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$\mathbb{Z}_n$ bundle gerbes [1]

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Abstract

A notion of $\mathbb{Z}_n$ bundle gerbes over a manifold is formulated. For an n root bundle gerbe the characteristic map is c_1 *mod* n, where c_1 is the Chern class of the associated $\mathbb{C}^\times$ bundle. $\mathbb{Z}_n$ bundle gerbes over a manifold M are classified completely by the cohomology $H^2(M;\mathbb{Z}_n)$. Bundle gerbes for a finitely generated abelian group are also considered.

1 Introduction

In order to study gerbes of J. Giraud [5] and J.-L. Brylinski [[3] 5.2.4. Definition p.196] with a band of the sheaf of discrete abelian group, we introduce a notion of $\mathbb{Z}_n$ bundle gerbe by the parallel argument of $\mathbb{C}^\times$ bundle gerbes due to M. K. Murray [7]. We generalize the base space of a $\mathbb{Z}_n$ bundle groupoid to a fibered space $\pi : Y \to M$ in order to get a notion of the $\mathbb{Z}_n$ bundle gerbe $G(n, M)$ over M. Then its triviality and a product operation of two $\mathbb{Z}_n$ bundle gerbes are defined. The main results of the present paper are the followings.

Theorem 2.1. *For a manifold M, we have a map $c : \{G(n, M)\} \to H^2(M;\mathbb{Z}_n)$ from the set of isomorphism classes of $\mathbb{Z}_n$ bundle gerbes to the 2-cohomology of M with coefficients in $\mathbb{Z}_n$. c is precisely the obstruction to a $\mathbb{Z}_n$ bundle gerbe being trivial.*

Just as the arguments by M. K. Murray and D. Stevenson [8], it is shown that $\mathbb{Z}_n$ bundle gerbes are gerbes. Firstly, n root bundle gerbes $G(n\surd, M)$ are defined by choosing a $\mathbb{C}^\times$ bundle $Y^{\mathbb{C}^\times}$ as the fibered space $\pi : Y \to M$ and relating the $\mathbb{Z}_n$ bundle groupoid over a fiber to the central extension

$$1 \to \mathbb{Z}_n \xrightarrow{\iota} \mathbb{C}^\times \xrightarrow{(\ )^n} \mathbb{C}^\times \to 1.$$

We denote the Chern class $c_1(Y^{\mathbb{C}^\times}) \in H^2(M;\mathbb{Z})$ by $c_1(G(n\surd, M))$. Then we have

Theorem 4.1. *For any n root bundle gerbe $G(n\surd, M)$ we get*

$$c(G(n\surd, M)) = c_1(G(n\surd, M)) \ mod\ n.$$

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The Maslov gerbe of A. Weinstein [11] is a $\mathbb{Z}_2$ bundle gerbe, more explicitly a square root bundle gerbe.

Secondly, besides n root bundle gerbes, projective $\mathbb{Z}_n$ bundle gerbes $G(PU(n), M)$ are defined by choosing a $PU(n)$ bundle $Y^{PU(n)}$ as the fibered space $\pi : Y \to M$ and relating the $\mathbb{Z}_n$ bundle groupoid over a fiber to the central extension

$$1 \to \mathbb{Z}_n \to SU(n) \xrightarrow{\rho} PU(n) \to 1,$$

where $\rho : SU(n) \subset U(n) \to PU(n) \cong U(n)/U(1)$ is the quotient map. The notion of stable isomorphism of two Z_n bundle gerbes is introduced and the class of gerbes stably equivalent (just Morita equivalent) to $G(n, M)$ is denoted by $G_S(n, M)$. The map c of Theorem 2.1 induces an injective homomorphism $c_S : \{G_S(n, M)\} \to H^2(M; \mathbb{Z}_n)$, called characteristic homomorphism.

Theorem 5.3. *Suppose that M is a manifold. Then the characteristic homomorphism restricted to $\{G_S(PU(n), M)\}$ is an isomorphism onto $Tor(H^3(M; \mathbb{Z}), \mathbb{Z}_n)$. Moreover the c_S is an isomorphism $\{G_S(n, M)\} \cong H^2(M; \mathbb{Z}_n)$.*

For a finitely generated abelian group D, one can consider D bundle gerbes and by the above theorem, the Morita equivalence classes $\{G_S(D, M)\}$ over a manifold M are completely classified by $H^2(M; D)$.

In Section 2, we define $\mathbb{Z}_n$ bundle gerbes $G(n, M)$ over M and prove Theorem 2.1. We obtain the characteristic homomorphism c_S of Morita equivalence classes $\{G_S(n, M)\}$ to $H^2(M; \mathbb{Z}_n)$. In Section 3, we explain that $\mathbb{Z}_n$ bundle gerbes are gerbes in the sense of Giraud. In Section 4, we examine structures of the principal $\mathbb{C}^\times$ bundles associated with n root bundle gerbes and prove Theorem 4.1. In Section 5, we examine structures of the principal $PU(n)$ bundles associated to projective $\mathbb{Z}_n$ bundle gerbes by the relation to Azumaya bundles of V. Mathai, R. B. Melrose and I. M. Singer [9], and P. Donovan and M. Karoubi [4]. Then we prove Theorem 5.3. In the last section, we show briefly the conclusion on the Morita equivalence classes of D bundle gerbes over a manifold for a finitely generated abelian group D.

