principal $PU(\mathcal{H})$ -bundle over X , we define the *twisted K -group* $K_P(X)$ to be the fiberwise homotopy classes of sections of the associated bundle $P \times_{Ad} \mathcal{F}(\mathcal{H})$ over X , where $PU(\mathcal{H})$ acts on the space $\mathcal{F}(\mathcal{H})$ of Fredholm operators on \mathcal{H} by adjoint.

In the above definition, the topologies on $PU(\mathcal{H})$ and $\mathcal{F}(\mathcal{H})$ are understood to come from the operator norm. Notice that we can also use the compact-open topology in the sense of [2].

If P is a trivial bundle, then $K_P(X)$ is exactly the homotopy classes of continuous functions $X \rightarrow \mathcal{F}(\mathcal{H})$. Thus, in this case, we recover the K -group of X by the well-known fact that $\mathcal{F}(\mathcal{H})$ is a classifying space of K -theory [1].

$PU(\mathcal{H})$ -bundles over X are classified by $H^3(X, \mathbb{Z})$: since $U(\mathcal{H})$ is contractible by Kuiper's theorem, $PU(\mathcal{H})$ is homotopy equivalent to the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, so that the classifying space $BPU(\mathcal{H})$ is homotopy equivalent to

$K(\mathbb{Z}, 3)$. If P and P' are isomorphic $PU(\mathcal{H})$ -bundles, then the twisted K -groups $K_P(X)$ and $K_{P'}(X)$ are also (non-canonically) isomorphic. So we often speak of “twisted K -theory twisted by a class in $H^3(X, \mathbb{Z})$ ”.

We will call the cohomology class corresponding to P the *Dixmier-Douady class*, and denote it by $\delta(P) \in H^3(X, \mathbb{Z})$. For later convenience, we recall here the construction of $\delta(P)$: take an open cover $\mathcal{U} = \{U_\alpha\}$ of X so that:

- there are local sections $s_\alpha : U_\alpha \rightarrow P|_{U_\alpha}$;
- there are lifts $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow U(\mathcal{H})$ of the transition functions $\bar{g}_{\alpha\beta} : U_{\alpha\beta} \rightarrow PU(\mathcal{H})$.

Here we write $U_{\alpha\beta}$ for the overlap $U_\alpha \cap U_\beta$, and the transition function is defined by the relation $s_\alpha \bar{g}_{\alpha\beta} = s_\beta$. Because of the cocycle condition for $\{\bar{g}_{\alpha\beta}\}$, we can find a map $c_{\alpha\beta\gamma} : U_{\alpha\beta\gamma} \rightarrow U(1)$ such that $g_{\alpha\beta} g_{\beta\gamma} = c_{\alpha\beta\gamma} g_{\alpha\gamma}$. These maps comprise a Čech 2-cocycle $(c_{\alpha\beta\gamma}) \in \check{Z}^2(\mathcal{U}, \underline{U(1)})$ with its coefficients in the sheaf of germs of $U(1)$ -valued functions, which represents $\delta(P)$ through the isomorphism $\check{H}^2(X, \underline{U(1)}) \cong H^3(X, \mathbb{Z})$.

2.2 Twisted vector bundles

For a $PU(\mathcal{H})$ -bundle P over X , a *twisted vector bundle* consists essentially of the data $(\mathcal{U}, E_\alpha, \phi_{\alpha\beta})$:

- an open cover $\mathcal{U} = \{U_\alpha\}$ of X ;
- vector bundles E_α over U_α ;
- isomorphisms of vector bundles $\phi_{\alpha\beta} : E_\beta|_{U_{\alpha\beta}} \rightarrow E_\alpha|_{U_{\alpha\beta}}$ over $U_{\alpha\beta}$ satisfying the “twisted cocycle condition” on $U_{\alpha\beta\gamma}$:

$$\phi_{\alpha\beta} \phi_{\beta\gamma} = c_{\alpha\beta\gamma} \phi_{\alpha\gamma},$$

where $c_{\alpha\beta\gamma}$ is as in the previous subsection.

