Travaux mathématiques, Volume 19 (2011), 7–26, © Université du Luxembourg

Quantization of universal Teichmüller space

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Abstract

In the first part of the paper we describe the complex geometry of the universal Teichmüller space $\mathcal{T}$ which may be realized as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. The quotient $\mathcal{S}$ of the diffeomorphism group of the circle modulo Möbius transformations is treated as a regular part of $\mathcal{T}$. In the second part we consider the quantization of universal Teichmüller space $\mathcal{T}$. We explain first how to quantize the regular part $\mathcal{S}$ by embedding it into a Hilbert–Schmidt Siegel disc. This quantization method, however, does not apply to the whole universal Teichmüller space $\mathcal{T}$. For its quantization we use an approach, similar to the "quantized calculus" of Connes and Sullivan.

1 Introduction

The universal Teichmüller space $\mathcal{T}$, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It can be defined as the space of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ) modulo Möbius transformations. The space $\mathcal{T}$ has a natural complex structure, induced by embedding of $\mathcal{T}$ into the complex Banach space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ. It also contains all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, as complex submanifolds. The space $\mathcal{S} := \text{Diff}_+(S^1)/\text{Möb}(S^1)$ of normalized diffeomorphisms of the circle may be considered as a "regular" part of $\mathcal{T}$.

Our motivation to study $\mathcal{T}$ comes from the string theory. Physicists have noticed that the space $\Omega_d := C_0^\infty(S^1, \mathbb{R}^d)$ of smooth loops in the d-dimensional vector space $\mathbb{R}^d$ may be identified with the phase space of the theory of smooth bosonic closed strings. By this identification the standard symplectic form (of type "$dp \wedge dq$") on the phase space translates into a natural symplectic form ω on Ω_d. This form has a remarkable property that it can be extended to the Sobolev completion of Ω_d, coinciding with the space $V_d := H_0^{1/2}(S^1, \mathbb{R}^d)$ of half-differentiable vector-functions on S^1. Moreover, V_d is the largest space among all Sobolev spaces $H_0^s(S^1, \mathbb{R}^d)$ on which ω can be correctly defined. In other words, V_d is a natural phase space, "chosen" by the form ω itself. From that point of view, it seems more reasonable to consider V_d as the phase space of bosonic string theory, rather than Ω_d. In these lectures we set $d = 1$ for simplicity and study the space $V := V_1 = H_0^{1/2}(S^1, \mathbb{R})$.

According to Nag–Sullivan [7], there is a natural group, attached to the space $V = H_0^{1/2}(S^1, \mathbb{R})$, namely the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle. Again one can say that the space V itself chooses the "right" group to be acted on. The group $\mathrm{QS}(S^1)$ acts on V by reparametrization of loops and this action is symplectic with respect to the form ω. The universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$ can be identified by this action with the space of complex structures on V which can be obtained from a reference complex structure by the action of reparametrization group $\mathrm{QS}(S^1)$. It is well known that such a space plays a crucial role in quantization which is the main subject of the second part of our lectures.

In these lectures we try to define what is the quantum counterpart of the space $\mathcal{T}$, provided with the action of the group $\mathrm{QS}(S^1)$. In order to explain the arising difficulties we consider first an analogous problem for the regular part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$ of $\mathcal{T}$, provided with the action of the group $\mathrm{Diff}_+(S^1)$. This space can be quantized, using an embedding of $\mathcal{S}$ into the Hilbert-Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Under this embedding the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on the Siegel disc by operator fractional-linear transformations. There is a holomorphic Fock bundle $\mathcal{F}$ over $\mathcal{D}_{\mathrm{HS}}$, provided with a projective action of $\mathrm{Sp}_{\mathrm{HS}}(V)$, which covers its action on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action is a projective representation of the Hilbert–Schmidt symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the fibre F_0 of the Fock bundle $\mathcal{F}$. This defines the Dirac quantization of the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Its restriction to $\mathcal{S}$ gives a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the group $\mathrm{Diff}_+(S^1)$ in the Fock space F_0 which defines the Dirac quantization of the space $\mathcal{S}$.

However, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T}$. By this reason we choose another approach to this problem, based on Connes quantization. Briefly, the idea is the following. The $\mathrm{QS}(S^1)$-action on the Sobolev space V, mentioned above, cannot be differentiated in the classical sense (in particular, there is no Lie algebra, associated to $\mathrm{QS}(S^1)$). However, one can define a quantized infinitesimal version of this action by associating with any quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ a quantum differential $d^q f$ which is an integral operator on V with kernel, given essentially by the finite-difference derivative of f. In these terms the quantization of $\mathcal{T}$ is given by a representation of the algebra of derivations of V, generated by quantum differentials $d^q f$, in the Fock space F_0.

2 Universal Teichmüller space

2.1 Definition of universal Teichmüller space

2.1.1 Quasiconformal maps

Let $w : D \to w(D)$ be a homeomorphism of the domain $D \subset \overline{\mathbb{C}}$ in the extended complex plane (Riemann sphere) $\overline{\mathbb{C}}$ onto domain $w(D) \subset \overline{\mathbb{C}}$ which has locally integrable derivatives (in generalized sense).