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2 $\mathbb{Z}_n$ bundle gerbes

Let $\mathbb{Z}_n$ denote the cyclic group of order n and let $p : P \to X$ be a principal $\mathbb{Z}_n$ bundle over a space X. Let $\mathbb{Z}_n$ acts on $P \times P$ to the right by $(u_1, u_2)g = (u_1 g, u_2 g)$. We denote the orbit of $(u_1, u_2)g$ by $< u_1, u_2 >$ and the set of orbits by $(P \times P)/\mathbb{Z}_n$. Then one gets a groupoid $(P \times P)/\mathbb{Z}_n \rightrightarrows X$ with respect to the following structure: The source and target projections are $\alpha < u_1, u_2 >= p(u_2)$ and $\beta < u_1, u_2 >= p(u_1)$; the object inclusion map $x = \tilde{x} =< u, u >$ where u is any element of $p^{-1}(x)$; and the partial multiplication $P \circ P \to P$ is defined by $< u_1, u_2' >< u_2, u_3 >=< u_1, u_3 \delta(u_2', u_2) >$, where $\delta : P \times P \to \mathbb{Z}_n$ is the map $(ug, u) \mapsto g$. The inverse of $< u_1, u_2 >$ is $< u_2, u_1 >$. This is the groupoid associated to $p : P \to X$ (cf. K. Mackenzie [6] (p.5-p.6)), which we call a $\mathbb{Z}_n$ *groupoid*.

Let $\pi : Y \to M$ be a fibration over a manifold M. We consider a $\mathbb{Z}_n$ bundle $P \to Y^{[2]} = Y \times_M Y$, such that for each $m \in M$, the restriction bundle $P|_{Y_m^2}$ is identified with the groupoid space $P_m \otimes P_m = (P_m \times P_m)/\mathbb{Z}_n$ associated with the $\mathbb{Z}_n$ bundle $P_m \to Y_m$ where $\pi^{-1}(m) = Y_m$ and $P_m = d_m^{-1}(P)$ with the diagonal map $d_m : Y_m \to Y^{[2]}$, $y \mapsto (y, y)$. The groupoid product $P_m \circ P_m \to P_m$ is naturally extended to a $\mathbb{Z}_n$ bundle isomorphism $P \circ P \to P$ covering the product $(y_1, y_2)(y_2, y_3) = (y_1, y_3)$ in $Y^{[2]}$.

A $\mathbb{Z}_n$ *bundle gerbe* $G(n, M)$ over M is defined to be a choice of a fibration $\pi : Y \to M$ and a $\mathbb{Z}_n$ bundle $P \to Y^{[2]}$ with a product, that is, a $\mathbb{Z}_n$ bundle isomorphism $P \circ P \to P$ covering the product $(y_1, y_2)(y_2, y_3) = (y_1, y_3)$. The product is associative whenever triple products are defined. Just as for $\mathbb{Z}_n$ groupoids, a $\mathbb{Z}_n$ bundle gerbe has an inverse and an identity denoted by the same symbols. Let $Q \to Y$ be a principal $\mathbb{Z}_n$ bundle. A $\mathbb{Z}_n$ bundle gerbe P is defined by $P_{(x,y)} = Aut_{\mathbb{Z}_n}(Q_x, Q_y) = Q_x^* \otimes Q_y$ where Q^* is the inverse bundle of Q. Then P is called the *trivial* bundle gerbe. We also have $P = \pi_1^{-1} Q^* \otimes \pi_2^{-1} Q$.

If (P, Y, M) and (Q, X, M) are $\mathbb{Z}_n$ bundle gerbes over M, we can form a fiber product $Y \times_M X \to M$ and then form a $\mathbb{Z}_n$ bundle $P \otimes Q$ over $(Y \times_M X)^{[2]}$ which is the *product* of the gerbes (P, Y, M) and (Q, X, M). For triple $\mathbb{Z}_n$ bundle gerbes, this product is associative.

Let $P \to Y^{[2]}$ be a $\mathbb{Z}_n$ bundle gerbe. Choose an open cover $\{U_\alpha\}$ of M such that over each U_α there is a section s_α of Y. Then on the overlap $U_\alpha \cap U_\beta$ we have a map $(s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$ defined by $(s_\alpha, s_\beta)(x) = (s_\alpha(x), s_\beta(x))$. As examples of Y, we mention a $\mathbb{C}^\times$ bundle in Section 4 and a $PU(n)$ bundle in M. F. Atiyah [1].

Theorem 2.1. *For a manifold M, we have a map $c : \{G(n, M)\} \to H^2(M; \mathbb{Z}_n)$ from the set of isomorphism classes of $\mathbb{Z}_n$ bundle gerbes to the 2-cohomology of M with coefficients in $\mathbb{Z}_n$. c is precisely the obstruction to a $\mathbb{Z}_n$ bundle gerbe being trivial.*

Proof. By a parallel argument to the $\mathbb{C}^\times$ bundle gerbe in [7], we define a "characteristic map" from $\{G(n,M)\}$ to $H^2(M;\mathbb{Z}_n)$ as follows: Let $P_{\alpha\beta}$ be the pull-back of P via the map $(s_\alpha, s_\beta) : U_\alpha \cap U_\beta \to Y^{[2]}$. The product in P gives an isomorphism $P_{\alpha\beta} \otimes P_{\beta\gamma} \cong P_{\alpha\gamma}$. Choose sections $\sigma_{\alpha\beta}$ of each $P_{\alpha\beta}$. Then the product gives a $\mathbb{Z}_n$ valued function

$$g_{\alpha\beta\gamma} : U_\alpha \cap U_\beta \cap U_\gamma \to \mathbb{Z}_n$$

defined by $\sigma_{\alpha\beta}\sigma_{\beta\gamma} = \sigma_{\alpha\gamma}g_{\alpha\beta\gamma}$.