In the rigorous definition of twisted vector bundles, we have to include the choices of the local sections s_α and the lifts $g_{\alpha\beta}$. Though it is crucial to specify these choices in considering isomorphism classes of twisted vector bundles, we omit them for simplicity.

The isomorphism classes of twisted vector bundles $\text{Vect}_P(X)$ gives rise to a semi-group by the direct sum of local vector bundles. Let $K(\text{Vect}_P(X))$ denote the group given by applying the Grothendieck construction to $\text{Vect}_P(X)$. Then the following fact is known. (See [5, 8, 12, 13, 16].)

Proposition 2.2. *For a $PU(\mathcal{H})$ -bundle whose Dixmier-Douady class $\delta(P)$ is of finite order, there exists an isomorphism:*

$$K_P(X) \longrightarrow K(\text{Vect}_P(X)).$$

Instead of twisted vector bundles, we can use bundle gerbe K -modules to obtain an equivalent result [4, 7].

The rank of a twisted vector bundle is a multiple of the order of $\delta(P)$. This can be seen readily as follows. Suppose that a twisted vector bundle has a finite rank r . Taking the determinant of the twisted cocycle condition, we have:

$$\det\phi_{\alpha\beta}\det\phi_{\beta\gamma} = c_{\alpha\beta\gamma}^r \det\phi_{\alpha\gamma}.$$

Hence $(c_{\alpha\beta\gamma}^r) \in \check{Z}^2(\mathcal{U}, U(1))$ is a coboundary and $r\delta(P) = 0$.

Because of the above remark, there are no no-trivial twisted vector bundles in the case where $\delta(P)$ is infinite order. So we cannot use twisted vector bundles of finite dimensions to realize $K_P(X)$ generally. In spite of this fact, collections of locally defined vector bundles seem to have the potential in defining $K_P(X)$ by means of finite-dimensional objects. An approach is to use the usual technique proving the isomorphism $K(X) \cong [X, \mathcal{F}(\mathcal{H})]$. In this approach, however, some complications prevent us from transparent management, in particular, in giving equivalence relation. The usage of Furuta's generalized vector bundle provides a more efficient approach, which we explain in the next section.

3 Furuta's generalized vector bundle

In this section, we explain a generalization of the notion of vector bundles introduced by M. Furuta [10]. We call the generalized vector bundles *vectorial bundles* for short. This notion is closely related to a finite-dimensional approximation of Fredholm operators. Applying these notions, we approach to our problem of defining twisted K -theory finite-dimensionally.

3.1 Approximation of a Fredholm operator

We begin with the simplest situation. A \mathbb{Z}_2 -graded vectorial bundle over a single point is a pair (E, h) consisting of:

- a \mathbb{Z}_2 -graded Hermitian vector space $E = E^0 \oplus E^1$ of finite rank; and
- a Hermitian map $h : E \rightarrow E$ of degree 1.

By using a \mathbb{Z}_2 -graded vectorial bundle over a point, we can approximate a single Fredholm operator as follows. Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a Fredholm operator. For simplicity, we assume the kernel or cokernel of A is non-trivial. We define the \mathbb{Z}_2 -graded Hilbert space $\hat{\mathcal{H}}$ by $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$, and the self-adjoint Fredholm operator $\hat{A} : \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}}$ of degree 1 by $\hat{A} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$. By the assumption, the spectrum $\sigma(\hat{A}^2)$ of the non-negative operator \hat{A}^2 contains 0. Since \hat{A}^2 is also Fredholm, $0 \in \sigma(\hat{A}^2)$ is a discrete spectrum. Hence there is a positive number μ such that:

- $\mu \notin \sigma(\hat{A}^2)$;
- the subset $\sigma(\hat{A}^2) \cap [0, \mu)$ consists of a finite number of eigenvalues;
- the eigenspace $\text{Ker}(\hat{A}^2 - \lambda)$ is finite dimensional for $\lambda \in \sigma(\hat{A}^2) \cap [0, \mu)$.