Definition 2.1. The homeomorphism w is called *quasiconformal* if there exist a function $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty =: k < 1$ such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{2.1}$$

is satisfied for almost all $z \in D$. The function μ is called the *Beltrami differential* of w and the constant k is often indicated in the name of k-*quasiconformal* maps.

Remark 2.1. For $k = 0$ the equation (2.1) reduces to the Cauchy–Riemann equation and so determines a conformal map $w : D \to w(D)$. Such a map sends infinitesimally small circles, centered at a point $z \in D$, again into infinitesimally small circles, centered at $w(z)$. While in the case of a general smooth quasiconformal map w such a map sends infinitesimally small circles, centered at $z \in D$, into infinitesimally small ellipses, centered at $w(z)$, with eccentricity (the ratio of the large axis to the small one) being uniformly bounded (w.r. to $z \in D$) by a common constant $K < \infty$. This constant K is related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k} \geq 1 .$$

The least possible constant K is called the *maximal dilatation* of w and is also sometimes indicated in the name of K-*quasiconformal* maps.

Remark 2.2. The term "Beltrami differential" for μ is motivated by the behavior of μ under conformal changes of variable. Namely, according to (2.1), the function μ should transform under a conformal change $z \mapsto f(z)$ as

$$\mu\left(f(z)\right) = \mu(z)\,\frac{\overline{f'(z)}}{f'(z)},$$

i.e. as a $(-1, 1)$-differential.

Remark 2.3. Quasiconformal maps $w : D \to D$ form a group, i.e. the composition of quasiconformal maps and the inverse of a quasiconformal map are again quasiconformal.

Theorem 2.1 (uniqueness theorem)**.** *Suppose that quasiconformal maps* $w_1, w_2 : D \to D'$ *satisfy the same Beltrami equation in* D *(i.e. have the same Beltrami differential in* D*). Then the maps*

$$w_1 \circ w_2^{-1} \quad \text{and} \quad w_2 \circ w_1^{-1}$$

are conformal. The composition $f \circ w$ *of a quasiconformal map* $w : D \to D'$ *with a conformal map* $f : D' \to D''$ *satisfy the same Beltrami equation in* D *as* w*.*

Remark 2.4. A quasiconformal map $w : D \to D'$ is always extended to a homeomorphism $w : \overline{D} \to \overline{D'}$ of the closures which is Hölder-continuous up to the boundary.

Theorem 2.2 (existence theorem)**.** *For any function* $\mu \in L^\infty(\overline{\mathbb{C}})$ *with* $\|\mu\|_\infty < 1$ *there exists a solution* w *of the Beltrami equation in* $\overline{\mathbb{C}}$*. Any other solution* $\tilde{w}$ *of this equation has the form* $\tilde{w} = w \circ f$ *where* f *is a fractional-linear transform.*

Remark 2.5. In Theorem 2.2 we have restricted ourselves to the case $D = \overline{\mathbb{C}}$ since the case of a general domain $D \subset \overline{\mathbb{C}}$ is easily reduced to the case of the extended complex plane. Indeed, given a Beltrami differential $\mu \in L^\infty(D)$ with norm $\|\mu\|_\infty < 1$ we can always extend it (e.g. by zero outside D) to the whole $\overline{\mathbb{C}}$, preserving the inequality $\|\mu\|_\infty < 1$, and then apply the above theorem to get a solution of Beltrami equation in $\overline{\mathbb{C}}$. Its restriction to D yields a solution of Beltrami equation in D, defined up to conformal maps, according to the uniqueness theorem.

2.1.2 Quasisymmetric homeomorphisms

Definition 2.2. A homeomorphism $f : S^1 \to S^1$ of the unit circle S^1, preserving its orientation, is called *quasisymmetric* if it extends to a quasiconformal homeomorphism $w : \Delta \to \Delta$ of the unit disc Δ. The set of all quasisymmetric homeomorphisms of S^1 is a group, denoted by $\mathrm{QS}(S^1)$.

Definition 2.3. The *universal Teichmüller space* $\mathcal{T}$ is the quotient

$$\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

where $\mathrm{M\ddot{o}b}(S^1)$ denotes the Möbius group of fractional-linear automorphisms of the unit disc Δ, restricted to the unit circle S^1.

Remark 2.6. One can avoid taking the quotient by Möbius group in the definition of $\mathcal{T}$ by considering only *normalized* quasisymmetric homeomorphisms, leaving three fixed points in the circle, say $\pm 1, i$, invariant.

Remark 2.7. Any orientation-preserving diffeomorphism in $\mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$ which is quasiconformal, according to Remark 2.1. So $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$, and we have the following chain of embeddings

$$\mathrm{Möb}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}_+(S^1) \ .$$

Hence,

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1).$$

The space $\mathcal{S}$ can be otherwise defined as the space of normalized diffeomorphisms of S^1 and will be considered as a "regular" part of $\mathcal{T}$.

