

# Singular Poisson-Kähler geometry of stratified Kähler spaces and quantization

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## Abstract

In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. In certain situations the difficulties can be overcome by means of Kähler quantization on *stratified Kähler spaces*. Such a space is a stratified symplectic space together with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion. Holomorphic quantization on a stratified Kähler space then yields a costratified Hilbert space, a quantum object having the classical singularities as its shadow. Given a Kähler manifold with a hamiltonian action of a compact Lie group that also preserves the complex structure, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

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## 1 Introduction

In the presence of classical phase space singularities the standard methods are insufficient to attack the problem of quantization. Ordinary Schrödinger quantization leads to a Hilbert space whose elements are classes of  $L^2$ -functions, and incorporating singularities here directly seems at present out of sight since we do not know how to handle the singularities in terms of classes of functions. However, Hilbert spaces of holomorphic functions are typically spaces whose points are ordinary functions rather than classes of functions, and we know well how we can understand singularities in terms of ordinary functions. We will show here that, in certain situations, by means of a suitable Kähler quantization procedure on *stratified Kähler spaces*, we can overcome the difficulties at the quantum level arising from classical phase space singularities. A stratified Kähler space is a stratified symplectic space endowed with a complex analytic structure which is compatible with the stratified symplectic structure; in particular each stratum is a Kähler manifold in an obvious fashion. Kähler quantization then yields a Hilbert space whose points are holomorphic functions (or more generally holomorphic sections of a holomorphic line bundle); the resulting quantum Hilbert space actually acquires more structure which, in turn, has the classical singularities as its shadow, as we will explain shortly.

Examples of stratified Kähler spaces abound: Symplectic reduction, applied to Kähler manifolds, yields a particular class of examples; this includes adjoint and generalized adjoint quotients of complex semisimple Lie groups which, in turn, underly certain lattice gauge theories. Other examples come from certain moduli spaces of holomorphic vector bundles on a Riemann surface and variants thereof; in physics language, these are spaces of conformal blocks. Still other examples arise from the closures of holomorphic nilpotent orbits. Symplectic reduction carries a Kähler manifold to a stratified Kähler space in such a way that the sheaf of germs of polarized functions coincides with the ordinary sheaf of germs of holomorphic functions. Projectivization of the closures of holomorphic nilpotent orbits yields exotic stratified Kähler structures on complex projective spaces and on certain complex projective varieties including complex projective quadrics. Other physical examples are reduced spaces relative to a constant value of angular momentum.

In the presence of singularities, a naive approach to quantization might consist in restriction of the quantization problem to a smooth open dense part, the “top stratum”. However this naive procedure may lead to a loss of information and in fact to inconsistent results. To explore the potential impact of classical phase space singularities on quantum problems, we developed the notion of *costratified Hilbert space*. This is the appropriate quantum state space over a stratified space; a costratified Hilbert space consists of a system of Hilbert spaces, one for each stratum which arises from quantization on the closure of that stratum, the stratification is reflected in certain bounded linear operators between these Hilbert spaces reversing the partial ordering among the strata, and the linear operators are compatible with the quantizations. The notion of costratified Hilbert space is, perhaps, the *quantum structure having the classical singularities as its shadow*. Within the framework of holomorphic quantization, a suitable quantization procedure on stratified Kähler spaces leads to costratified Hilbert spaces. Given a Kähler manifold with a hamiltonian action of a compact Lie group that also preserves the complex structure, reduction after quantization then coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the invariant unreduced and reduced quantum observables as well.

We illustrate the approach with a certain concrete model: In a particular case, we describe a quantum (lattice) gauge theory which incorporates certain classical singularities. The reduced phase space is a stratified Kähler space; we make explicit the requisite singular holomorphic quantization procedure and spell out the resulting costratified Hilbert space. In particular, certain tunneling probabilities between the strata emerge, we will explain how the energy eigenstates can be determined, and we will explore the corresponding expectation values of the orthoprojectors onto the subspaces associated with the strata in the strong and weak coupling approximations.

The physics described in the present lecture notes was worked out in research collaboration with my physics friends G. Rudolph and M. Schmidt [28], [29]. I am much indebted to them for having taught me the relevant physics.

## 2 Physical systems with classical phase space singularities

### 2.1 An example of a classical phase space singularity

In  $\mathbb{R}^3$  with coordinates  $x, y, r$ , consider the semicone  $N$  given by the equation  $x^2 + y^2 = r^2$  and the inequality  $r \geq 0$ . We refer to this semicone as the *exotic plane* with a single vertex. The semicone  $N$  is the classical reduced *phase space* of a single particle moving in ordinary affine space of dimension  $\geq 2$  with angular momentum zero. In Section 7 below we will actually justify this claim. The reduced Poisson algebra  $(C^\infty N, \{ \cdot, \cdot \})$  may be described in the following fashion: Let  $x$  and  $y$  be the ordinary coordinate functions in the plane, and consider the algebra  $C^\infty N$  of smooth functions in the variables  $x, y, r$  subject to the relation  $x^2 + y^2 = r^2$ . Define the Poisson bracket  $\{ \cdot, \cdot \}$  on this algebra by

$$\{x, y\} = 2r, \quad \{x, r\} = 2y, \quad \{y, r\} = -2x,$$

and endow  $N$  with the complex structure having  $z = x + iy$  as holomorphic coordinate. The Poisson bracket is then *defined at the vertex* as well, away from the vertex the Poisson structure is an ordinary *symplectic* Poisson structure, and the complex structure does *not* “see” the vertex. At the vertex, the radius function  $r$  is *not* a smooth function of the variables  $x$  and  $y$ . Thus the vertex is a singular point for the Poisson structure whereas it is *not* a singular point for the complex analytic structure. The Poisson and complex analytic structure combine to a “stratified Kähler structure”. Below we will explain what this means.

### 2.2 Lattice gauge theory

Let  $K$  be a compact Lie group, let  $\mathfrak{k}$  denote its Lie algebra, and let  $K^\mathbb{C}$  be the complexification of  $K$ . Endow  $\mathfrak{k}$  with an invariant inner product. The polar decomposition of the complex group  $K^\mathbb{C}$  and the inner product on  $\mathfrak{k}$  induce a diffeomorphism

$$(2.1) \quad T^*K \cong TK \longrightarrow K \times \mathfrak{k} \longrightarrow K^\mathbb{C}$$

in such a way that the complex structure on  $K^\mathbb{C}$  and the cotangent bundle symplectic structure on  $T^*K$  combine to  $K$ -bi-invariant Kähler structure. When we then build a lattice gauge theory from a configuration space  $Q$  which is the product  $Q = K^\ell$  of  $\ell$  copies of  $K$ , we arrive at the (unreduced) momentum phase

space

$$T^*Q = T^*K^\ell \cong (K^\mathbb{C})^\ell,$$

and reduction modulo the  $K$ -symmetry given by conjugation leads to a reduced phase space of the kind

$$T^*K^\ell // K \cong (K^\mathbb{C})^\ell // K^\mathbb{C}$$

which necessarily involves singularities in a sense to be made precise, however. Here  $T^*K^\ell // K$  denotes the symplectic quotient whereas  $(K^\mathbb{C})^\ell // K^\mathbb{C}$  refers to the complex algebraic quotient (geometric invariant theory quotient). The special case  $\ell = 1$ , that of a single spatial plaquette—a quotient of the kind  $K^\mathbb{C} // K^\mathbb{C}$  is referred to in the literature as an *adjoint quotient*—, is mathematically already very attractive and presents a host of problems which we have elaborated upon in [28]. To explain how in this particular case the structure of the reduced phase space can be unravelled, following [28], we proceed as follows:

Pick a maximal torus  $T$  of  $K$ , denote the rank of  $T$  by  $r$ , and let  $W$  be the Weyl group of  $T$  in  $K$ . Then, as a space,  $T^*T$  is diffeomorphic to the complexification  $T^\mathbb{C}$  of the torus  $T$  and  $T^\mathbb{C}$ , in turn, amounts to a product  $(\mathbb{C}^*)^r$  of  $r$  copies of the space  $\mathbb{C}^*$  of non-zero complex numbers. Moreover, the reduced phase space  $\mathcal{P}$  comes down to the space  $T^*T/W \cong (\mathbb{C}^*)^r/W$  of  $W$ -orbits in  $(\mathbb{C}^*)^r$  relative to the action of the Weyl group  $W$ .