Let $0 = \lambda_1 < \lambda_2 < \dots < \lambda_n < \mu$ be the distinct eigenvalues in $\sigma(\hat{A}^2) \cap [0, \mu)$. Then we have the following orthogonal decomposition of $\hat{\mathcal{H}}$:

$$\hat{\mathcal{H}} = (\hat{\mathcal{H}}, \hat{A})_{\lambda_1} \oplus (\hat{\mathcal{H}}, \hat{A})_{\lambda_2} \oplus \dots \oplus (\hat{\mathcal{H}}, \hat{A})_{\lambda_n} \oplus \hat{\mathcal{H}}',$$

where $(\hat{\mathcal{H}}, \hat{A})_{\lambda} = \text{Ker}(\hat{A}^2 - \lambda)$ is the eigenspace of \hat{A}^2 with its eigenvalue λ , and $\hat{\mathcal{H}}'$ is the orthogonal complement. Notice that \hat{A} preserves each eigenspace as well as the orthogonal complement. More precisely, \hat{A} restricts to the trivial map on $(\hat{\mathcal{H}}, \hat{A})_{\lambda_1} \cong \text{Ker}A \oplus \text{ker}A^*$, while \hat{A} induces isomorphisms on $(\hat{\mathcal{H}}, \hat{A})_{\lambda_i}$ for $i > 1$ and $\hat{\mathcal{H}}'$.

Now, cutting off the infinite-dimensional part $\hat{\mathcal{H}}'$, we define E and h by $E = \bigoplus_i (\hat{\mathcal{H}}, \hat{A})_{\lambda_i}$ and $h = \hat{A}|_E$. The pair (E, h) is nothing but a \mathbb{Z}_2 -graded vectorial bundle over a point.

As a finite-dimensional approximation of a single Fredholm operator, we obtained a \mathbb{Z}_2 -graded vectorial bundle over a single point. As will be explained in Subsection 3.3, a similar approximation is possible for a family of Fredholm operators parameterized by X . The resulting object is a \mathbb{Z}_2 -graded vectorial bundle over X .

3.2 Definition of vectorial bundle

We now introduce \mathbb{Z}_2 -graded vectorial bundles:

Definition 3.1. Let $(\mathcal{U}, (E_{\alpha}, h_{\alpha}), \phi_{\alpha\beta})$ be the following data:

- an open cover $\mathcal{U} = \{U_{\alpha}\}$ of X ;
- \mathbb{Z}_2 -graded Hermitian vector bundles E_{α} over U_{α} ;
- Hermitian maps $h_{\alpha} : E_{\alpha} \rightarrow E_{\alpha}$ of degree 1;
- vector bundle maps $\phi_{\alpha\beta} : E_{\beta}|_{U_{\alpha\beta}} \rightarrow E_{\alpha}|_{U_{\alpha\beta}}$ of degree 0 over $U_{\alpha\beta}$ such that $h_{\alpha}\phi_{\alpha\beta} = \phi_{\alpha\beta}h_{\beta}$.

A \mathbb{Z}_2 -graded vectorial bundle over X is defined to be data $(\mathcal{U}, (E_{\alpha}, h_{\alpha}), \phi_{\alpha\beta})$ satisfying the following conditions:

$$\begin{aligned} \phi_{\alpha\alpha} &\doteq 1 && \text{on } U_{\alpha}, \\ \phi_{\alpha\beta}\phi_{\beta\gamma} &\doteq \phi_{\alpha\gamma} && \text{on } U_{\alpha\beta\gamma}. \end{aligned}$$

In the above definition, the symbol \doteq stands for an equivalence relation. The meaning of the first condition $\phi_{\alpha\alpha} \doteq 1$ is as follows.

For any point $x \in U_{\alpha}$, there are a neighborhood $V \subset U_{\alpha}$ of x and a positive number μ such that: for all $y \in V$ and $v \in (E_{\alpha}, h_{\alpha})_{y, < \mu}$ we have $\phi_{\alpha\alpha}(v) = v$.