By making use of comutativity of $\mathbb{Z}_n$ action with $\sigma_{\alpha\beta}$'s one gets

$$\begin{aligned}
g_{\beta\gamma\delta}g^{-1}_{\alpha\gamma\delta}g_{\alpha\beta\delta}g^{-1}_{\alpha\beta\gamma} &= (\sigma^{-1}_{\beta\delta}\sigma_{\beta\gamma}\sigma_{\gamma\delta})(\sigma^{-1}_{\alpha\delta}\sigma_{\alpha\gamma}\sigma_{\gamma\delta})^{-1}(\sigma^{-1}_{\alpha\delta}\sigma_{\alpha\beta}\sigma_{\beta\delta})(\sigma^{-1}_{\alpha\gamma}\sigma_{\alpha\beta}\sigma_{\beta\gamma})^{-1} \\
&= (\sigma^{-1}_{\beta\delta}\sigma_{\beta\gamma}\sigma_{\gamma\delta})(\sigma^{-1}_{\gamma\delta}\sigma^{-1}_{\alpha\gamma}\sigma_{\alpha\delta})(\sigma^{-1}_{\alpha\delta}\sigma_{\alpha\beta}\sigma_{\beta\delta})(\sigma^{-1}_{\beta\gamma}\sigma^{-1}_{\alpha\beta}\sigma_{\alpha\gamma}) \\
&= (\sigma^{-1}_{\beta\delta}\sigma_{\beta\gamma}\sigma^{-1}_{\beta\gamma}\sigma^{-1}_{\alpha\beta}\sigma_{\alpha\gamma})\sigma^{-1}_{\alpha\gamma}\sigma_{\alpha\beta}\sigma_{\beta\delta} \\
&= \sigma^{-1}_{\beta\delta}\sigma_{\beta\delta} \\
&= 1.
\end{aligned}$$

Therefore $g = \{g_{\alpha\beta\gamma}\}$ is a Čech cocycle of the open cover of M with respect to $\mathbb{Z}_n$.

Let $P \to Y^{[2]}$ and $P' \to (Y')^{[2]}$ be isomorphic $\mathbb{Z}_n$ bundle gerbes over M. Since $Y \to M$ and $Y' \to M$ sre isomorphic too, one can regards $\{s_\alpha\}$ and $\{s'_\alpha\}$ are the same upto a bundle ismorphism. Let $\sigma'_{\alpha\beta}$ be a section of $P'_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ and let $h_{\alpha\beta}$ denote $\sigma_{\alpha\beta}(\sigma'_{\alpha\beta})^{-1}$. The cocycle $g'_{\alpha\beta\gamma}$ is defined by $\sigma'_{\alpha\beta}\sigma'_{\beta\gamma} = \sigma'_{\alpha\gamma}g'_{\alpha\beta\gamma}$. Then we have

$$\begin{aligned}
\sigma_{\alpha\beta}\sigma_{\beta\gamma}(\sigma'_{\beta\gamma})^{-1}(\sigma'_{\alpha\beta})^{-1} &= (\sigma_{\alpha\gamma}g_{\alpha\beta\gamma})(\sigma'_{\alpha\gamma}g'_{\alpha\beta\gamma})^{-1} \\
&= \sigma_{\alpha\gamma}(\sigma'_{\alpha\gamma})^{-1}g_{\alpha\beta\gamma}(g'_{\alpha\beta\gamma})^{-1}.
\end{aligned}$$

The last equation shows that

$$\partial h = g(g')^{-1},$$

for $h = \{h_{\alpha\beta}\}$, that is,

$$[g] = [g'] \in H^2(M;\mathbb{Z}_n).$$

Therefore the map $c : \{G(n,M)\} \to H^2(M;\mathbb{Z}_n)$ is well defined.

To prove the second part of the theorem, suppose that P is trivial, say $P = \pi_1^{-1}Q^* \otimes \pi_2^{-1}Q$ for some bundle $Q \to Y$. We can define $Q_\alpha = s^*_\alpha(Q)$ and we have a canonical isomorhpism $P_{\alpha\beta} = Q^*_\alpha \otimes Q_\beta$ commuting with products. If we choose a section δ_α of Q and define $\sigma_{\alpha\beta} = (\delta_\alpha)^{-1} \otimes \delta_\beta$ we obtain a trivial cocycle g.