Viewed as the orbit space  $T^*T/W$ , via singular Marsden-Weinstein reduction, the reduced phase space  $\mathcal{P}$  inherits a stratified symplectic structure. That is to say: (i) The algebra  $C^\infty(T^\mathbb{C})^W$  of ordinary smooth  $W$ -invariant functions on  $T^\mathbb{C}$  inherits a Poisson bracket and thus furnishes a Poisson algebra of continuous functions on  $\mathcal{P}$ ; (ii) for each stratum, the Poisson structure yields an ordinary symplectic Poisson structure on that stratum; and (iii) the restriction mapping from  $C^\infty(T^\mathbb{C})^W$  to the algebra of ordinary smooth functions on that stratum is a Poisson map.

Viewed as the orbit space  $T^\mathbb{C}/W$ , the reduced phase space  $\mathcal{P}$  acquires a complex analytic structure. The complex analytic structure and the Poisson structure combine to a *stratified Kähler structure* on  $\mathcal{P}$  [20], [24], [25]. The precise meaning of the term “stratified Kähler structure” is that the Poisson structure satisfies (ii) and (iii) above and that the Poisson and complex structures satisfy the additional compatibility condition that, for each stratum, necessarily a complex manifold, the symplectic and complex structures on that stratum combine to an ordinary Kähler structure.

In Section 12 below we will discuss a model that originates, in the hamiltonian approach, from lattice gauge theory with respect to the group  $K$ . The (classical unreduced) Hamiltonian  $H: T^*K \rightarrow \mathbb{R}$  of this model is given by

$$(2.2) \quad H(x, Y) = -\frac{1}{2}|Y|^2 + \frac{\nu}{2}(3 - \operatorname{Re} \operatorname{tr}(x)), \quad x \in K, Y \in \mathfrak{k}.$$

Here  $\nu = 1/g^2$ , where  $g$  is the coupling constant, the notation  $|\cdot|$  refers to the norm defined by the inner product on  $\mathfrak{k}$ , and the trace refers to some representation of  $K$ ; below we will suppose  $K$  to be realized as a closed subgroup of some unitary group. Moreover, the lattice spacing is here set equal to 1. The Hamiltonian  $H$  is manifestly gauge invariant.

### 2.3 The canoe

We will now explore the following special case:

$$K = \mathrm{SU}(2), \quad K^{\mathbb{C}} = \mathrm{SL}(2, \mathbb{C}), \quad W \cong \mathbb{Z}/2.$$

A maximal torus  $T$  in  $\mathrm{SU}(2)$  is simply a copy of the circle group  $S^1$ , the space  $T^*T \cong T^{\mathbb{C}}$  is a copy of the space  $\mathbb{C}^*$  of non-zero complex numbers, and the  $W$ -invariant holomorphic map

$$(2.3) \quad f: \mathbb{C}^* \longrightarrow \mathbb{C}, \quad f(z) = z + z^{-1}$$

induces a complex analytic isomorphism  $\mathcal{P} \longrightarrow \mathbb{C}$  from the reduced space

$$\mathcal{P} = T^*K//K \cong T^*T/W \cong \mathbb{C}^*/W$$

onto a single copy  $\mathbb{C}$  of the complex line.

**Remark.** More generally, for  $K = \mathrm{SU}(n)$ , complex analytically,  $T^*K//K$  comes down to  $(n-1)$ -dimensional complex affine space  $\mathbb{C}^{n-1}$ . Indeed,  $K^{\mathbb{C}} = \mathrm{SL}(n, \mathbb{C})$ , having  $(\mathbb{C}^*)^{n-1}$  as a maximal complex torus. Realize this torus as the subspace of  $(\mathbb{C}^*)^n$  which consists of all  $(z_1, \dots, z_n)$  such that  $z_1 \dots z_n = 1$ . Then the elementary symmetric functions  $\sigma_1, \dots, \sigma_{n-1}$  yield the map

$$\begin{aligned} (\sigma_1, \dots, \sigma_{n-1}): (\mathbb{C}^*)^{n-1} &\longrightarrow \mathbb{C}^{n-1}, \\ \mathbf{z} = (z_1, \dots, z_n) &\longmapsto (\sigma_1(\mathbf{z}), \dots, \sigma_{n-1}(\mathbf{z})) \end{aligned}$$

which, in turn, induces the complex analytic isomorphism

$$\mathrm{SL}(n, \mathbb{C})//\mathrm{SL}(n, \mathbb{C}) \cong (\mathbb{C}^*)^{n-1}/W \cong \mathbb{C}^{n-1}$$

from the quotient onto a copy of  $\mathbb{C}^{n-1}$ . We note that, more generally, when  $K$  is a general connected and simply connected Lie group of rank  $r$  (say), in view of an observation of Steinberg's [44], the fundamental characters  $\chi_1, \dots, \chi_r$  of  $K^{\mathbb{C}}$  furnish a map from  $K^{\mathbb{C}}$  onto  $r$ -dimensional complex affine space  $\mathbb{A}^r$  which identifies the complex adjoint quotient  $K^{\mathbb{C}}//K^{\mathbb{C}}$  with  $\mathbb{A}^r$ . As a *stratified Kähler space*, the quotient has considerably more structure, though. We explain this in the sequel for the special case under consideration.

Thus we return to the special case  $K = \mathrm{SU}(2)$ : In view of the realization of the complex analytic structure via the holomorphic map  $f: \mathbb{C}^* \rightarrow \mathbb{C}$  given by  $f(z) = z + z^{-1}$  spelled out above, complex analytically, the quotient  $\mathcal{P}$  is just a copy  $\mathbb{C}$  of the complex line, and we will take  $Z = z + z^{-1}$  as a holomorphic coordinate on the quotient. On the other hand, in terms of the notation

$$\begin{aligned} z &= x + iy, \quad Z = X + iY, \quad r^2 = x^2 + y^2, \\ X &= x + \frac{x}{r^2}, \quad Y = y - \frac{y}{r^2}, \quad \tau = \frac{y^2}{r^2}, \end{aligned}$$

the real structure admits the following description: In the case at hand, the algebra written above as  $C^\infty(T^\mathbb{C})^W$  comes down to the algebra  $C^\infty(\mathcal{P})$  of continuous functions on  $\mathcal{P} \cong \mathbb{C}$  which are smooth functions in three variables (say)  $X, Y, \tau$ , subject to certain relations; the notation  $C^\infty(\mathcal{P})$  is common for such an algebra of continuous functions even though the elements of this algebra are not necessarily ordinary smooth functions. To explain the precise structure of the algebra  $C^\infty(\mathcal{P})$ , consider ordinary real 3-space with coordinates  $X, Y, \tau$  and, in this 3-space, let  $C$  be the real semi-algebraic set given by

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau, \quad \tau \geq 0.$$

As a space,  $C$  can be identified with  $\mathcal{P}$ . Further, a real analytic change of coordinates, spelled out in Section 7 of [25], actually identifies  $C$  with the familiar *canoe*. The algebra  $C^\infty(\mathcal{P})$  is that of Whitney-smooth functions on  $C$ , that is, continuous functions on  $C$  that are restrictions of smooth functions in the variables  $X, Y, \tau$  or, equivalently, smooth functions in the variables  $X, Y, \tau$ , where two functions are identified whenever they coincide on  $C$ . The Poisson brackets on  $C^\infty(\mathcal{P})$  are determined by the formulas

$$\begin{aligned} \{X, Y\} &= X^2 + Y^2 + 4(2\tau - 1), \\ \{X, \tau\} &= 2(1 - \tau)Y, \\ \{Y, \tau\} &= 2\tau X. \end{aligned}$$

On the subalgebra of  $C^\infty(\mathcal{P})$  which consists of real polynomial functions in the variables  $X, Y, \tau$ , the relation

$$Y^2 = (X^2 + Y^2 + 4(\tau - 1))\tau$$

is defining. The resulting *stratified Kähler* structure on  $\mathcal{P} \cong \mathbb{C}$  is *singular* at  $-2 \in \mathbb{C}$  and  $2 \in \mathbb{C}$ , that is, the Poisson structure *vanishes* at either of these two points. Further, at  $-2 \in \mathbb{C}$  and  $2 \in \mathbb{C}$ , the function  $\tau$  is *not* an ordinary smooth function of the variables  $X$  and  $Y$ , viz.

$$\tau = \frac{1}{2} \sqrt{Y^2 + \frac{(X^2 + Y^2 - 4)^2}{16}} - \frac{X^2 + Y^2 - 4}{8},$$

whereas away from  $-2 \in \mathbb{C}$  and  $2 \in \mathbb{C}$ , the Poisson structure is an ordinary symplectic Poisson structure. This makes explicit, in the case at hand, the singular character of the reduced space  $\mathcal{P}$  as a stratified Kähler space which, as a complex analytic space, is just a copy of  $\mathbb{C}$ , though and, as such, has *no* singularities, i. e. is an ordinary complex manifold.