Since quasisymmetric homeomorphisms of S^1 were defined via quasiconformal maps of Δ, i.e. in terms of solutions of Beltrami equation in Δ, one can expect that there should be a definition of $\mathcal{T}$ directly in terms of Beltrami differentials.

Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It can be identified (as a set) with the unit ball in the complex Banach space $L^\infty(\Delta)$. Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it to a Beltrami differential $\check{\mu}$ on the extended complex plane $\overline{\mathbb{C}}$ by setting $\check{\mu}$ equal to zero outside the unit disc Δ. Then we can apply the existence Theorem 2.2 for quasiconformal maps on the extended complex plane $\overline{\mathbb{C}}$ and obtain a normalized quasiconformal homeomorphism w^μ, satisfying Beltrami equation (2.1) on $\overline{\mathbb{C}}$ with potential $\check{\mu}$. This homeomorphism is conformal on the exterior $\Delta_- := \overline{\mathbb{C}} \setminus \overline{\Delta}$ of the closed unit disc $\overline{\Delta}$ on $\overline{\mathbb{C}}$ and fixes the points $\pm 1, -i$.

Introduce an equivalence relation between Beltrami differentials in Δ by identifying two Beltrami differentials μ and ν for which the corresponding conformal maps coincide: $w^\mu|_{\Delta_-} \equiv w^\nu|_{\Delta_-}$. The universal Teichmüller space $\mathcal{T}$ coincides with the quotient

$$\mathcal{T} = B(\Delta)/\sim$$

of the space $B(\Delta)$ of Beltrami differentials modulo the introduced equivalence relation.

2.2 Complex structure of universal Teichmüller space

We introduce a complex structure on the universal Teichmüller space $\mathcal{T}$, using its embedding into the space of holomorphic quadratic differentials.

Consider an arbitrary point $[\mu]$ of $\mathcal{T}$, represented by the quasiconformal map w^μ. Its restriction to Δ_- is a conformal map so we can take its Schwarzian $S(w^\mu|_{\Delta_-})$.

Digression 1. Recall that the *Schwarzian* of a conformal map f is defined by

$$S(f) := \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 .$$

A characteristic property of Schwarzian is its invariance under fractional-linear maps

$$S\left(\frac{af+b}{cf+d}\right) = S(f).$$

By taking the Schwarzian $S(w^\mu|_{\Delta_-})$, we get a holomorphic quadratic differential in the disc Δ_- (the latter fact follows from the transformation properties of Beltrami differentials, prescribed by Beltrami equation (2.1)). Moreover, the image of this map does not depend on the choice of Beltrami differential μ in the class $[\mu]$. Composing this map with a standard fractional-linear isomorphism $\Delta_- \to \Delta$, we obtain an embedding

$$\Psi : \mathcal{T} \longrightarrow B_2(\Delta)\ , \quad [\mu] \longmapsto \psi(\mu), \tag{2.2}$$

having its image in the space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ.

The space $B_2(\Delta)$ of holomorphic quadratic differentials in Δ is a complex Banach space, provided with a natural hyperbolic norm, given by

$$\|\psi\|_2 := \sup_{z\in\Delta}(1-|z|^2)^2|\psi(z)|$$

for a quadratic differential ψ. It can be proved (cf. [5]) that $\|\psi[\mu]\|_2 \le 6$ for any Beltrami differential $\mu \in B(\Delta)$.

The constructed map $\Psi : \mathcal{T} \to B_2(\Delta)$, called the *Bers embedding*, is a homeomorphism of $\mathcal{T}$ onto an open bounded connected contractible subset in $B_2(\Delta)$, containing the ball of radius 1/2, centered at the origin (cf. [5]).

Using the constructed embedding (2.2), we can introduce a complex structure on the universal Teichmüller space $\mathcal{T}$ by pulling it back from the complex Banach space $B_2(\Delta)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold.

2.3 Classical Teichmüller spaces

The universal Teichmüller space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$ as complex submanifolds. In particular, it is true for all Teichmüller spaces of compact Riemann surfaces of genus g. This property motivates the use of the term "universal" in the name of $\mathcal{T}$.

Let X be a compact Riemann surface of genus $g > 1$, uniformized by the unit disc Δ. Such a surface can be represented as the quotient

$$X = \Delta/G$$

where G is a discrete (Fuchsian) subgroup of $\text{Möb}(\Delta)$.

Definition 2.4. A quasisymmetric homeomorphism $f : S^1 \to S^1$ is called *G-invariant* if

$$f \circ g \circ f^{-1} \in \text{Möb}(S^1) \text{ for any } g \in G \iff fGf^{-1} \subset \text{Möb}(S^1).$$

Denote by $\text{QS}(S^1)^G$ the subgroup of G-invariant quasisymmetric homeomorphisms in $\text{QS}(S^1)$. Then by definition

$$T(G) := \text{QS}(S^1)^G/\text{Möb}(S^1).$$

The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

Remark 2.8. According to definition of $T(G)$, due to Teichmüller, the space $T(G)$ parameterizes different complex structures on the Riemann surface X/Δ which can be obtained from the original complex structure by a quasiconformal deformation.