If g is trivial, say $g_{\alpha\beta\gamma} = \rho_{\alpha\beta}\rho_{\beta\gamma}\rho_{\gamma\alpha}$, where ρ is $\mathbb{Z}_n$ valued function. One can replace $\sigma_{\alpha\beta}$ by $\sigma_{\alpha\beta}\rho^{-1}_{\alpha\beta}$ and assume without loss ofgenerality that $g \equiv 1$, that is $g_{\alpha\beta\gamma} = 1$. Let $Y_\alpha = \pi^{-1}(U_\alpha)$. Define a principal $\mathbb{Z}_n$ bundle Q_α over Y_α by defining its fiber at y to be $(Q_\alpha)_y = P_{(y,s_\alpha(\pi(y)))}$. The $\sigma_{\alpha\beta}$ are elements of

$$P_{(s_\alpha(\pi(y)),s_\beta(\pi(y)))} = P^*_{(y,s_\alpha(\pi(y)))} \otimes P_{(y,s_\beta(\pi(y)))}$$

$$= \quad (Q^*_\alpha)_y \otimes (Q_\beta)_y.$$

The $\sigma_{\alpha\beta}$ therefore define automorphisms between Q_α and Q_β over $Y_\alpha \cap Y_\beta$. Piecing together, we get a bundle Q over all Y, which trivializes the gerbe P over Y. $\square$

We call the map $c : \{G(n, M)\} \to H^2(M; \mathbb{Z}_n)$ a *characteristic map* and call $c(G(n, M)) \in H^2(M; \mathbb{Z}_n)$ the *characteristic class* of $G(n, M)$. Two $\mathbb{Z}_n$ bundle gerbes $P = (P, Y, M)$ and $Q = (Q, Z, M)$ are called *stably isomorphic* if there are trivial bundle gerbes T_1 and T_2 such that $P \otimes T_1 = Q \otimes T_2$. We see directly that the stable isomorphism is an equivalence relation and product operations are compatible with the equivalence. Let $G_S(n, M)$ denote the stable equivalence class of $\mathbb{Z}_n$ bundle gerbes over M.

Corollary 2.2. *The map* $c : \{G(n, M)\} \to H^2(M; \mathbb{Z}_n)$ *induces an injective homomorphism* $c_S : \{G_S(n, M)\} \to H^2(M; \mathbb{Z}_n)$.

Proof. Since the trivial $\mathbb{Z}_n$ bundle gerbe goes to zero by the homomorphism c and c is additive over tensor products, one gets $c(P) = c(P \otimes T_1) = c(Q \otimes T_2) = c(Q)$. Therefore, c induces a homomorphism $c_S : \{G_S(n, M)\} \to H^2(M; \mathbb{Z}_n)$. If $c(P) = c(Q)$ then we $c(P \otimes Q^*) = c(P) - c(Q) = 0$. Hence $P \otimes Q^*$ is trivial by Theorem 2.1. We see that $Q \otimes Q^*$ is also trivial and that $P \otimes (Q^* \otimes Q) = Q \otimes (P \otimes Q^*)$, so P and Q are stably isomorphic, that is, the homomorphism c_S is injective. $\square$

Remark 2.3. For a trivial bundle gerbe (T, X, M) the map $P \otimes T \to P,\ u \otimes v \mapsto u$ over the Morita morphism $[(Y \times_M X)^{[2]} \rightrightarrows Y \times_M X] \to [Y^{[2]} \rightrightarrows Y]$, $((y_{m,1}, y_{m,2}), (x_{m,1}, x_{m,2})) \mapsto (y_{m,1}, y_{m,2})$ $(m \in M)$ is $\mathbb{Z}_n$ equivariant, therefore the map $P \otimes T \to P$ is a $\mathbb{Z}_n$ eqivariant Morita morphism of Lie groupoids. Hence the stable equivalence of $\mathbb{Z}_n$ bundle gerbes is Morita equivalence of gerbes as central extensions of a groupoid in the sense of J.-L. Tu, P. Xu and C. Laurent-Gengoux [10]. It is easy to see that the converse is true and $G_S(n, M)$ is the set of Morita equivalence classes of $\mathbb{Z}_n$ bundle gerbes.

Remark 2.4. For any abelian group A, A bundle gerbes $G(A, M)$ over M, Morita equivalence classes $G_S(A, M)$ and the injective homomorphism c_S can be defined by replacing $\mathbb{Z}_n$ by A, in the arguments in the above. In the last section we extend our results to the D bundle gerbes for a finitely generated abelian group D.

3 Relationship with $\mathbb{Z}_n$ gerbes

We construct $\mathbb{Z}_n$ gerbes $\mathcal{G}(n, M)$ in the sense of J. Giraud [5] (cf. J.-L. Brylinski [[3] 5.2.4 Definition, p.196] from $\mathbb{Z}_n$ bundle gerbes along the way to get $\mathbb{C}^\times$ gerbes by M. K. Murray and P. Stevenson [8]. Let (P, Y, M) be a $\mathbb{Z}_n$ bundle gerbe over a manifold M. For any open set U in M, we define a category $\mathcal{G}_n(U)$ as follows. The objects of $\mathcal{G}_n(U)$ are the set of all trivializations of the restriction of (P, Y) to U. That is all pairs (Q, f) where Q is a $\mathbb{Z}_n$ bundle over $Y_U = \pi^{-1}(U) \subset Y$ and $f : \pi_1^{-1}(P)^* \otimes \pi_2^{-1}(P) \to Q|_{Y_U^{[2]}}$ is an isomorphism of $\mathbb{Z}_n$ bundle gerbes. The morphisms between two objects (Q, f) and (P, g) are all ismorphisms of $\mathbb{Z}_n$ bundle gerbes which commute with f and g.