For later reference, we will now describe the stratification of the reduced configuration space  $\mathcal{X} \cong T/W$  and that of the reduced phase space  $\mathcal{P} \cong (T \times \mathfrak{t})/W$ . The stratifications we will use arise from the  $W$ -orbit type decompositions: We will not make precise the notion of stratification and that of stratified space, see e. g. [10].

The torus  $T$  amounts to the complex unit circle and its Lie algebra  $\mathfrak{t}$  to the imaginary axis. The Weyl group  $W = S_2$  acts on  $T$  by complex conjugation and on  $\mathfrak{t}$  by reflection. Hence the reduced configuration space  $\mathcal{X} \cong T/W$  is homeomorphic to the closed interval  $[-1, 1]$  and the reduced phase space  $\mathcal{P} \cong (T \times \mathfrak{t})/W$  to the well-known canoe, see Figure 1.

Let

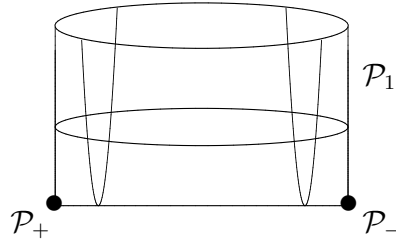
$$\mathcal{X}_+ = \{1\}, \quad \mathcal{X}_- = \{-1\}, \quad \mathcal{X}_0 = \mathcal{X}_+ \cup \mathcal{X}_- = \{-1, 1\}, \quad \mathcal{X}_1 = ]-1, 1[$$

so that the orbit type decomposition of  $\mathcal{X}$  relative to the  $W$ -action has the form  $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_0$ . The “piece”  $\mathcal{X}_1$  (the open interval) is connected; it is the “top” stratum, the open, connected and dense stratum. In particular, the restriction to the pre-image of  $\mathcal{X}_1$  of the orbit projection is a  $W$ -covering projection. The lower stratum  $\mathcal{X}_0$  decomposes into the two connected components  $\mathcal{X}_+$  and  $\mathcal{X}_-$ ; the single point in  $\mathcal{X}_+$  arises from a fixed point of the  $W$ -action, and the same is true of  $\mathcal{X}_-$ . Likewise the orbit type decomposition of  $\mathcal{P}$  relative to the  $W$ -action has the form  $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_0$ . Here the “piece”  $\mathcal{P}_1$  is the “top” stratum, i. e. the open, connected and dense stratum which is here 2-dimensional. As before, the restriction to the pre-image of  $\mathcal{P}_1$  of the orbit projection is a  $W$ -covering projection. Further,  $\mathcal{P}_0$  decomposes into two connected components  $\mathcal{P}_0 = \mathcal{P}_+ \cup \mathcal{P}_-$ , each containing a vertex of the canoe; each such vertex arises from a fixed point of the  $W$ -action. Under the identification of  $\mathcal{P}$  with the complex line  $\mathbb{C}$  described previously, the two vertices of the canoe correspond to the points  $2$  and  $-2$  of  $\mathbb{C}$  so that

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C}, \quad \mathcal{P}_- = \{-2\} \subseteq \mathbb{C}, \quad \mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 = \mathbb{C} \setminus \{2, -2\}.$$

A closer look reveals that we can see that decomposition of  $\mathbb{C}$  as arising from hyperbolic cosine, viewed as a holomorphic function: The two points  $2$  and  $-2$  are the focal points of the corresponding families of ellipses and hyperbolas in  $\mathbb{C}$ . Two of these ellipses and two of these hyperbolas are in fact indicated in Figure 1. We will come back to the stratifications in Sections 10 and 13 below.

**Remark 2.1.** In the case under discussion ( $K = \mathrm{SU}(2)$ ), as a stratified symplectic space,  $\mathcal{P}$  is isomorphic to the reduced phase space of a spherical pendulum, reduced at vertical angular momentum  $0$  (whence the pendulum is constrained to move in a plane), see [8].

Figure 1: The reduced phase space  $\mathcal{P}$  for  $K = \mathrm{SU}(2)$ .

### 3 Stratified Kähler spaces

In the presence of singularities, restricting quantization to a smooth open dense stratum, sometimes referred to as “top stratum”, can result in a loss of information and may in fact lead to *inconsistent* results. To develop a satisfactory notion of Kähler quantization in the presence of singularities, on the classical level, we isolated a notion of “Kähler space with singularities”; we refer to such a space as a *stratified Kähler space*. Ordinary *Kähler quantization* may then be extended to a *quantization scheme over stratified Kähler spaces*.

We will now explain the concept of a *stratified Kähler space*. In [20] we introduced a general notion of stratified Kähler space and that of complex analytic stratified Kähler space as a special case. We do not know whether the two notions really differ. For the present paper, the notion of complex analytic stratified Kähler space suffices. To simplify the terminology somewhat, “stratified Kähler space” will always mean “complex analytic stratified Kähler space”.

We recall first that, given a stratified space  $N$ , a *stratified symplectic structure on  $N$*  is a Poisson algebra  $(C^\infty N, \{ \cdot, \cdot \})$  of continuous functions on  $N$  which, on each stratum, amounts to an ordinary smooth symplectic Poisson algebra. The functions in  $C^\infty N$  are not necessarily ordinary smooth functions. Restriction of the functions in  $C^\infty N$  to a stratum is required to yield the compactly supported functions on that stratum, and these suffice to generate a symplectic Poisson algebra on the stratum.

Next we recall that a *complex analytic space* (in the sense of GRAUERT) is a topological space  $X$ , together with a sheaf of rings  $\mathcal{O}_X$ , having the following property: The space  $X$  can be covered by open sets  $Y$ , each of which embeds into the polydisc  $U$  in some  $\mathbb{C}^n$  (the number  $n$  may vary as  $U$  varies) as the zero set of a finite system of holomorphic functions  $f_1, \dots, f_q$  defined on  $U$ , such that the restriction  $\mathcal{O}_Y$  of the sheaf  $\mathcal{O}_X$  to  $Y$  is isomorphic as a sheaf to the quotient sheaf  $\mathcal{O}_U / (f_1, \dots, f_q)$ ; here  $\mathcal{O}_U$  is the sheaf of germs of holomorphic functions on  $U$ . The sheaf  $\mathcal{O}_X$  is then referred to as the *sheaf of holomorphic functions on  $X$* . See [11] for a development of the general theory of complex analytic spaces.

**Definition 3.1.** A *stratified Kähler space* consists of a complex analytic space  $N$ , together with

- (i) a complex analytic stratification (a not necessarily proper refinement of the standard complex analytic stratification), and with
- (ii) a stratified symplectic structure  $(C^\infty N, \{ \cdot, \cdot \})$  which is compatible with the complex analytic structure

The two structures being *compatible* means the following:

- (i) For each point  $q$  of  $N$  and each holomorphic function  $f$  defined on an open neighborhood  $U$  of  $q$ , there is an open neighborhood  $V$  of  $q$  with  $V \subset U$  such that, on  $V$ ,  $f$  is the restriction of a function in  $C^\infty(N)$ ;
- (ii) on each stratum, the symplectic structure determined by the symplectic Poisson structure (on that stratum) combines with the complex analytic structure to a Kähler structure.

EXAMPLE 1: The *exotic plane*, endowed with the structure explained in Subection 2.1 above, is a stratified Kähler space. Here the radius function  $r$  is *not* an ordinary smooth function of the variables  $x$  and  $y$ . Thus the stratified symplectic structure cannot be given in terms of ordinary smooth functions of the variables  $x$  and  $y$ .

This example generalizes to an entire class of examples: The *closure of a holomorphic nilpotent orbit* (in a hermitian Lie algebra) inherits a stratified Kähler structure [20]. Angular momentum zero reduced spaces are special cases thereof; see Section 7 below for details.

*Projectivization* of the closure of a holomorphic nilpotent orbit yields what we call an *exotic projective variety*. This includes complex quadrics, SEVERI and SCORZA varieties and their *secant* varieties [20], [22]. In physics, spaces of this kind arise as reduced classical phase spaces for systems of harmonic oscillators with zero angular momentum and constant energy. We shall explain some of the details in Section 7 below.