2.4 Grassmann realization

2.4.1 Sobolev space of half-differentiable functions

Definition 2.5. The *Sobolev space of half-differentiable functions* on S^1 is a Hilbert space

$$V := H_0^{1/2}(S^1, \mathbb{R}),$$

consisting of functions $f \in L_0^2(S^1, \mathbb{R})$ with zero average over the circle, which have Fourier decompositions

$$f(z) = \sum_{k \neq 0} f_k z^k \ , \quad f_k = \bar{f}_{-k} \ , \quad z = e^{i\theta} \ ,$$

and a finite Sobolev norm

$$\|f\|_{1/2}^2 = \sum_{k \neq 0} |k||f_k|^2 = 2\sum_{k>0} k|f_k|^2 < \infty \ . \tag{2.3}$$

Properties of $V = H_0^{1/2}(S^1, \mathbb{R})$:

1. *Symplectic structure*: define a 2-form ω on V by the formula

$$\omega(\xi, \eta) = 2 \,\text{Im} \sum_{k>0} k\xi_k \bar{\eta}_k$$

for vectors $\xi, \eta \in V$ with Fourier series

$$\xi(z) = \sum_{k\neq 0} \xi_k z^k, \quad \eta(z) = \sum_{k\neq 0} \eta_k z^k.$$

This form, which is correctly defined on V due to condition (2.3), determines a symplectic form on V. Moreover, $H_0^{1/2}(S^1, \mathbb{R})$ is the largest Hilbert space in the scale of Sobolev spaces $H_0^s(S^1, \mathbb{R})$, $s \in \mathbb{R}$, on which this form is correctly defined.

2. *Complex structure*: the Sobolev space V has a complex structure J^0, defined by the formula

$$\xi(z) = \sum_{k\neq 0} \xi_k z^k \longmapsto (J^0\xi)(z) = -i \sum_{k>0} \xi_k z^k + i \sum_{k<0} \xi_k z^k$$

for a vector $\xi(z) = \sum_{k\neq 0} \xi_k z^k \in V$.

3. *Riemannian metric*: the introduced symplectic and complex structures on V are compatible with each other in the sense that they generate together a Riemannian metric, defined by

$$g^0(\xi, \eta) = \omega(\xi, J^0\eta) = 2\,\mathrm{Re} \sum_{k>0} k\xi_k \bar{\eta}_k.$$

In other words, V has the structure of a Kähler Hilbert space.

4. *Complexification* of V, equal to

$$V^{\mathbb{C}} = H_0^{1/2}(S^1, \mathbb{C}),$$

is a complex Hilbert space with a Kähler metric, given by the Hermitian extension of the Riemannian metric g^0 on V to $V^{\mathbb{C}}$. The space $V^{\mathbb{C}}$ is decomposed into the direct sum

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

of $(\mp i)$-eigenspaces of the complex structure operator $J^0 \in \mathrm{End}\, V^{\mathbb{C}}$. More explicitly,

$$W_+ = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k>0} f_k z^k\}, \quad W_- = \{f \in V^{\mathbb{C}} : \ f(z) = \sum_{k<0} f_k z^k\}.$$

This splitting is orthogonal with respect to Hermitian inner product on $V^{\mathbb{C}}$.

2.4.2 QS-action on the Sobolev space V

With any homeomorphism $h : S^1 \to S^1$, preserving the orientation, we can associate a "change-of-variable" operator

$$T_h : L_0^2(S^1, \mathbb{R}) \to L_0^2(S^1, \mathbb{R}),$$

defined by

$$T_h(\xi) := \xi \circ h - \frac{1}{2\pi} \int_0^{2\pi} \xi\left(h(\theta)\right) d\theta.$$

This operator has the following remarkable property.

Theorem 2.3 (Nag–Sullivan [7]). *(i) The operator T_h acts on V, i.e. $T_h : V \to V$, if and only if $h \in QS(S^1)$.*

(ii) The operator T_h with $h \in QS(S^1)$ acts symplectically on V, i.e. it preserves symplectic form ω. Moreover, its complex-linear extension to $V^{\mathbb{C}}$ preserves the subspace W_+ if and only if $h \in \text{Möb}(S^1)$. In the latter case, T_h acts as a unitary operator on W_+.

Remark 2.9. We have pointed out in the previous subsection that the Sobolev space V is the largest Hilbert space in the scale of Sobolev spaces, on which the form ω is correctly defined. In other words, this space is "chosen" by symplectic form ω itself. According to Theorem 2.3, the space V also "chooses" the reparametrization group $QS(S^1)$ in the sense that it is the largest reparametrization group, leaving V invariant. So we get a natural phase space (V, ω) together with a natural group $QS(S^1)$ of its canonical transformations.

According to Theorem 2.3, we have an embedding

$$\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1) \longrightarrow \text{Sp}(V)/\text{U}(W_+). \tag{2.4}$$

Here, $\text{Sp}(V)$ is the symplectic group of V, consisting of bounded linear symplectic operators on V, and $\text{U}(W_+)$ is its subgroup, consisting of unitary operators (i.e. the operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+).