Theorem 3.1. *$\mathbb{Z}_n$ bundle gerbes are gerbes.*

Proof. For every open set U, we have a groupoid $\mathcal{G}_n(U)$ which is possibly trivial one. The restriction functor is exctly the trivialization over Y_U to Y_V if $V \subset U$. This makes $\mathcal{G}(n, M)$ a presheaf of groupoids. To show $\mathcal{G}(n, M)$ is a sheaf of groupoids, we need to check two patching conditions on objects and morphisms as in [[3] 5.2.1. Definition (2), p.191]. Assume that we have an open cover $\{U_\alpha\}$ of an open set U. First consider two trivializations (Q_i, f_i) $i = 1, 2$ in $\mathcal{G}_n(U)$ with morphisms $\phi_\alpha : Q_1|_{U_\alpha} \to Q_2|_{U_\alpha}$ for each α agreeing on overlaps. Then these clearly patch together to yield a global morphism ϕ and as the ϕ_α commute with the f_i so also does ϕ. Second assume that we have trvializations (Q_α, f_α) in each $\mathcal{G}_n(U_\alpha)$ and morphisms $\phi_{\alpha\beta} : Q_\alpha|_{U_\alpha \cap U_\beta} \to Q_\beta|_{U_\alpha \cap U_\beta}$ satisfying $\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = 1$. Then by cluching (Q_α, f_α), we can get a global trivialization $(Q, f) \in \mathcal{G}_n(U)$ whose restriction to each U_α is (Q_α, f_α). Hence $\mathcal{G}(n, M)$ is a sheaf of groupoids.

Next, we cosider the conditions of a gerbe in [[3] 5.2.4. Definition, p.196]. For the first condition (G1), assume that $\mathcal{G}_n(U)$ is non-empty. Let (Q, f) be an object in $\mathcal{G}_n(U)$ and consider the automorphisms of (Q, f). If we think of Q first as $\mathbb{Z}_n$ bundle on Y_U then the group of all automorphisms is the group of all maps from Y_U to $\mathbb{Z}_n$. However if we require that they also commute with f, they have to be maps that are constant on the fiber of $\pi : Y \to M$. Hence they are the group of all maps from U into $\mathbb{Z}_n$. Therefore (G1) is satisfied.

For the second condition(G2), let (Q, f) and (R, g) be objects in $\mathcal{G}_n(U)$ and let $z \in U$. We have $Q \otimes R^* = \pi^{-1}(T)$ for some $\mathbb{Z}_n$ bundle T over U. Choosing a contractible neighborhood V of z, we can trivialize T and this induces an isomorphism from $Q|_V$ to $R|_V$ as required. Finally, the third condition (G3) that we can cover M by open sets U such that $\mathcal{G}_n(U)$ is non-empty follows from the fact that we can cover M by open sets over which Y has sections and hence we can trvialize the bundle gerbe locally. □

4 Chern class of $G(n\surd, M)$

We consider a bundle gerbe $G(n, M) = (P, Y, M)$ where the fibered space Y is a $\mathbb{C}^\times$ bundle $Y = Y^{\mathbb{C}^\times} \to M$ and the $\mathbb{Z}_n$ groupoid over each fiber $Y_m \cong \mathbb{C}^\times$ $(m \in M)$

is the gauge groupoid of the central extension

$$1 \to \mathbb{Z}_n \xrightarrow{\iota} \mathbb{C}^\times \xrightarrow{(\ )^n} \mathbb{C}^\times \to 1.$$

We call the $\mathbb{Z}_n$ bundle gerbe $G(n, M) = (P, Y^{\mathbb{C}^\times}, M)$ an n *root bundle gerbe* and denote it by $G(n\surd, M)$. We define the first Chern class $c_1(G(n\surd, M))$ by $c_1(Y^{\mathbb{C}^\times}) \in H^2(M; \mathbb{Z})$.

Theorem 4.1. *For any n root bundle gerbe $G(n\surd, M)$, we get*

$$c(G(n\surd, M)) = c_1(G(n\surd, M)) \ mod \ n.$$

Proof. For sufficiently fine open cover $\mathcal{U} = \{U_\alpha\}$, we choose coordinate transformations $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^\times$ of the local triviality $\{(Y^{\mathbb{C}^\times})_{U_\alpha}\}$. $\phi = \{\phi_{\alpha\beta}\}$ represents the element of the sheaf cohomology $H^1(M; \underline{\mathbb{C}}^\times)$ corresponding to $Y^{\mathbb{C}^\times}$. In the cohomology exact sequence with respect to the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \stackrel{exp \cdot 2\pi i}{\to} \mathbb{C}^\times \to 1,$$

we have the connecting homomorphism $\partial^* : H^1(M; \underline{\mathbb{C}}^\times) \to H^2(M; \mathbb{Z})$ and $\partial^*[\phi] = c_1(Y^{\mathbb{C}^\times})$. From the definition of ∂^*, one gets

$$c_1(Y^{\mathbb{C}^\times}) = [g_{\alpha\beta\gamma}]$$

which is regarded as a value of $log(\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta})$ and is an integer, since $\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta} = 1$. Let $\theta_{\alpha\beta}$ denote the least non-negative value of the imaginary part of $log\phi_{\alpha\beta}$ and set $\sigma_{\alpha\beta} = n\theta_{\alpha\beta} \ mod \ n$. By the exact sequence

$$1 \to \mathbb{Z}_n \xrightarrow{\iota} \mathbb{C}^\times \xrightarrow{(\ )^n} \mathbb{C}^\times \to 1,$$

it follows that $\sigma_{\alpha\beta}$ is an $\{\mathbb{R} \ mod \ n\}$ valued function and

$$\begin{aligned} \sigma_{\alpha\gamma}^{-1}\sigma_{\alpha\beta}\sigma_{\beta\gamma} &= log(\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta}\phi_{\beta\gamma}) \ mod \ n \\ &= log(\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta}) \ mod \ n. \end{aligned}$$