Here $(E_\alpha, h_\alpha)_{y, < \mu}$ is the subspace in the fiber of E_α at y given by the direct sum of eigenspaces of $(h_\alpha)_y^2$ whose eigenvalues are less than μ :

$$(E_\alpha, h_\alpha)_{y, < \mu} = \bigoplus_{\lambda < \mu} \text{Ker}((h_\alpha)_y^2 - \lambda) = \bigoplus_{\lambda < \mu} \{v \in (E_\alpha)_y \mid (h_\alpha)_y^2 v = \lambda v\}.$$

The meaning of the second condition $\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq \phi_{\alpha\gamma}$ is now obvious.

Definition 3.2. Let $\mathbb{E} = (\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ and $\mathbb{E}' = (\mathcal{U}', (E'_\alpha, h'_\alpha), \phi'_{\alpha\beta})$ be \mathbb{Z}_2 -graded vectorial bundles over X .

(a) A set (f_α) of vector bundle maps $f_\alpha : E_\alpha \rightarrow E'_\alpha$ of degree 0 such that $f_\alpha h_\alpha = h'_\alpha f_\alpha$ on U_α is said to be a *homomorphism* from \mathbb{E} to \mathbb{E}' , if we have $f_\alpha \phi_{\alpha\beta} \doteq \phi'_{\alpha\beta} f_\beta$ on $U_{\alpha\beta}$.

(b) A homomorphism $(f_\alpha) : \mathbb{E} \rightarrow \mathbb{E}'$ is said to be an *isomorphism*, if there exists a homomorphism $(f'_\alpha) : \mathbb{E}' \rightarrow \mathbb{E}$ such that $f_\alpha f'_\alpha \doteq 1$ and $f'_\alpha f_\alpha \doteq 1$ on U_α .

In the above definition of homomorphism, \mathbb{E} and \mathbb{E}' share the same open cover. In the case where they have different open covers \mathcal{U} and \mathcal{U}' respectively, it suffices to take a common refinement of \mathcal{U} and \mathcal{U}' .

Definition 3.3. A *homotopy* between \mathbb{Z}_2 -graded vectorial bundles \mathbb{E} and \mathbb{E}' over X is defined to be a \mathbb{Z}_2 -graded vectorial bundle $\tilde{\mathbb{E}}$ over $X \times [0, 1]$ such that \mathbb{E} and \mathbb{E}' are isomorphic to $\tilde{\mathbb{E}}|_{X \times \{0\}}$ and $\tilde{\mathbb{E}}|_{X \times \{1\}}$, respectively.

We write $KF(X)$ for the set of homotopy classes of isomorphism classes of \mathbb{Z}_2 -graded vectorial bundles. The set $KF(X)$ gives rise to a group by means of the direct sum of vector bundles given locally.

A \mathbb{Z}_2 -graded (ordinary) vector bundle E gives an example of a \mathbb{Z}_2 -graded vectorial bundle by setting $\mathcal{U} = \{X\}$ and $h = 0$. This construction induces a well-defined homomorphism $K(X) \rightarrow KF(X)$. In [10], Furuta proved:

Proposition 3.4. *The homomorphism $K(X) \rightarrow KF(X)$ is an isomorphism.*

3.3 Approximation of a family of Fredholm operators

As a family version of the construction in Subsection 3.1, we can show:

Lemma 3.5. *Let $A = \{A_x\} : X \rightarrow \mathcal{F}(\mathcal{H})$ be a continuous map. For any point $p \in X$, there are a neighborhood U_p of p and a positive number μ_p such that the following family of vector spaces gives rise to a vector bundle over U_p :*

$$\bigcup_{x \in U_p} (\hat{\mathcal{H}}, \hat{A}_x)_{< \mu_p} = \bigcup_{x \in U_p} \bigoplus_{\lambda < \mu_p} \text{Ker}(\hat{A}_x^2 - \lambda).$$

A key to this lemma is that eigenvalues of A_x is continuous in x .