EXAMPLE 2: Moduli spaces of semistable holomorphic vector bundles or, more generally, moduli spaces of semistable principal bundles on a non-singular complex projective curve carry stratified Kähler structures [20]. These spaces arise as moduli spaces of homomorphisms or more generally twisted homomorphisms from fundamental groups of surfaces to compact connected Lie groups as well. In conformal field theory, they occur as spaces of *conformal blocks*. The construction of the moduli spaces as complex projective varieties goes back to [37] and [42]; see [43] for an exposition of the general theory. Atiyah and Bott [6] initiated another approach to the study of these moduli spaces by identifying them with moduli spaces of projectively flat constant central curvature connections on principal bundles over Riemann surfaces, which they analyzed by methods of gauge theory. In particular, by applying the method of symplectic reduction to the action of the infinite-dimensional group of gauge transformations on the infinite-dimensional symplectic manifold of all connections on a principal bundle, they showed that an invariant inner product on the Lie algebra of the Lie group in question induces a

natural symplectic structure on a certain smooth open stratum which, together with the complex analytic structure, turns that stratum into an ordinary Kähler manifold. This infinite-dimensional approach to moduli spaces has roots in quantum field theory. Thereafter a finite-dimensional construction of the moduli space as a symplectic quotient arising from an ordinary finite-dimensional Hamiltonian  $G$ -space for a compact Lie group  $G$  was developed; see [17], [18] and the literature there; this construction exhibits the moduli space as a stratified symplectic space. The stratified Kähler structure mentioned above combines the complex analytic structure with the stratified symplectic structure; it includes the Kähler manifold structure on the open and dense stratum.

An important special case is that of the moduli space of semistable rank 2 degree zero vector bundles with trivial determinant on a curve of genus 2. As a space, this is just ordinary complex projective 3-space, but the stratified symplectic structure involves more functions than just ordinary smooth functions. The complement of the space of stable vector bundles is a *Kummer surface*. See [16], [18] and the literature there.

Any ordinary Kähler manifold is plainly a stratified Kähler space. This kind of example generalizes in the following fashion: For a Lie group  $K$ , we will denote its Lie algebra by  $\mathfrak{k}$  and the dual thereof by  $\mathfrak{k}^*$ . The next result says that, roughly speaking, Kähler reduction, applied to an ordinary Kähler manifold, yields a stratified Kähler structure on the reduced space.

**Theorem 3.2** ([20]). *Let  $N$  be a Kähler manifold, acted upon holomorphically by a complex Lie group  $G$  such that the action, restricted to a compact real form  $K$  of  $G$ , preserves the Kähler structure and is hamiltonian, with momentum mapping  $\mu: N \rightarrow \mathfrak{k}^*$ . Then the reduced space  $N_0 = \mu^{-1}(0)/K$  inherits a stratified Kähler structure.*

For intelligibility, we explain briefly how the structure on the reduced space  $N_0$  arises. Details may be found in [20]: Define  $C^\infty(N_0)$  to be the quotient algebra  $C^\infty(N)^K/I^K$ , that is, the algebra  $C^\infty(N)^K$  of smooth  $K$ -invariant functions on  $N$ , modulo the ideal  $I^K$  of functions in  $C^\infty(N)^K$  that vanish on the zero locus  $\mu^{-1}(0)$ . The ordinary smooth symplectic Poisson structure  $\{\cdot, \cdot\}$  on  $C^\infty(N)$  is  $K$ -invariant and hence induces a Poisson structure on the algebra  $C^\infty(N)^K$  of smooth  $K$ -invariant functions on  $N$ . Furthermore, Noether's theorem entails that the ideal  $I^K$  is a Poisson ideal, that is to say, given  $f \in C^\infty(N_0)^K$  and  $h \in I^K$ , the function  $\{f, h\}$  is in  $I^K$  as well. Consequently the Poisson bracket  $\{\cdot, \cdot\}$  descends to a Poisson bracket  $\{\cdot, \cdot\}_0$  on  $C^\infty(N_0)$ . Relative to the orbit type stratification, the Poisson algebra  $(C^\infty N_0, \{\cdot, \cdot\}_0)$  turns  $N_0$  into a stratified symplectic space.

The inclusion of  $\mu^{-1}(0)$  into  $N$  passes to a homeomorphism from  $N_0$  onto the categorical  $G$ -quotient  $N//G$  of  $N$  in the category of complex analytic varieties. The stratified symplectic structure combines with the complex analytic structure on  $N//G$  to a stratified Kähler structure. When  $N$  is complex algebraic, the complex algebraic  $G$ -quotient coincides with the complex analytic  $G$ -quotient.

Thus, in view of Theorem 3.2, examples of stratified Kähler spaces abound.

EXAMPLE 3: Adjoint quotients of complex reductive Lie groups, see (2.2) above.

**Remark 3.3.** In [6], ATIYAH AND BOTT raised the issue of *determining the singularities* of moduli spaces of semistable holomorphic vector bundles or, more generally, of moduli spaces of semistable principal bundles on a non-singular complex projective curve. The stratified Kähler structure which we isolated on a moduli space of this kind, as explained in Example 2 above, actually determines the singularity structure; in particular, near any point, the structure may be understood in terms of a suitable local model. The appropriate notion of singularity is that of singularity in the sense of stratified Kähler spaces; this notion depends on the entire structure, not just on the complex analytic structure. Indeed, the examples spelled out above (the exotic plane with a single vertex, the exotic plane with two vertices, the 3-dimensional complex projective space with the Kummer surface as singular locus, etc.) show that a point of a stratified Kähler space may well be a singular point without being a complex analytic singularity.

## 4 Quantum theory and classical singularities

According to DIRAC, the *correspondence* between a classical theory and its quantum counterpart should be based on an analogy between their mathematical structures. An interesting issue is then that of the role of singularities in quantum problems. Singularities are known to arise in classical phase spaces. For example, in the hamiltonian picture of a theory, reduction modulo gauge symmetries leads in general to singularities on the classical level. This leads to the question what the significance of singularities on the quantum side might be. Can we ignore them, or is there a quantum structure which has the classical singularities as its shadow? As far as know, one of the first papers in this topic is that of EMMRICH AND RÖMER [9]. This paper indicates that wave functions may “congregate” near a *singular* point, which goes counter to the sometimes quoted statement that *singular points in a quantum problem are a set of measure zero so cannot possibly be important*. In a similar vein, ASOREY ET AL observed that vacuum nodes correspond to the chiral gauge orbits of reducible gauge fields with non-trivial magnetic monopole components [4]. It is also noteworthy that in classical mechanics and in classical field theories singularities in the solution spaces are the *rule rather than the exception*. This is in particular true for Yang-Mills theories and for Einstein’s gravitational theory where singularities occur even at some of the most interesting and physically relevant solutions, namely at the symmetric ones. It is still not understood what role these singularities might have in quantum gravity. See, for example, ARMS, MARSDEN AND MONCRIEF [2], [3] and the literature there.

## 5 Correspondence principle and Lie-Rinehart algebras

To make sense of the *correspondence principle* in certain *singular* situations, one needs a tool which, for the stratified symplectic Poisson algebra on a stratified symplectic space, serves as a *replacement* for the tangent bundle of a smooth symplectic manifold. This replacement is provided by an appropriate *Lie-Rinehart algebra*. This Lie-Rinehart algebra yields in particular a satisfactory generalization of the Lie algebra of smooth vector fields in the smooth case. This enables us to put *flesh on the bones of Dirac's correspondence principle in certain singular situations*.

A *Lie-Rinehart algebra* consists of a commutative algebra and a Lie algebra with additional structure which generalizes the mutual structure of interaction between the algebra of smooth functions and the Lie algebra of smooth vector fields on a smooth manifold. More precisely:

**Definition 5.1.** A *Lie-Rinehart algebra* consists of a commutative algebra  $A$  and a Lie-algebra  $L$  such that  $L$  acts on  $A$  by derivations and that  $L$  has an  $A$ -module structure, and these are required to satisfy

$$\begin{aligned} [\alpha, a\beta] &= \alpha(a)\beta + a[\alpha, \beta], \\ (a\alpha)(b) &= a(\alpha(b)), \end{aligned}$$

where  $a, b \in A$  and  $\alpha, \beta \in L$ .

**Definition 5.2.** An  $A$ -module  $M$  which is also a left  $L$ -module is called a *left  $(A, L)$ -module* provided

$$(5.1) \quad \alpha(ax) = \alpha(a)x + a\alpha(x)$$

$$(5.2) \quad (a\alpha)(x) = a(\alpha(x))$$

where  $a \in A$ ,  $x \in M$ ,  $\alpha \in L$ .