Therefore one can use $\sigma_{\alpha\beta}$ to define the characteristic class c of $\mathbb{Z}_n$ bundle gerbe $G(n\surd, M)$ in Theorem 2.1, that is,

$$\begin{aligned} c(G(n\surd, M)) &= [log(\phi_{\beta\gamma}\phi_{\alpha\gamma}^{-1}\phi_{\alpha\beta})] \ mod \ n \\ &= \partial^*[\phi] \ mod \ n \\ &= c_1(Y^{\mathbb{C}^\times}) \ mod \ n. \end{aligned}$$

□

A. Weinstein [11] considered a square root $\sqrt{\lambda}$ of a complex line bundle λ as $\mathbb{Z}_2$ gerbe to formulate the notion of Maslov gerbe. For each open set $U \subseteq M$, $\sqrt{\lambda}(U)$ is the groupoid whose objects are pair (τ, ι) consisting of a line bundle τ and an isomorphism ι from the tensor square τ^2 to the restriction $\lambda|_U$. A morphism from (τ, ι) to (τ', ι') is a bundle isomorphism $\sigma : \tau \to \tau'$ such that $\iota'\sigma^2\iota^{-1}$ is the identity automorphism of $\lambda|_U$ where σ^2 is the tensor square of σ. Any two objects in $\sqrt{\lambda}(U)$ are isomorphic and the automorphism group of (τ, ι) may be identified with the continuous (hence locally constant) functions on U with values in $\mathbb{Z}_2$.

Proposition 4.2. *A 2 root bundle gerbe $G(\sqrt{\ }, M)$ over M is a square root $\sqrt{\lambda}$ of a line bundle λ over M.*

Proof. Let λ be the line bundle over M associated with the $\mathbb{C}^\times$ bundle $Y = Y^{\mathbb{C}^\times}$. For an open set $U \subset M$, the objects of $\mathcal{G}_2(U)$ with $G(\sqrt{\ }, M)$ are the set of all trivializations of the restriction of (P, Y, M) to U, that is, all pairs (τ, f) where τ is a $\mathbb{Z}_2$ bundle over $Y_U = \pi^{-1}(U) \subset Y$ and $f : \pi_1^{-1}(P)^* \otimes \pi_2^{-1}(P) \to \tau|_{Y_U^{[2]}}$ is an isomorphism of $\mathbb{Z}_2$ bundle gerbes. $\mathcal{G}_2(U)$ is either empty or a groupoid and is non-empty if Y admits a section over U.

Since the trivialization (τ, f) of the (P, Y, M) to U means the compatibility of $P|_{Y_U^{[2]}} \to Y_U^{[2]}$ with the exact sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{\iota} \mathbb{C}^\times \xrightarrow{(\ )^2} \mathbb{C}^\times \to 1,$$

that is, τ has a structure of $\mathbb{Z}_2$ central extension of $\mathbb{C}^\times$ bundle Y_U, we have an isomorphism $\iota = f^{-1} : \tau^2 \xrightarrow{\cong} Y|_U = \lambda_U$. For another trivialization (τ', f') of the restriction of (P, Y, M) to U, we get an isomorphism $\iota' = f'^{-1} : \tau'^2 \xrightarrow{\cong} \lambda_U$ and a morphism between two objects (τ, f) and (τ', f') defines a bundle isomorphism $\sigma \xrightarrow{\cong} \tau'$ such that $\iota'\sigma^2\iota^{-1}$ is the identity automorphism of $\lambda|_U$. Therefore P is the bundle gerbe $\sqrt{\lambda}$. □

5 Projective $\mathbb{Z}_n$ bundle gerbes

Let $PU(n)$ denote the projective unitary group, which is isomorphic to $U(n)/U(1)$. Besides n root bundle gerbes in the previous section, we consider other $\mathbb{Z}_n$ bundle gerbe $G(n, M) = (P, Y, M)$ where the fibered space Y is a principal $PU(n)$ bundle $Y^{PU(n)} \to M$ and the gauge groupoid of the central extension

$$1 \to \mathbb{Z}_n \to SU(n) \xrightarrow{\rho} PU(n) \to 1,$$

where $\rho : SU(n) \subset U(n) \to PU(n) \cong U(n)/U(1)$ is the quotient map. We call the $\mathbb{Z}_n$ bundle gerbes $(P, Y^{PU(n)}, M)$ a *projective* $\mathbb{Z}_n$ *bundle gerbe* and denote it by $G(PU(n), M)$.

In order to classify the projective $\mathbb{Z}_n$ bundle gerbes, we use the notion of an Azumaya bundle due to V. Mathai, R. B. Melrose and I. M. Singer [[9] p.344], and P. Donovan and M. Karoubi [[4] p.12]. An Azumaya algebra of rank n is an algebra isomorphic to to the algebra of $n \times n$ matrices $M(n, \mathbb{C})$, (although, in general, the Azumaya algebra is defined as a central separable algebra over a commutative ring in [2]). An *Azumaya bundle* over a manifold M is a vector bundle with fibers which are Azumaya algebras and which has local trivialization reducing these algebras to $M(n, \mathbb{C})$.