Digression 2. Recall the definition of symplectic group $\text{Sp}(V)$. In terms of decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ can be written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} .$$

Such an operator belongs to the symplectic group $\text{Sp}(V)$ if it has the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with components, satisfying the relations

$$\bar{a}^t a - b^t \bar{b} = 1, \quad \bar{a}^t b = b^t \bar{a}$$

where a^t, b^t denote the transposed operators $a^t : W_- \to W_-$, $b^t : W_- \to W_+$. The unitary group $\mathrm{U}(W_+)$ is embedded into $\mathrm{Sp}(V)$ as a subgroup, consisting of diagonal block matrices of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} .$$

The space

$$\mathcal{J}(V) := \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

on the right hand side of (2.4), can be identified with the space of complex structures on V, compatible with ω. Indeed, any such structure, given by a linear operator J on V with $J^2 = -I$, determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{2.5}$$

of $V^{\mathbb{C}}$ into the direct sum of $(\pm i)$-eigenspaces, isotropic with respect to ω. Conversely, any decomposition (2.5) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, equal to iI on W and $-iI$ on $\overline{W}$, which is compatible with ω. Moreover, a complex structure J, obtained from a reference complex structure J^0 by the action of an element A of $\mathrm{Sp}(V)$, is equivalent to J^0 if and only if $A \in \mathrm{U}(W_+)$. Hence,

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{J}(V) .$$

The space on the right can be, in its turn, identified with the *Siegel disc* $\mathcal{D}$, defined as the set

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\}.$$

The symmetricity of Z means that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$ with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1}$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The isotropy subgroup at $Z = 0$ coincides with the set of operators $A \in \mathrm{Sp}(V)$ such that $b = 0$, i.e. with $\mathrm{U}(W_+)$. So the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

can be identified with the Siegel disc $\mathcal{D}$.

It can be proved (cf. [7]) that the constructed embedding of universal Teichmüller space $\mathcal{T}$ into the Siegel disc $\mathcal{D} = \mathrm{Sp}(V)/\mathrm{U}(W_+)$ is an equivariant holomorphic map of Banach manifolds.

Restriction of this map to the regular part $\mathcal{S}$ of universal Teichmüller space yields an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+) \tag{2.6}$$

where the *Hilbert–Schmidt subgroup* $\mathrm{Sp}_{\mathrm{HS}}(V)$ of $\mathrm{Sp}(V)$ consists of bounded linear operators $A \in \mathrm{Sp}(V)$, having block representations of the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

where b is a Hilbert–Schmidt operator.

Digression 3. Recall that a linear bounded operator $T : H_1 \to H_2$ from a Hilbert space H_1 to a Hilbert space H_2 is called Hilbert–Schmidt if there exists an orthonormal basis $\{e_i\}$ in H_1 such that the Hilbert–Schmidt norm

$$\|T\|_2 := \left(\sum_{i=0}^{\infty} \|Te_i\|_{H_2}^2 \right)^{1/2}$$

is finite. If this is true for some orthonormal basis $\{e_i\}$ in H_1 then it is true for any orthonormal basis in H_1 and the value of the norm $\|T\|_2$ does not depend on the choice of this basis.

We identify, as above, the right hand side of (2.6) with a subspace $\mathcal{J}_{\mathrm{HS}}(V)$ of the space $\mathcal{J}(V)$ of compatible complex structures on V. As before, the space $\mathcal{J}_{\mathrm{HS}}(V)$ of Hilbert–Schmidt complex structures on V can be realized as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\}.$$

The embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is an equivariant holomorphic map of Banach manifolds.

3 Quantization of Universal Teichmüller Space

3.1 Dirac quantization

3.1.1 Definition

We start by recalling a general definition of quantization of finite-dimensional classical systems, due to Dirac. A *classical system* is given by a pair $(M, \mathcal{A})$ where M is the phase space of the system and $\mathcal{A}$ is its algebra of observables.

The *phase space* M is a smooth symplectic manifold of even dimension $2n$, provided with symplectic 2-form ω. Locally, it is equivalent to the standard model, given by symplectic vector space $M_0 := \mathbb{R}^{2n}$ together with standard symplectic form ω_0, given in canonical coordinates (p_i, q_i), $i = 1, \dots, n$, on $\mathbb{R}^{2n}$ by

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

The *algebra of observables* $\mathcal{A}$ is a Lie subalgebra of the Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M, provided with the Poisson bracket, determined by symplectic 2-form ω. In particular, in the case of standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ one can take for $\mathcal{A}$ the Heisenberg algebra, generated by coordinate functions p_i, q_i, $i = 1, \dots, n$, and 1, satisfying the commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0,$$
$$\{p_i, q_j\} = \delta_{ij} \quad \text{for } i, j = 1, \dots, n.$$

Remark 3.1. One of usual ways to produce algebras of observables is to consider a Lie group Γ of symplectomorphisms of a symplectic manifold (M, ω) and take for $\mathcal{A}$ its Lie algebra $\mathrm{Lie}(\Gamma)$, consisting of Hamiltonian vector fields X_f on M. If M is simply connected then $\mathcal{A}$ can be identified with the dual algebra of functions f, generating Hamiltonian vector fields from $\mathrm{Lie}(\Gamma)$.