By means of the lemma, the family of Fredholm operators $A : X \rightarrow \mathcal{F}(\mathcal{H})$ yields a \mathbb{Z}_2 -graded vectorial bundle $(\{U_p\}_{p \in X}, (E_{U_p}, h_{U_p}), \phi_{U_p U_q})$, where the \mathbb{Z}_2 -graded vector bundle E_{U_p} is that in Lemma 3.5, the Hermitian map h_{U_p} is given by restricting the Fredholm operator: $h_{U_p}|_x = \hat{A}_x|_{E_{U_p}}$, and the map of vector bundles $\phi_{U_p U_q} : E_{U_q} \rightarrow E_{U_p}$ is the following composition of the natural inclusion and the orthogonal projection:

$$\bigcup_{x \in U_p \cap U_q} (\hat{\mathcal{H}}, \hat{A}_x)_{< \mu_q} \longrightarrow (U_p \cap U_q) \times \hat{\mathcal{H}} \longrightarrow \bigcup_{x \in U_p \cap U_q} (\hat{\mathcal{H}}, \hat{A}_x)_{< \mu_p}.$$

The construction above induces a well-defined homomorphism

$$\alpha : [X, \mathcal{F}(\mathcal{H})] \longrightarrow KF(X).$$

This homomorphism is compatible with the isomorphism $\text{ind} : [X, \mathcal{F}(\mathcal{H})] \rightarrow K(X)$ in [1]. Namely, the following diagram is commutative:

$$\begin{array}{ccc} [X, \mathcal{F}(\mathcal{H})] & \xlongequal{\quad} & [X, \mathcal{F}(\mathcal{H})] \\ \text{ind} \downarrow & & \downarrow \alpha \\ K(X) & \longrightarrow & KF(X). \end{array}$$

The compatibility follows from the fact that one can realize any vector bundle $E \rightarrow X$ as $E = \bigcup_{x \in X} \text{Ker} \hat{A}_x$ by taking $A : X \rightarrow \mathcal{F}(\mathcal{H})$ such that $\sigma(\hat{A}_x^2) = \{0, 1\}$. (See the proof of the surjectivity of ind in [1].)

3.4 Twisted vectorial bundle and twisted K -theory

We now apply vectorial bundles and finite dimensional approximations explained so far to twisted K -theory.

Recall that twisted vector bundles are defined by “twisting” the ordinary cocycle condition for vector bundles. In a similar way, for a $PU(\mathcal{H})$ -bundle P , we define a *twisted \mathbb{Z}_2 -graded vectorial bundle* by replacing the “cocycle condition” $\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq \phi_{\alpha\gamma}$ in Definition 3.1 by the “twisted cocycle condition”:

$$\phi_{\alpha\beta}\phi_{\beta\gamma} \doteq c_{\alpha\beta\gamma}\phi_{\alpha\gamma}.$$

A twisted \mathbb{Z}_2 -graded vectorial bundle can be constructed from a section $\mathbb{A} : X \rightarrow P \times_{Ad} \mathcal{F}(\mathcal{H})$. The section gives a set of maps $\{A_p : W_p \rightarrow \mathcal{F}(\mathcal{H})\}_{p \in X}$ such that $A_p = g_{pq} A_q g_{pq}^{-1}$, where W_p is an open set containing p and $g_{pq} : W_p \cap W_q \rightarrow U(\mathcal{H})$ is a lift of transition function of P . Now, we use Lemma 3.5 to define a Hermitian vector bundle over $U_p \subset W_p$ by $E_{U_p} = \bigcup_{x \in U_p} (\hat{\mathcal{H}}, (\hat{A}_p)_x)_{< \mu_p}$. The

map A_p also defines a Hermitian map h_{U_p} on E_{U_p} by restriction. If we define $\phi_{U_p U_q} : E_{U_q} \rightarrow E_{U_p}$ by the following composition:

$$\bigcup_{x \in U_{pq}} (\hat{\mathcal{H}}, (\hat{A}_q)_x)_{<\mu_q} \rightarrow U_{pq} \times \hat{\mathcal{H}} \xrightarrow{\text{id} \times g_{pq}} U_{pq} \times \hat{\mathcal{H}} \rightarrow \bigcup_{x \in U_{pq}} (\hat{\mathcal{H}}, (\hat{A}_p)_x)_{<\mu_p},$$

then $(\{U_p\}, (E_{U_p}, h_{U_p}), \phi_{U_p U_q})$ is a twisted \mathbb{Z}_2 -graded vectorial bundle.