Proposition 5.1. *An Azumaya bundle $\mathcal{A}$ of rank n over a manifold M defines a $\mathbb{Z}_n$ bundle gerbe $G(PU(n), M)$ and conversely.*

Proof. The Azumaya algebra $M(n, \mathbb{C})$ is the algebra $End(\mathbb{C}^n)$ of linear endomorphisms. Since the $\mathbb{C}$ algebra $M(n, \mathbb{C})$ has the $\mathbb{C}$ automorphism group $PGL(n, \mathbb{C}) = PU(n)$, a $PU(n)$ bundle $Y^{PU(n)}(\mathcal{A})$ is associated to $\mathcal{A}$ and the local trivializations give local lifts of coordinate transformations to $SU(n)$ with respect to the projection ρ:

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}_n & \to & SU(n) & \xrightarrow{\rho} & PU(n) & \to & 1 \\ & & \downarrow & & i \downarrow & & \| & & \\ 1 & \to & U(1) & \to & U(n) & \xrightarrow{\rho} & PU(n) & \to & 1. \end{array}$$

Hence $\mathcal{A}$ determines a family of $\mathbb{Z}_n$ groupoids over $PU(n)$ bundle parametrized by the manifold M, which gives exactly a $\mathbb{Z}_n$ bundle P over $(Y^{PU(n)}(\mathcal{A}))^{[2]}$. Therefore $\mathcal{A}$ defines a projective $\mathbb{Z}_n$ bundle gerbe

$$G(PU(n), M) = (P, Y^{PU(n)}(\mathcal{A}), M)$$

over M. The converse follows almost directly. □

Remark 5.2. A projective vector bundle data of a full trivialization of the Azumaya bundle $\mathcal{A}$ of rank n over M [[9] p.350] defines a projective $\mathbb{Z}_n$ bundle gerbe and conversely.

Two Azumaya bundles $\mathcal{E}$ and $\mathcal{F}$ over M are said to be *equivalent* if there are vector bundles E and F over M such that $\mathcal{E} \otimes End(E)$ is isomorphic to $\mathcal{F} \otimes End(F)$.

All equivalence classes of Azumaya bundles over M is called the Brauer group of M and is denoted by $Br(M)$. By the way to prove Serre's theorem [[4] Theorem 8, p.12], we have the isomorphism $Br(M) \cong tor(H^3(M;\mathbb{Z}))$. Any nonzero element of $Tor(H^3(M;\mathbb{Z}),\mathbb{Z}_n) \subset tor(H^3(M;\mathbb{Z}))$ is represented by an Azumaya bundle of a rank n' dividing n and so it represented by an Azumaya bundle of rank n. Let $Br(n,M)$ denote the set of all equivalence classes of Azumaya bundles of rank n. Since $Br(n,M)$ corresponds to $Tor(H^3(M;\mathbb{Z}),\mathbb{Z}_n)$ under the Serre's isomorphism, it is a subgroup of $Br(M)$. From Proposition 5.1 it follows that $Br(n,M) = \{G_S(PU(n),M)\}$.

Let $\partial^*_{SU(n)} : H^1(M;\underline{SU(n)}) \to H^2(M;\mathbb{Z}_n)$ denote the connecting homomorphism in the sheaf cohomology exact sequence with respect to the short exact sequence $1 \to \mathbb{Z}_n \to SU(n) \xrightarrow{\rho} PU(n) \to 1$. By the definition of the characteristic homomorphism $c_S : \{G_S(PU(n),M)\} \to H^2(M;\mathbb{Z}_n)$, we have a commutative diagram

$$\begin{array}{ccc} B(n,M) = \{G_S(PU(n),M)\} & \xrightarrow{c_S} & H^2(M;\mathbb{Z}_n) \\ {\scriptstyle q}\uparrow & \nearrow {\scriptstyle \partial^*_{SU(n)}} & \\ H^1(M;\underline{PU(n)}) & & \end{array}$$

where q denotes the quotient map by the equivalence class of Azumaya bundles.

From the commutative diagram of short exact sequences,

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z} & \to & \mathbb{Z} & \to & \mathbb{Z}_n & \to & 0 \\ & & {\scriptstyle n\cdot}\downarrow & & \cap & & \cap & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{R} & \to & S^1 & \to & 0, \end{array}$$

and by identifying S^1 with $U(1)$, the diagram,

$$\begin{array}{ccc} H^2(M;\mathbb{Z}_n) & & \\ i_* \downarrow & \searrow \partial^*_n & \\ H^2(M;\underline{U(1)}) & \xrightarrow{\partial^*} & H^3(M;\mathbb{Z}) \end{array}$$

is commutative, where ∂^* and ∂^*_n is the connectiong homomorphism with respect to the upper and the lower short exact sequence. Now, we examine the surjectivity of c_S.