We will now explain briefly the Lie-Rinehart algebra associated with a Poisson algebra; more details may be found in [14], [15], and [23]. Thus, let  $(A, \{\cdot, \cdot\})$  be a Poisson algebra. Let  $D_A$  the the  $A$ -module of formal differentials of  $A$  the elements of which we write as  $du$ , for  $u \in A$ . For  $u, v \in A$ , the association

$$(du, dv) \longrightarrow \pi(du, dv) = \{u, v\}$$

yields an  $A$ -valued  $A$ -bilinear skew-symmetric 2-form  $\pi = \pi_{\{\cdot, \cdot\}}$  on  $D_A$ , referred to as the *Poisson 2-form* associated with the Poisson structure  $\{\cdot, \cdot\}$ . The adjoint

$$\pi^\sharp: D_A \longrightarrow \text{Der}(A) = \text{Hom}_A(D_A, A)$$

of  $\pi$  is a morphism of  $A$ -modules, and the formula

$$[adu, bdv] = a\{u, b\}dv + b\{a, v\}du + abd\{u, v\}$$

yields a Lie bracket  $[\cdot, \cdot]$  on  $D_A$ .

**Theorem 5.3** ([14]). *The  $A$ -module structure on  $D_A$ , the bracket  $[\cdot, \cdot]$ , and the morphism  $\pi^\sharp$  of  $A$ -modules turn the pair  $(A, D_A)$  into a Lie-Rinehart algebra.*

We will write the resulting Lie-Rinehart algebra as  $(A, D_{\{\cdot, \cdot\}})$ . For intelligibility we recall that, given a Lie-Rinehart algebra  $(A, L)$ , the Lie algebra  $L$  together with the additional  $A$ -module structure on  $L$  and  $L$ -module structure on  $A$  is referred to as an  $(\mathbb{R}, A)$ -Lie algebra. Thus  $D_{\{\cdot, \cdot\}}$  is an  $(\mathbb{R}, A)$ -Lie algebra.

When the Poisson algebra  $A$  is the algebra of smooth functions  $C^\infty(M)$  on a symplectic manifold  $M$ , the  $A$ -dual  $\text{Der}(A) = \text{Hom}_A(D_A, A)$  of  $D_A$  amounts to the  $A$ -module  $\text{Vect}(M)$  of smooth vector fields, and

$$(5.3) \quad (\pi^\sharp, \text{Id}): (D_A, A) \longrightarrow (\text{Vect}(M), C^\infty(M))$$

is a morphism of Lie-Rinehart algebras, where  $(\text{Vect}(M), C^\infty(M))$  carries its ordinary Lie-Rinehart structure. The  $A$ -module morphism  $\pi^\sharp$  is plainly surjective, and the kernel consists of those formal differentials which “vanish at each point of”  $M$ .

We return to our general Poisson algebra  $(A, \{\cdot, \cdot\})$ . The Poisson 2-form  $\pi_{\{\cdot, \cdot\}}$  determines an *extension*

$$(5.4) \quad 0 \longrightarrow A \longrightarrow \bar{L}_{\{\cdot, \cdot\}} \longrightarrow D_{\{\cdot, \cdot\}} \longrightarrow 0$$

of  $(\mathbb{R}, A)$ -Lie algebras which is central as an extension of ordinary Lie algebras; in particular, on the kernel  $A$ , the Lie bracket is trivial. Moreover, as  $A$ -modules,

$$(5.5) \quad \bar{L}_{\{\cdot, \cdot\}} = A \oplus D_{\{\cdot, \cdot\}},$$

and the Lie bracket on  $\bar{L}_{\{\cdot, \cdot\}}$  is given by

$$(5.6) \quad [(a, du), (b, dv)] = (\{u, b\} + \{a, v\} - \{u, v\}, d\{u, v\}), \quad a, b, u, v \in A.$$

Here we have written “ $\bar{L}$ ” rather than simply  $L$  to indicate that the extension (5.4) represents the *negative* of the class of  $\pi_{\{\cdot, \cdot\}}$  in Poisson cohomology  $H_{\text{Poisson}}^2(A, A)$ , cf. [14]. When  $(A, \{\cdot, \cdot\})$  is the smooth symplectic Poisson algebra of an ordinary smooth symplectic manifold, (perhaps) up to sign, the class of  $\pi_{\{\cdot, \cdot\}}$  comes essentially down to the cohomology class represented by the symplectic structure.

The following concept was introduced in [15].

**Definition 5.4.** Given an  $(A \otimes \mathbb{C})$ -module  $M$ , we refer to an  $(A, \overline{L}_{\{\cdot, \cdot\}})$ -module structure

$$(5.7) \quad \chi: \overline{L}_{\{\cdot, \cdot\}} \longrightarrow \text{End}_{\mathbb{R}}(M)$$

on  $M$  as a *prequantum module structure* for  $(A, \{\cdot, \cdot\})$  provided

- (i) the values of  $\chi$  lie in  $\text{End}_{\mathbb{C}}(M)$ , that is to say, for  $a \in A$  and  $\alpha \in D_{\{\cdot, \cdot\}}$ , the operators  $\chi(a, \alpha)$  are complex linear transformations, and
- (ii) for every  $a \in A$ , with reference to the decomposition (5.5), we have

$$(5.8) \quad \chi(a, 0) = i a \text{Id}_M.$$

A pair  $(M, \chi)$  consisting of an  $(A \otimes \mathbb{C})$ -module  $M$  and a prequantum module structure will henceforth be referred to as a *prequantum module* (for  $(A, \{\cdot, \cdot\})$ ).

*Prequantization* now proceeds in the following fashion, cf. [14]: The assignment to  $a \in A$  of  $(a, da) \in \overline{L}_{\{\cdot, \cdot\}}$  yields a morphism  $\iota$  of real Lie algebras from  $A$  to  $\overline{L}_{\{\cdot, \cdot\}}$ ; thus, for any prequantum module  $(M, \chi)$ , the composite of  $\iota$  with  $-i\chi$  is a representation  $a \mapsto \widehat{a}$  of the  $A$  underlying real Lie algebra having  $M$ , viewed as a complex vector space, as its representation space; this is a representation by  $\mathbb{C}$ -linear operators so that any constant acts by multiplication, that is, for any real number  $r$ , viewed as a member of  $A$ ,

$$(5.9) \quad \widehat{r} = r \text{Id}$$

and so that, for  $a, b \in A$ ,

$$(5.10) \quad \widehat{\{a, b\}} = i [\widehat{a}, \widehat{b}] \quad (\text{the Dirac condition}).$$

More explicitly, these operators are given by the formula

$$(5.11) \quad \widehat{a}(x) = \frac{1}{i} \chi(0, da)(x) + ax, \quad a \in A, \quad x \in M.$$

In this fashion, prequantization, that is to say, the first step in the realization of the correspondence principle in one direction, can be made precise in certain singular situations.

When  $(A, \{\cdot, \cdot\})$  is the Poisson algebra of smooth functions on an ordinary smooth symplectic manifold, this prequantization factors through the morphism (5.3) of Lie-Rinehart algebras in such a way that, on the target, the construction comes down to the ordinary prequantization construction.

**Remark.** In the physics literature, Lie-Rinehart algebras were explored in a paper by KASTLER AND STORA under the name *Lie-Cartan pairs* [31].

## 6 Quantization on stratified Kähler spaces

In the paper [21] we have shown that the *holomorphic* quantization scheme may be extended to stratified Kähler spaces. We recall the main steps:

1) The notion of ordinary Kähler polarization generalizes to that of *stratified Kähler polarization*. This concept is defined in terms of the Lie-Rinehart algebra associated with the stratified symplectic Poisson structure; it specifies *polarizations on the strata* and, moreover, encapsulates the *mutual positions of polarizations on the strata*.

Under the circumstances of Theorem 3.2, *symplectic reduction carries a Kähler polarization preserved by the symmetries into a stratified Kähler polarization*.

2) The notion of prequantum bundle generalizes to that of *stratified prequantum module*. Given a stratified Kähler space, a stratified prequantum module is, roughly speaking, a system of prequantum modules in the sense of Definition 5.4, one for the closure of each stratum, together with appropriate morphisms among them which reflect the stratification.