Definition 3.1. The *Dirac quantization* of a classical system $(M, \mathcal{A})$ is an irreducible linear representation

$$r : \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the space of linear self-adjoint operators, acting on a complex Hilbert space H, called the *quantization space*. The map r should satisfy the condition

$$r(\{f, g\}) = \frac{1}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}$. We also impose on r the following normalization condition: $r(1) = I$.

Remark 3.2. For complexified algebras of observables $\mathcal{A}^{\mathbb{C}}$ or, more generally, complex involutive Lie algebras of observables (i.e. Lie algebras with conjugation) their Dirac quantization is given by an irreducible Lie-algebra representation

$$r : \mathcal{A}^{\mathbb{C}} \longrightarrow \operatorname{End} H \ ,$$

satisfying the normalization condition and the conjugation law: $r(\bar{f}) = r(f)^*$ for any $f \in \mathcal{A}$.

Remark 3.3. We are going to apply this definition of quantization to infinite-dimensional classical systems, in which both the phase space and algebra of observables are infinite-dimensional. For infinite-dimensional algebras of observables it is more natural to look for their projective representations. Using such a representation for an original algebra $\mathcal{A}$, we can construct the quantization of the extended system $(M, \tilde{\mathcal{A}})$ with $\tilde{\mathcal{A}}$ being a suitable central extension of $\mathcal{A}$.

3.1.2 Statement of the problem

We shall explain first how to quantize the regular part of universal Teichmüller space $\mathcal{T}$, represented by the classical system

$$(\mathcal{S}, \operatorname{Vect}(S^1))$$

where $\mathcal{S} = \operatorname{Diff}_+(S^1)/\text{Möb}(S^1)$ and $\operatorname{Vect}(S^1)$ is the Lie algebra of $\operatorname{Diff}_+(S^1)$, consisting of smooth vector fields on S^1.

To quantize this system, we first enlarge it to an extended system, using the embedding $\mathcal{S} \hookrightarrow \mathcal{J}_{\text{HS}}(V)$ from Subsection 2.4.2. This extended system is given by

$$(\mathcal{J}_{\text{HS}}(V), \text{sp}_{\text{HS}}(V))$$

where $\text{sp}_{\text{HS}}(V)$ is the Lie algebra of $\text{Sp}_{\text{HS}}(V)$.

3.2 Quantization of $\mathcal{S}$

3.2.1 Fock space

Fix a compatible complex structure $J \in \mathcal{J}(V)$, generating a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{3.1}$$

of $V^{\mathbb{C}}$ into the direct sum of $\pm i$-eigenspaces of J and provide $V^{\mathbb{C}}$ with a Hermitian inner product

$$< z, w >_J := \omega(z, Jw),$$

determined by J and symplectic form ω.

The Fock space $F(V^{\mathbb{C}}, J)$ is the completion of the algebra of symmetric polynomials on W with respect to a natural norm, generated by $< \cdot, \cdot >_J$. In more detail, denote by $S(W)$ the algebra of symmetric polynomials in variables $z \in W$. This algebra is provided with an inner product, generated by $< \cdot, \cdot >_J$. By definition, this inner product on monomials of the same degree is equal to

$$< z_1 \cdot \ldots \cdot z_n, z'_1 \cdot \ldots \cdot z'_n >_J = \sum_{\{i_1,\ldots,i_n\}} < z_1, z'_{i_1} >_J \cdot \ldots \cdot < z_n, z'_{i_n} >_J$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$. The inner product of monomials of different degrees is set to zero. The constructed inner product is extended to the whole algebra $S(W)$ by linearity. The completion $\widehat{S(W)}$ of $S(W)$ with respect to the introduced norm is called the *Fock space* of $V^{\mathbb{C}}$ with respect to complex structure J:

$$F_J = F(V^{\mathbb{C}}, J) := \widehat{S(W)}.$$

If $\{w_n\}$, $n = 1, 2, \ldots,$ is an orthonormal basis of W one can take for an orthonormal basis of F_J a family of homogeneous polynomials of the form

$$P_K(z) = \frac{1}{\sqrt{k!}} < z, w_1 >_J^{k_1} \cdot \ldots \cdot < z, w_n >_J^{k_n}, \quad z \in W, \tag{3.2}$$

where $K = (k_1, \ldots, k_n, 0, \ldots)$, $k_i \in \mathbb{N} \cup 0$, and $k! = k_1! \cdot \cdots \cdot k_n!$.

3.2.2 Symplectic group action on Fock spaces

We unify different Fock spaces F_J with $J \in \mathcal{J}_{\mathrm{HS}}(V)$ into a single *Fock bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{J}_{\mathrm{HS}}(V)} F_J \longrightarrow \mathcal{J}_{\mathrm{HS}}(V) = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+).$$

Theorem 3.1 (Shale–Berezin)**.** *The Fock bundle*

$$\mathcal{F} \longrightarrow \mathcal{J}_{HS}(V)$$

is a holomorphic Hermitian Hilbert-space bundle. The group $Sp_{HS}(V)$ acts projectively on $\mathcal{F}$ by unitary transformations and this action covers the natural action of $Sp_{HS}(V)$ on $\mathcal{J}_{HS}(V)$ by left translations.