Theorem 5.3. *Suppose that M is a manifold. Then the characteristic homomorphism restricted to $\{G_S(PU(n),M)\}$ is an isomorphism onto $Tor(H^3(M;\mathbb{Z}),\mathbb{Z}_n)$. Moreover the c_S is an isomorphism $\{G_S(n,M)\} \cong H^2(M;\mathbb{Z}_n)$.*

Proof. From the commutative short exact sequences

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{Z}_n & \to & SU(n) & \xrightarrow{\rho} & PU(n) & \to & 1 \\ & & \downarrow & & i\downarrow & & \| & & \\ 1 & \to & U(1) & \to & U(n) & \xrightarrow{\rho} & PU(n) & \to & 1, \end{array}$$

it follows the commutative diagram

$$\begin{array}{ccccc} B(n,M) = \{G_S(PU(n),M)\} & & \xrightarrow{c_S} & & H^2(M;\mathbb{Z}_n) \\ q\uparrow & & \partial^*_{SU(n)}\nearrow & & i_*\downarrow \\ H^1(M;\underline{PU(n)}) & & \xrightarrow{\partial^*_{U(n)}} & & H^2(M;\underline{U(1)}). \end{array}$$

by making use of Proposition 5.1, where $\partial^*_{U(n)}$ is the connecting homomorphism with respect to the lower short exact exact sequence. Since we have

$$\begin{aligned} \partial^*_n c_S q &= \partial^*_n \partial^*_{SU(n)} = \partial^* i_* \partial^*_{SU(n)} \\ &= \partial^* \partial^*_{U(n)}, \end{aligned}$$

the homomorphism $\partial^*_n c_S$ gives the Serre's isomorphism,

$$\{G_S(PU(n),M)\} \xrightarrow{\cong} Tor(H^3(M;\mathbb{Z}),\mathbb{Z}_n).$$

The second statement is proved as follows. It is well known that for any element $u \in H^2(M;\mathbb{Z}) \otimes \mathbb{Z}_n$, one can find a $\mathbb{C}^\times$ bundle $Y^{\mathbb{C}^\times}$ over M such that $u = c_1(Y^{\mathbb{C}^\times})\ mod\, n$. Let F be a $\mathbb{Z}_n$ bundle over $\mathbb{C}^\times$ and G_F the gauge groupoid of F. For a sufficiently fine open cover of M, one can construct naturally a $\mathbb{Z}_n$ bundle gerbe $G(n\surd, M) = (P, Y^{\mathbb{C}^\times}, M)$ with $u = c_S(G_S(n\surd, M))$. Therefore, we have $c_S\{G_S(n\surd, M)\} = H^2(M;\mathbb{Z}) \otimes \mathbb{Z}_n$ by Theorem 4.1. By the universal coefficient formula with respect to $\mathbb{Z}_n$, it follows that

$$\begin{aligned} & c_S(\{G_S(n\surd, M)\} \cdot \{G_S(PU(n),M)\}) \\ &\quad = H^2(M;\mathbb{Z}) \otimes \mathbb{Z}_n \oplus Tor(H^3(M;\mathbb{Z}),\mathbb{Z}_n) \\ &\quad = H^2(M;\mathbb{Z}_n). \end{aligned}$$

Hence the characteristic homomorphism $c_S : \{G_S(n,M)\} \to H^2(M;\mathbb{Z}_n)$ is surjective. Since the homomorphism c_S is injective by Corollary 2.2, it is an isomorphism. □

From Theorem 5.3, we obtain immediately,

Corollary 5.4. *Any $\mathbb{Z}_n$ bundle gerbe $G(n,M)$ over a manifold M is Morita equivalent to a product of an n root bundle gerbe $G(n\surd, M)$ and a projective $\mathbb{Z}_n$ bundle gerbe $G(PU(n),M)$.*

6 Bundle gerbes for finitely generated abelian groups

For any abelian group D, D *bundle gerbe* $G(D,M) = (P_D, Y, M)$ over a manifold M is defined by the precisely parallel argument to $G(n,M)$, where $Y \to M$ is a fibered space and P_D is a D bundle over $Y^{[2]}$. In the direct way as Section 2, we obtain a product of any two D bundle gerbes, the characteristic map $c : \{G(D,M)\} \to H^2(M;D)$ and stable equivalence classes $G_S(D,M)$ of $G(D,M)$, since there the argument uses the commutativity of $\mathbb{Z}_n$ essentially. The D bundle gerbe is a gerbe in the sense of [3] and [5] as in Section 3. The characteristic map c induces an injective homomorphism $c_S : \{G_S(D,M)\} \to H^2(M;D)$, which extends that in Section 2.

We consider a $\mathbb{Z}$ bundle gerbe $G(\infty,M) = (P_\infty, Y^{\mathbb{C}^\times}, M)$ where $Y^{\mathbb{C}^\times}$ is the gauge groupoid of the central extension

$$0 \to \mathbb{Z} \to \mathbb{C} \stackrel{exp\cdot 2\pi i}{\to} \mathbb{C}^\times \to 1.$$

Then the Chern class $c_1(G(\infty,M)) = c_1(Y^{\mathbb{C}^\times})$ is well defined and the following theorem is obtained by a more direct proof without the reduction modulo n in Theorem 4.1.

Proposition 6.1. *For any $\mathbb{Z}$ bundle gerbe $G(\infty,M)$, we have $c(G(\infty,M)) = c_1(G(\infty,M))$, and c_S is an isomorphism $\{G_S(\infty,M)\} \cong H^2(M;\mathbb{Z})$, that is, the equivalence classes of $\mathbb{Z}$ bundle gerbes correspond to $\mathbb{C}^\times$ bundle over M in one-to-one way.*

By the fundamental theorem of finitely generated abelian group, D is a direct product of a finite number of cyclic groups. By making use of the universal coefficient formula, Theorem 5.3 and Proposition 6.1 give rise to

Corollary 6.2. *Suppose that D is a finitely generated abelian group and M is a manifold. Then the group $\{G_S(D,M)\}$ of Morita equivalence classes of D bundle gerbes over M is isomorphic to $H^2(M;D)$ by the characteristic map c_S.*

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