3) The notion of quantum Hilbert space generalizes to that of *costratified quantum Hilbert space* in such a way that the costratified structure reflects the stratification on the classical level. *Thus the costratified Hilbert space structure is a quantum structure which has the classical singularities as its shadow*.

4) The main result says that  $[Q, R] = 0$ , that is, quantization commutes with reduction [21]:

**Theorem 6.1.** *Under the circumstances of Theorem 3.2, suppose that the Kähler manifold is quantizable (that is, suppose that the cohomology class of the Kähler form is integral). When a suitable additional condition is satisfied, reduction after quantization coincides with quantization after reduction in the sense that not only the reduced and unreduced quantum phase spaces correspond but the (invariant) unreduced and reduced quantum observables as well.*

What is referred to here as ‘suitable additional condition’ is a condition on the behaviour of the gradient flow. For example, when the Kähler manifold is compact, the condition will automatically be satisfied.

On the reduced level, the resulting classical phase space involves in general singularities and is a stratified Kähler space; the appropriate quantum phase space is then a costratified Hilbert space.

## 7 An illustration arising from angular momentum and holomorphic nilpotent orbits

Let  $s$  and  $\ell$  be non-zero natural numbers. The unreduced classical momentum phase space of  $\ell$  particles in  $\mathbb{R}^s$  is real affine space of real dimension  $2s\ell$ . For

example, for our solar system,  $s = 3$ , and  $\ell$  is the number of celestial bodies we take into account, that is, the sun, the planets, their moons, asteroids, etc., and the true physical phase space is the reduced space subject to the (physically reasonable) constraint that the total angular momentum of the solar system be constant and non-zero. The shifting trick reduces this case to that of total angular momentum zero relative to the planar orthogonal group. The subsequent discussion implies that the reduced phase space relative to the planar orthogonal group is the space of complex symmetric  $(\ell \times \ell)$ -matrices of rank at most equal to 2. The true reduced phase space we are looking for then fibers over a semisimple orbit in  $\mathfrak{sp}(\ell, \mathbb{R})$  with fiber the space of complex symmetric  $(\ell \times \ell)$ -matrices of rank at most equal to 2. The additional requirement that the total energy be constant then reduces the system by one more degree of freedom.

We return to the general case. Identify real affine space of real dimension  $2s\ell$  with the vector space  $(\mathbb{R}^{2s})^{\times \ell}$  as usual, endow  $\mathbb{R}^s$  with the standard inner product,  $\mathbb{R}^{2\ell}$  with the standard symplectic structure, and thereafter  $(\mathbb{R}^{2s})^{\times \ell}$  with the obvious induced inner product and symplectic structure. The isometry group of the inner product on  $\mathbb{R}^s$  is the orthogonal group  $O(s, \mathbb{R})$ , the group of linear transformations preserving the symplectic structure on  $\mathbb{R}^{2\ell}$  is the symplectic group  $Sp(\ell, \mathbb{R})$ , and the actions extend to linear  $O(s, \mathbb{R})$ - and  $Sp(\ell, \mathbb{R})$ -actions on  $(\mathbb{R}^{2s})^{\times \ell}$  in an obvious manner. As usual, denote the Lie algebras of  $O(s, \mathbb{R})$  and  $Sp(\ell, \mathbb{R})$  by  $\mathfrak{so}(s, \mathbb{R})$  and  $\mathfrak{sp}(\ell, \mathbb{R})$ , respectively.

The  $O(s, \mathbb{R})$ - and  $Sp(\ell, \mathbb{R})$ -actions on  $(\mathbb{R}^{2s})^{\times \ell}$  are hamiltonian. To spell out the  $O(s, \mathbb{R})$ -momentum mapping having the value zero at the origin, identify  $\mathfrak{so}(s, \mathbb{R})$  with its dual  $\mathfrak{so}(s, \mathbb{R})^*$  by interpreting  $a \in \mathfrak{so}(s, \mathbb{R})$  as the linear functional on  $\mathfrak{so}(s, \mathbb{R})$  which assigns  $\text{tr}(a^t x)$  to  $x \in \mathfrak{so}(s, \mathbb{R})$ ; here  ${}^t x$  refers to the transpose of the matrix  $x$ . We note that, for  $s \geq 3$ ,

$$(s - 2)\text{tr}(a^t b) = -\beta(a, b), \quad a, b \in \mathfrak{so}(s, \mathbb{R}),$$

where  $\beta$  is the KILLING form of  $\mathfrak{so}(s, \mathbb{R})$ . Moreover, for a vector  $\mathbf{x} \in \mathbb{R}^s$ , realized as a column vector, let  ${}^t \mathbf{x}$  be its transpose, so that  ${}^t \mathbf{x}$  is a row vector. With these preparations out of the way, the *angular momentum mapping*

$$\mu_O: (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{so}(s, \mathbb{R})$$

with reference to the origin is given by

$$\mu_O(\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_\ell, \mathbf{p}_\ell) = \mathbf{q}_1 {}^t \mathbf{p}_1 - \mathbf{p}_1 {}^t \mathbf{q}_1 + \dots + \mathbf{q}_\ell {}^t \mathbf{p}_\ell - \mathbf{p}_\ell {}^t \mathbf{q}_\ell.$$

Likewise, identify  $\mathfrak{sp}(\ell, \mathbb{R})$  with its dual  $\mathfrak{sp}(\ell, \mathbb{R})^*$  by interpreting  $a \in \mathfrak{sp}(\ell, \mathbb{R})$  as the linear functional on  $\mathfrak{sp}(\ell, \mathbb{R})$  which assigns  $\frac{1}{2}\text{tr}(ax)$  to  $x \in \mathfrak{sp}(\ell, \mathbb{R})$ ; we remind the reader that the *Killing* form  $\beta$  of  $\mathfrak{sp}(\ell, \mathbb{R})$  is given by

$$\beta(a, b) = 2(\ell + 1)\text{tr}(ab)$$

where  $a, b \in \mathfrak{sp}(\ell, \mathbb{R})$ . The  $\mathrm{Sp}(\ell, \mathbb{R})$ -momentum mapping

$$\mu_{\mathrm{Sp}}: (\mathbb{R}^{2s})^{\times \ell} \longrightarrow \mathfrak{sp}(\ell, \mathbb{R})$$

having the value zero at the origin is given by the assignment to

$$[\mathbf{q}_1, \mathbf{p}_1, \dots, \mathbf{q}_\ell, \mathbf{p}_\ell] \in (\mathbb{R}^s \times \mathbb{R}^s)^{\times \ell}$$

of

$$\begin{bmatrix} [\mathbf{q}_j \mathbf{p}_k] & -[\mathbf{q}_j \mathbf{q}_k] \\ [\mathbf{p}_j \mathbf{p}_k] & -[\mathbf{p}_j \mathbf{q}_k] \end{bmatrix} \in \mathfrak{sp}(\ell, \mathbb{R}),$$

where  $[\mathbf{q}_j \mathbf{p}_k]$  etc. denotes the  $(\ell \times \ell)$ -matrix having the inner products  $\mathbf{q}_j \mathbf{p}_k$  etc. as entries.

Consider the  $\mathrm{O}(s, \mathbb{R})$ -reduced space

$$N_0 = \mu_{\mathrm{O}}^{-1}(0)/\mathrm{O}(s, \mathbb{R}).$$

The  $\mathrm{Sp}(\ell, \mathbb{R})$ -momentum mapping induces an embedding of the reduced space  $N_0$  into  $\mathfrak{sp}(\ell, \mathbb{R})$ . We now explain briefly how the image of  $N_0$  in  $\mathfrak{sp}(\ell, \mathbb{R})$  may be described. More details may be found in [20], see also [22].