The infinitesimal version of this action yields a projective representation of symplectic Hilbert–Schmidt algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space $F_0 = F(V^{\mathbb{C}}, J^0)$, i.e. a quantization of the system

$$\left(\mathcal{J}_{\mathrm{HS}}, \widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}\right)$$

where $\widetilde{\mathrm{sp}_{\mathrm{HS}}(V)}$ is a central extension of Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

The restriction of the constructed Fock bundle $\mathcal{F}$ to the submanifold $\mathcal{S} \subset \mathcal{J}_{\mathrm{HS}}$ is a holomorphic Hermitian Hilbert-space bundle

$$\mathcal{F}_{\mathcal{S}} := \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1)$$

together with a projective unitary action of $\mathrm{Diff}_+(S^1)$, covering its action on $\mathcal{S}$ by left translations. The infinitesimal version of this action generates a projective unitary representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0, i.e. a quantization of the system

$$(\mathcal{S}, \mathrm{vir})$$

where vir is the *Virasoro algebra*, being a central extension of Lie algebra $\mathrm{Vect}(S^1)$.

3.3 Quantization of $\mathcal{T}$

3.3.1 Dirac versus Connes quantization

To quantize $\mathcal{S}$, we have used the fact that the symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ acts on the Fock bundle $\mathcal{F} \to \mathcal{J}_{\mathrm{HS}}(V)$. For the whole Teichmüller space $\mathcal{T}$ we still have the embedding

$$\mathcal{T} \longrightarrow \mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

but we cannot construct an $\mathrm{Sp}(V)$-action on $\mathcal{F}$, covering its action on $\mathcal{J}(V)$. This is forbidden by Shale–Berezin theorem. So we employ another approach for the quantization of $\mathcal{T}$, using Connes' definition of quantization.

Recall that in Dirac's approach we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M and the algebra of observables $\mathcal{A}$ which is a Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation r of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of functions $f, g \in \mathcal{A}$ into the commutator $\frac{1}{i}[r(f), r(g)]$ of the corresponding operators. In Connes' approach the algebra of observables $\mathfrak{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation π of $\mathfrak{A}$ in a Hilbert space H, sending the differential df of a function $f \in \mathfrak{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a self-adjoint symmetry operator S with $S^2 = I$.

In the following table we compare Connes and Dirac approaches to quantization:

	Dirac approach	Connes approach
Classical system	$(M, \mathcal{A})$ where: M – phase space $\mathcal{A}$ – involutive Lie algebra of observables	$(M, \mathfrak{A})$ where: M – phase space $\mathfrak{A}$ – involutive associative algebra of observables with differential d
Quantization	representation $r\colon \mathcal{A} \to \operatorname{End} H$, sending $\{f, g\} \mapsto \frac{1}{i}[r(f), r(g)]$	representation $\pi\colon \mathfrak{A} \to \operatorname{End} H$, sending $df \mapsto [S, \pi(f)]$, where $S = S^*$, $S^2 = I$

Remark 3.4. We can reformulate the Connes definition in terms of Lie algebras by switching to the algebra of derivations of associative algebra of observables $\mathfrak{A}$. Recall that the Lie algebra $\operatorname{Der}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ consists of linear maps $\mathfrak{A} \to \mathfrak{A}$, satisfying the Leibnitz rule. The Connes quantization means in these terms the construction of an irreducible representation of $\operatorname{Der}(\mathfrak{A})$ in the space $\operatorname{End} H$, considered as a Lie algebra with a Lie bracket, given by commutator.

Remark 3.5. If all observables are smooth functions on M, both approaches are equivalent to each other. Indeed, the differential df of a smooth observable f is symplectically dual to the Hamiltonian vector field X_f which establishes a relation between the associative algebra $\mathfrak{A} \ni f$ of functions f on M and the Lie algebra $\mathcal{A} \ni X_f$ of Hamiltonian vector fields X_f. A symmetry operator S is determined by a polarization $H = H_+ \oplus H_-$ of the quantization space H and related to the complex structure J (determined by the same polarization) by a simple formula $S = iJ$.

In the case when the algebra of observables $\mathcal{A}$ contains non-smooth functions, the Dirac approach formally cannot be applied. In Connes approach the differential df of a non-smooth observable $f \in \mathfrak{A}$ is also not defined but its quantum analogue

$$d^q f := [S, \pi(f)]$$

may still have sense, as it is demonstrated by the example in the next subsection.