Introducing isomorphisms and homotopies in a similar way, we obtain the group $KF_P(X)$ of homotopy classes of isomorphism classes of twisted \mathbb{Z}_2 -graded vectorial bundles. The above construction of twisted vectorial bundles induces the well-defined homomorphism

$$\alpha : K_P(X) \longrightarrow KF_P(X).$$

Since this map generalizes $\alpha : [X, \mathcal{F}(\mathcal{H})] \rightarrow KF(X)$, it is reasonable to expect that α gives rise to an isomorphism. In fact, we have:

Theorem 3.6. *For any $PU(\mathcal{H})$ -bundle P over a compact manifold X , the homomorphism $\alpha : K_P(X) \longrightarrow KF_P(X)$ is bijective.*

We sketch the proof of this result in the next subsection.

3.5 Sketch of the proof of Theorem 3.6

The fundamental idea to prove Theorem 3.6 is to construct a kind of generalized cohomology theory on CW complexes by means of $KF_P(X)$.

As is known [2, 7], the twisted K -group $K_P(X)$ fits into a certain generalized cohomology theory $\{K_P^{-n}(X, Y)\}_{n \in \mathbb{Z}}$. In particular, for a CW pair (X, Y) equipped with a $PU(\mathcal{H})$ -bundle $P \rightarrow X$, we have the long exact sequence:

$$\dots \rightarrow K_{P|_Y}^{-n-1}(Y) \xrightarrow{\delta_{-n-1}} K_P^{-n}(X, Y) \rightarrow K_P^{-n}(X) \rightarrow K_{P|_Y}^{-n}(Y) \xrightarrow{\delta_{-n}} \dots$$

Note that we can identify $K_P^1(X, Y)$ with $K_P^{-1}(X, Y) = K_{P \times I}(X \times I, Y \times I \cup X \times \partial I)$, and $\delta_0 : K_{P|_Y}^0(Y) \rightarrow K_P^1(X, Y)$ with the composition of the following maps:

$$K_{P|_Y}^0(Y) \xrightarrow{\beta} K_{P|_Y \times D^2}^0(Y \times D^2, Y \times S^1) = K_{P|_Y}^{-2}(Y) \xrightarrow{\delta_{-2}} K_P^{-1}(X, Y).$$

Here β induces the Bott periodicity, and is given by ‘‘multiplying’’ a map $T : D^2 \rightarrow \mathcal{F}(\mathcal{H})$ representing the generator of $K(D^2, S^1)$.

To construct a similar cohomology theory, we define $KF_P(X, Y)$ by using twisted \mathbb{Z}_2 -graded vectorial bundles on X whose support do not intersect Y . (The *support* of a twisted \mathbb{Z}_2 -graded vectorial bundle $\mathbb{E} = (\mathcal{U}, (E_\alpha, h_\alpha), \phi_{\alpha\beta})$ on X is

the closure of the points $x \in X$ such that $(h_\alpha)_x$ is not invertible for an α .) For $n \geq 0$, we put:

$$KF_P^{-n}(X, Y) = KF_{P \times I^n}(X \times I^n, Y \times I^n \cup X \times \partial I^n).$$

Then $KF_P^{-n}(X, Y)$ satisfies the (suitably modified) homotopy axiom and the excision axiom in the Eilenberg-Steenrod axioms. In a way parallel to the method in [1], we can also introduce a natural map $\delta_{-n} : KF_{P|Y}^{-n}(Y) \rightarrow KF_P^{-n+1}(X, Y)$, and obtain the long exact sequence for a pair:

$$\cdots \rightarrow KF_P^{-1}(X) \rightarrow KF_{P|Y}^{-1}(Y) \xrightarrow{\delta_{-1}} KF_P^0(X, Y) \rightarrow KF_P^0(X) \rightarrow KF_{P|Y}^0(Y).$$