Choose a positive complex structure  $J$  on  $\mathbb{R}^{2\ell}$  which is compatible with  $\omega$  in the sense that  $\omega(J\mathbf{u}, J\mathbf{v}) = \omega(\mathbf{u}, \mathbf{v})$  for every  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$ ; here ‘positive’ means that the associated real inner product  $\cdot$  on  $\mathbb{R}^{2\ell}$  given by  $\mathbf{u} \cdot \mathbf{v} = \omega(\mathbf{u}, J\mathbf{v})$  for  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2\ell}$  is positive definite. The subgroup of  $\mathrm{Sp}(\ell, \mathbb{R})$  which preserves the complex structure  $J$  is a maximal compact subgroup of  $\mathrm{Sp}(\ell, \mathbb{R})$ ; relative to a suitable orthonormal basis, this group comes down to a copy of the ordinary unitary group  $\mathrm{U}(\ell)$ . Furthermore, the complex structure  $J$  induces a CARTAN decomposition

$$(7.1) \quad \mathfrak{sp}(\ell, \mathbb{R}) = \mathfrak{u}(\ell) \oplus \mathfrak{p};$$

here  $\mathfrak{u}(\ell)$  is the Lie algebra of  $\mathrm{U}(\ell)$ , the symmetric constituent  $\mathfrak{p}$  decomposes as the direct sum

$$\mathfrak{p} \cong \mathrm{S}_{\mathbb{R}}^2[\mathbb{R}^\ell] \oplus \mathrm{S}_{\mathbb{R}}^2[\mathbb{R}^\ell]$$

of two copies of the real vector space  $\mathrm{S}_{\mathbb{R}}^2[\mathbb{R}^\ell]$  of real symmetric  $(\ell \times \ell)$ -matrices, and the complex structure  $J$  induces a complex structure on  $\mathrm{S}_{\mathbb{R}}^2[\mathbb{R}^\ell] \oplus \mathrm{S}_{\mathbb{R}}^2[\mathbb{R}^\ell]$  in such a way that the resulting complex vector space is complex linearly isomorphic to the complex vector space  $\mathrm{S}_{\mathbb{C}}^2[\mathbb{C}^\ell]$  of complex symmetric  $(\ell \times \ell)$ -matrices in a canonical fashion. We refer to a nilpotent orbit  $\mathcal{O}$  in  $\mathfrak{sp}(\ell, \mathbb{R})$  as being *holomorphic* if the orthogonal projection from  $\mathfrak{sp}(\ell, \mathbb{R})$  to  $\mathrm{S}_{\mathbb{C}}^2[\mathbb{C}^\ell]$ , restricted to  $\mathcal{O}$ , is a diffeomorphism from  $\mathcal{O}$  onto its image in  $\mathrm{S}_{\mathbb{C}}^2[\mathbb{C}^\ell]$ . The diffeomorphism from a holomorphic nilpotent orbit  $\mathcal{O}$  onto its image in  $\mathrm{S}_{\mathbb{C}}^2[\mathbb{C}^\ell]$  extends to a homeomorphism from the closure  $\overline{\mathcal{O}}$  onto its image in  $\mathrm{S}_{\mathbb{C}}^2[\mathbb{C}^\ell]$ , and the closures of the holomorphic nilpotent orbits constitute an ascending sequence

$$(7.2) \quad 0 \subseteq \overline{\mathcal{O}}_1 \subseteq \dots \subseteq \overline{\mathcal{O}}_k \subseteq \dots \subseteq \overline{\mathcal{O}}_\ell \subseteq \mathfrak{sp}(\ell, \mathbb{R}), \quad 1 \leq k \leq \ell,$$

such that the orthogonal projection from  $\mathfrak{sp}(\ell, \mathbb{R})$  to  $S_{\mathbb{C}}^2[\mathbb{C}^\ell]$ , restricted to  $\overline{\mathcal{O}}_\ell$ , is a homeomorphism from  $\overline{\mathcal{O}}_\ell$  onto  $S_{\mathbb{C}}^2[\mathbb{C}^\ell]$ . For  $1 \leq k \leq \ell$ , this orthogonal projection, restricted to  $\overline{\mathcal{O}}_k$ , is a homeomorphism from  $\overline{\mathcal{O}}_k$  onto the space of complex symmetric  $(\ell \times \ell)$ -matrices of rank at most equal to  $k$ ; in particular, each space of the kind  $\overline{\mathcal{O}}_k$  is a *stratified* space, the stratification being given by the rank of the corresponding complex symmetric  $(\ell \times \ell)$ -matrices.

The Lie bracket of the Lie algebra  $\mathfrak{sp}(\ell, \mathbb{R})$  induces a Poisson bracket on the algebra  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R})^*)$  of smooth functions on the dual  $\mathfrak{sp}(\ell, \mathbb{R})^*$  of  $\mathfrak{sp}(\ell, \mathbb{R})$  in a canonical fashion. Via the identification of  $\mathfrak{sp}(\ell, \mathbb{R})$  with its dual, the Lie bracket on  $\mathfrak{sp}(\ell, \mathbb{R})$  induces a Poisson bracket  $\{\cdot, \cdot\}$  on  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$ . Indeed, the assignment to  $a \in \mathfrak{sp}(\ell, \mathbb{R})$  of the linear function

$$f_a: \mathfrak{sp}(\ell, \mathbb{R}) \longrightarrow \mathbb{R}$$

given by  $f_a(x) = \frac{1}{2}\text{tr}(ax)$  induces a linear isomorphism

$$(7.3) \quad \mathfrak{sp}(\ell, \mathbb{R}) \longrightarrow \mathfrak{sp}(\ell, \mathbb{R})^*;$$

let

$$[\cdot, \cdot]^*: \mathfrak{sp}(\ell, \mathbb{R})^* \otimes \mathfrak{sp}(\ell, \mathbb{R})^* \longrightarrow \mathfrak{sp}(\ell, \mathbb{R})^*$$

be the bracket on  $\mathfrak{sp}(\ell, \mathbb{R})^*$  induced by the Lie bracket on  $\mathfrak{sp}(\ell, \mathbb{R})$ . The Poisson bracket  $\{\cdot, \cdot\}$  on the algebra  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  is given by the formula

$$\{f, h\}(x) = [f'(x), h'(x)]^*(x), \quad x \in \mathfrak{sp}(\ell, \mathbb{R}).$$

The isomorphism (7.3) induces an embedding of  $\mathfrak{sp}(\ell, \mathbb{R})$  into  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$ , and this embedding is plainly a morphism

$$\delta: \mathfrak{sp}(\ell, \mathbb{R}) \longrightarrow C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$$

of Lie algebras when  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  is viewed as a real Lie algebra via the Poisson bracket. In the literature, a morphism of the kind  $\delta$  is referred to as a *comomentum* mapping.

Let  $\mathcal{O}$  be a holomorphic nilpotent orbit. The embedding of  $\overline{\mathcal{O}}$  into  $\mathfrak{sp}(\ell, \mathbb{R})$  induces a map from the algebra  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  of ordinary smooth functions on  $\mathfrak{sp}(\ell, \mathbb{R})$  to the algebra  $C^0(\overline{\mathcal{O}})$  of continuous functions on  $\overline{\mathcal{O}}$ , and we denote the image of  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  in  $C^0(\overline{\mathcal{O}})$  by  $C^\infty(\overline{\mathcal{O}})$ . By construction, each function in  $C^\infty(\overline{\mathcal{O}})$  is the restriction of an ordinary smooth function on the ambient space  $\mathfrak{sp}(\ell, \mathbb{R})$ . Since each stratum of  $\overline{\mathcal{O}}$  is an ordinary smooth closed submanifold of  $\mathfrak{sp}(\ell, \mathbb{R})$ , the functions in  $C^\infty(\overline{\mathcal{O}})$ , restricted to a stratum of  $\overline{\mathcal{O}}$ , are ordinary smooth functions on that stratum. Hence  $C^\infty(\overline{\mathcal{O}})$  is a *smooth structure* on  $\overline{\mathcal{O}}$ . The algebra  $C^\infty(\overline{\mathcal{O}})$  is referred to as the algebra of WHITNEY-smooth functions on  $\overline{\mathcal{O}}$ , relative to the embedding of  $\overline{\mathcal{O}}$  into the affine space  $\mathfrak{sp}(\ell, \mathbb{R})$ . Under the identification (7.3), the orbit  $\mathcal{O}$  passes to a *coadjoint* orbit. Consequently, under the surjection

$C^\infty(\mathfrak{sp}(\ell, \mathbb{R})) \rightarrow C^\infty(\overline{\mathcal{O}})$ , the Poisson bracket  $\{\cdot, \cdot\}$  on the algebra  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  descends to a Poisson bracket on  $C^\infty(\overline{\mathcal{O}})$ , which we still denote by  $\{\cdot, \cdot\}$ , with a slight abuse of notation. This Poisson algebra turns  $\overline{\mathcal{O}}$  into a stratified symplectic space. Combined with the complex analytic structure coming from the projection from  $\overline{\mathcal{O}}$  onto the corresponding space of complex symmetric  $(\ell \times \ell)$ -matrices, in this fashion, the space  $\overline{\mathcal{O}}$  acquires a *stratified Kähler space* structure. The composite of the above comomentum mapping  $\delta$  with the projection from  $C^\infty(\mathfrak{sp}(\ell, \mathbb{R}))$  to  $C^\infty(\overline{\mathcal{O}})$  yields an embedding

$$(7.4) \quad \delta_{\mathcal{O}}: \mathfrak{sp}(\ell, \mathbb{R}) \longrightarrow C^\infty(\overline{\mathcal{O}})$$

which is still a morphism of Lie algebras and therefore a comomentum mapping in the appropriate sense.