3.3.2 Example

Suppose that $\mathfrak{A}$ is the algebra $L^\infty(S^1, \mathbb{C})$ of bounded functions on the circle S^1. Any function $f \in \mathfrak{A}$ determines a bounded multiplication operator in the Hilbert space $H = L^2(S^1, \mathbb{C})$ by the formula

$$M_f : v \in H \longmapsto fv \in H.$$

A symmetry operator S in H is given by the *Hilbert transform* $S : L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$:

$$(Sf)(e^{i\varphi}) = \frac{1}{2\pi} P.V. \int_0^{2\pi} K(\varphi, \psi) f(e^{i\psi}) d\psi$$

where the integral is taken in the principal value sense and the kernel is given by

$$K(\varphi, \psi) = 1 + i \cot \frac{\varphi - \psi}{2} \ . \tag{3.3}$$

Note that for φ, close to ψ, this kernel behaves asymptotically like $2/(\varphi - \psi)$.

The differential *df* of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense but its quantum analogue

$$d^q f := [S, M_f]$$

is a bounded operator in H. Moreover, $d^q f$ for $f \in H$ is a Hilbert–Schmidt operator, given by

$$d^q f(v)(e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} k(\varphi, \psi) v(e^{i\psi}) d\psi \tag{3.4}$$

with kernel

$$k(\varphi, \psi) := K(\varphi, \psi)(f(\varphi) - f(\psi)) \ ,$$

where $K(\varphi, \psi)$ is defined by (3.3). The kernel $k(\varphi, \psi)$ for φ, close to ψ, behaves asymptotically like

$$\frac{f(e^{i\varphi}) - f(e^{i\psi})}{\varphi - \psi}.$$

Using this fact, it can be checked that the quasiclassical limit of $d^q f$, arranged by taking the limit $\varphi \to \psi$, coincides (up to a constant) with the multiplication operator $v \mapsto f'v$. So the quantization means in this case simply the replacement of the derivative by its finite-difference analogue.

3.3.3 Quantization of the universal Teichmüller space

We apply these ideas to the universal Teichmüller space $\mathcal{T}$. In Subsection 2.4.2 we have defined a natural action of the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of S^1 on the Sobolev space V. As we have remarked, this action does not admit the differentiation, so classically there is no Lie algebra, associated with $\mathrm{QS}(S^1)$. In other words, there is no classical algebra of observables, associated to $\mathcal{T}$. (The situation is similar to that in the example above.) However, we shall construct a *quantum algebra of observables*, associated to $\mathcal{T}$.

For that we define a quantum infinitesimal version of $\mathrm{QS}(S^1)$-action on V, given by the integral operator $d^q f$, defined by formula (3.4). Then we extend this operator $d^q f$ to the Fock space F_0 by defining it first on elements of the basis (3.2) of F_0 with the help of Leibnitz rule, and then extending to the whole symmetric algebra $S(W_+)$ by linearity. The completion of the obtained operator yields an operator $d^q f$ on F_0. The operators $d^q f$ with $f \in \mathrm{QS}(S^1)$, constructed in this way, generate a *quantum Lie algebra* $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$. We consider it as a quantum Lie algebra of observables, associated with $\mathcal{T}$. We can also consider the constructed Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ as a replacement of the (non-existing) classical Lie algebra of the group $\mathrm{QS}(S^1)$.

Compare now the main steps of Connes quantization of $\mathcal{T}$ with the analogous steps in Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$.

In the case of $\mathcal{J}_{\mathrm{HS}}$:

1. we start with the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on $\mathcal{J}_{\mathrm{HS}}$;
2. then, using Shale theorem, extend this action to a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces $F(V, J)$;
3. an infinitesimal version of this action yields a projective unitary representation of symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0.

In the case of $\mathcal{T}$:

1. we have an action of $\mathrm{QS}(S^1)$ on the space V; however, in contrast with Dirac quantization of $\mathcal{J}_{\mathrm{HS}}$, the step (2) in case of $\mathcal{T}$ is impossible, since by Shale theorem we cannot extend the action of $\mathrm{QS}(S^1)$ to Fock spaces $F(V, S)$;
2. we define instead a quantized infinitesimal action of $\mathrm{QS}(S^1)$ on V, given by quantum differentials $d^q f$;
3. extending operators $d^q f$ to the Fock space F_0, we obtain a quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$, generated by extended operators $d^q f$ on F_0.

Conclusion. The Connes quantization of the universal Teichmüller space $\mathcal{T}$ consists of two steps:

1. The first step ("the first quantization") is the construction of quantized infinitesimal $\mathrm{QS}(S^1)$-action on V, given by quantum differentials $d^q f$ with $f \in \mathrm{QS}(S^1)$.

2. The second step ("the second quantization") is the extension of quantum differentials $d^q f$ to the Fock space F_0. The extended operators $d^q f$ with $f \in \mathrm{QS}(S^1)$ generate the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated with $\mathcal{T}$.

Note that the correspondence principle for the constructed Connes quantization of $\mathcal{T}$ means that this quantization, being restricted to $\mathcal{S}$, coincides with Dirac quantization of $\mathcal{S}$.

4 Acknowledgements

While preparing this paper, the author was partly supported by the RFBR grants 08-01-00014, 09-02-91349, by the program of Support of Scientific Schools (grant NSH-7675.2010.1), and by the Scientific Program of RAS "Nonlinear Dynamics".

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