To extend this sequence, we put $KF_P^1(X, Y) = KF_P^{-1}(X, Y)$ and define $\delta_0 : KF_{P|Y}^0(Y) \rightarrow KF_P^1(X, Y)$ to be the composition of:

$$KF_{P|Y}^0(Y) \xrightarrow{\beta} KF_{P|Y \times D^2}^0(Y \times D^2, Y \times S^1) = KF_{P|Y}^{-2}(Y) \xrightarrow{\delta_{-2}} KF_P^{-1}(X, Y),$$

where β is given by tensoring a vector bundle representing the generator of $K(D^2, S^1)$. Then the composition of $KF_P^0(X) \rightarrow KF_{P|Y}^0(Y) \rightarrow KF_P^1(X, Y)$ is trivial. (This sequence is not yet shown to be exact at this stage.)

The spaces $X \times I^n$ and $Y \times I^n \cup X \times \partial I^n$ in the definition of $KF_P^{-n}(X, Y)$ are used in that of $K_P^{-n}(X, Y)$. Hence the finite-dimensional approximation induces the natural homomorphism $\alpha_{-n} : K_P^{-n}(X, Y) \rightarrow KF_P^{-n}(X, Y)$ for $n \geq -1$. We can readily see that δ_{-n} ($n \geq 1$) commutes with α_{-n} , since δ_{-n} is essentially defined by an inclusion map of spaces. If X is compact, then β commutes with α_{-n} , so that δ_0 does. The key to this fact is that the compactness allows us to choose a map $T : D^2 \rightarrow \mathcal{F}(\mathcal{H})$ realizing the generator of $K(D^2, S^1)$ in a way appropriate for the finite-dimensional approximation.

Now, for a finite CW complex X and a $PU(\mathcal{H})$ -bundle $P \rightarrow X$, we can prove the bijectivity of $\alpha_{-n} : K_P^{-n}(X) \rightarrow KF_P^{-n}(X)$ ($n \geq 0$) by the induction on the number of cells in X . Notice that, if P is trivial, then an argument by using Proposition 3.4 implies the bijectivity of α_{-n} ($n \geq 0$). Thus, α_{-n} ($n \geq 0$) is bijective in the case where X is a point and has only one cell. If the number of the cells in X is $r > 1$, then we can express X as $X = Y \cup e^q$, where $Y \subset X$ is a subcomplex with $(r-1)$ cells, and $e^q \subset X$ is a cell of dimension q . We here make use of the commutative diagram ($n \geq 0$):

$$\begin{array}{ccccccccc} K_{P|Y}^{-n-1}(Y) & \rightarrow & K_P^{-n}(X, Y) & \rightarrow & K_P^{-n}(X) & \rightarrow & K_{P|Y}^{-n}(Y) & \rightarrow & K_P^{-n+1}(X, Y) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ KF_{P|Y}^{-n-1}(Y) & \rightarrow & KF_P^{-n}(X, Y) & \rightarrow & KF_P^{-n}(X) & \rightarrow & KF_{P|Y}^{-n}(Y) & \rightarrow & KF_P^{-n+1}(X, Y). \end{array}$$

We can assume that the first and the fourth columns are bijective in the induction. The excision axiom implies that $K_P^{-n}(X, Y) \cong K_P^{-n}(D^q, S^{q-1})$ and $KF_P^{-n}(X, Y) \cong KF_P^{-n}(D^q, S^{q-1})$. Since any $PU(\mathcal{H})$ -bundle over D^q is trivial, the second and fifth columns are also bijective. Thus, so is the third column. (The exactness at $KF_{P|Y}^{-n}(Y)$ is not necessary in the five-term lemma.)

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Kiyonori Gomi
Graduate school of Mathematical Sciences, The University of Tokyo,
Komaba 3-8-1, Meguro-Ku, Tokyo, 153-8914 Japan.
e-mail: kgomi@ms.u-tokyo.ac.jp

Current address:
The Department of Mathematics, The University of Texas at Austin
1 University Station C1200, Austin, TX 78712-0257 USA
e-mail: kgomi@math.utexas.edu