The  $\mathrm{Sp}(\ell, \mathbb{R})$ -momentum mapping induces an embedding of the reduced space  $N_0$  into  $\mathfrak{sp}(\ell, \mathbb{R})$  which identifies  $N_0$  with the closure  $\overline{\mathcal{O}}_{\min(s, \ell)}$  of the holomorphic nilpotent orbit  $\mathcal{O}_{\min(s, \ell)}$  in  $\mathfrak{sp}(\ell, \mathbb{R})$ . In this fashion, the reduced space  $N_0$  inherits a stratified Kähler structure. Since the  $\mathrm{Sp}(\ell, \mathbb{R})$ -momentum mapping induces an identification of  $N_0$  with  $\overline{\mathcal{O}}_s$  for every  $s \leq \ell$  in a compatible manner, the ascending sequence (7.2), and in particular the notion of holomorphic nilpotent orbit, is actually independent of the choice of complex structure  $J$  on  $\mathbb{R}^{2\ell}$ . For a single particle, i. e.  $\ell = 1$ , the description of the reduced space  $N_0$  comes down to that of the semicone given in Section 2.1 above.

Thus, when the number  $\ell$  of particles is at most equal to the (real) dimension  $s$  of the space  $\mathbb{A}^s$  in which these particles move, as a space, the reduced space  $N$  amounts to a copy of complex affine space of dimension  $\frac{\ell(\ell+1)}{2}$  and hence to a copy of real affine space of dimension  $\ell(\ell+1)$ . When the number  $\ell$  of particles exceeds the (real) dimension  $s$  of the space in which the particles move, as a space, the reduced space  $N$  amounts to a copy of the complex affine variety of complex symmetric matrices of rank at most equal to  $s$ .

## 8 Quantization in the situation of the previous class of examples

In the situation of the previous section, we will now explain briefly the quantization procedure developed in [21]. Suppose that  $s \leq \ell$  (for simplicity), let  $m = s\ell$ , and endow the affine coordinate ring of  $\mathbb{C}^m$ , that is, the polynomial algebra  $\mathbb{C}[z_1, \dots, z_m]$ , with the inner product  $\cdot$  given by the standard formula

$$(8.1) \quad \psi \cdot \psi' = \int \psi \overline{\psi'} e^{-\frac{z\bar{z}}{2}} \varepsilon_m, \quad \varepsilon_m = \frac{\omega^m}{(2\pi)^m m!},$$

where  $\omega$  refers to the symplectic form on  $\mathbb{C}^m$ . Furthermore, endow the polynomial algebra  $\mathbb{C}[z_1, \dots, z_m]$  with the induced  $\mathrm{O}(s, \mathbb{R})$ -action. By construction, the affine

complex coordinate ring  $\mathbb{C}[\overline{\mathcal{O}}_s]$  of  $\overline{\mathcal{O}}_s$  is canonically isomorphic to the algebra

$$\mathbb{C}[z_1, \dots, z_m]^{O(s, \mathbb{R})}$$

of  $O(s, \mathbb{R})$ -invariants in  $\mathbb{C}[z_1, \dots, z_m]$ . The restriction of the inner product  $\cdot$  to  $\mathbb{C}[\overline{\mathcal{O}}_s]$  turns  $\mathbb{C}[\overline{\mathcal{O}}_s]$  into a pre-Hilbert space, and HILBERT space completion yields a HILBERT space which we write as  $\widehat{\mathbb{C}}[\overline{\mathcal{O}}_s]$ . This is the Hilbert space which arises by *holomorphic quantization* on the stratified Kähler space  $\overline{\mathcal{O}}_s$ ; see [21] for details. On this Hilbert space, the elements of the Lie algebra  $\mathfrak{u}(\ell)$  of the unitary group  $U(\ell)$  act in an obvious fashion; indeed, the elements of  $\mathfrak{u}(\ell)$ , viewed as functions in  $C^\infty(\overline{\mathcal{O}}_s)$ , are classical observables which are directly quantizable, and quantization yields the obvious  $\mathfrak{u}(\ell)$ -representation on  $\widehat{\mathbb{C}}[\overline{\mathcal{O}}_s]$ . This construction may be carried out for any  $s \leq \ell$  and, for each  $s \leq \ell$ , the resulting quantizations yields a *costratified Hilbert space* of the kind

$$\mathbb{C} \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}}_1] \longleftarrow \dots \longleftarrow \widehat{\mathbb{C}}[\overline{\mathcal{O}}_s].$$

Here each arrow is just a restriction mapping and is actually a morphism of representations for the corresponding quantizable observables, in particular, a morphism of  $\mathfrak{u}(\ell)$ -representations; each arrow amounts essentially to an orthogonal projection. Plainly, the costratified structure integrates to a costratified  $U(\ell)$ -representation, i. e. to a corresponding system of  $U(\ell)$ -representations. The resulting costratified quantum phase space for  $\overline{\mathcal{O}}_s$  is a kind of *singular* Fock space. This quantum phase space is entirely given in terms of *data on the reduced level*.

Consider the unreduced classical harmonic oscillator energy  $E$  which is given by  $E = z_1 \bar{z}_1 + \dots + z_m \bar{z}_m$ ; it quantizes to the Euler operator (quantized harmonic oscillator hamiltonian). For  $s \leq \ell$ , the reduced classical phase space  $Q_s$  of  $\ell$  harmonic oscillators in  $\mathbb{R}^s$  with total angular momentum zero and fixed energy value which is here encoded in the even number  $2k$  fits into an ascending sequence

$$(8.2) \quad Q_1 \subseteq \dots \subseteq Q_s \subseteq \dots \subseteq Q_\ell \cong \mathbb{C}\mathbb{P}^d$$

of stratified Kähler spaces where

$$\mathbb{C}\mathbb{P}^d = \mathbb{P}(S^2[\mathbb{C}^\ell]), \quad d = \frac{\ell(\ell+1)}{2} - 1.$$

The sequence (8.2) arises from the sequence (7.2) by *projectivization*. The parameter  $k$  (energy value  $2k$ ) is encoded in the Poisson structure. Let  $\mathcal{O}(k)$  be the  $k$ 'th power of the hyperplane bundle on  $\mathbb{C}\mathbb{P}^d$ , let

$$\iota_{Q_s}: Q_s \longrightarrow Q_\ell \cong \mathbb{C}\mathbb{P}^d$$

be the inclusion, and let  $\mathcal{O}_{Q_s}(k) = \iota_{Q_s}^* \mathcal{O}(k)$ . The quantum *Hilbert* space amounts now to the space of holomorphic sections of  $\iota_{Q_s}^* \mathcal{O}(k)$ , and the resulting *costratified quantum Hilbert space* has the form

$$\Gamma^{\text{hol}}(\mathcal{O}_{Q_1}(k)) \longleftarrow \dots \longleftarrow \Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k)).$$

Each vector space  $\Gamma^{\text{hol}}(\mathcal{O}_{Q_{s'}}(k))$  ( $1 \leq s' \leq s$ ) is a finite-dimensional representation space for the quantizable observables in  $C^\infty(Q_s)$ , in particular, a  $\mathfrak{u}(\ell)$ -representation, and this representation integrates to a  $U(\ell)$ -representation, and each arrow is a morphism of representations; similarly as before, these arrows are just restriction maps.

We will now give a description of the decomposition of the space

$$\Gamma^{\text{hol}}(\mathcal{O}_{Q_\ell}(k)) = S_{\mathbb{C}}^k[\mathfrak{p}^*]$$

of homogeneous degree  $k$  polynomial functions on  $\mathfrak{p} = S_{\mathbb{C}}^2[\mathbb{C}^\ell]$  into its *irreducible*  $U(\ell)$ -representations in terms of highest weight vectors. To this end we note that coordinates  $x_1, \dots, x_\ell$  on  $\mathbb{C}^\ell$  give rise to coordinates of the kind  $\{x_{i,j} = x_{j,i}; 1 \leq i, j \leq \ell\}$  on  $S_{\mathbb{C}}^2[\mathbb{C}^\ell]$ , and the determinants

$$\delta_1 = x_{1,1}, \quad \delta_2 = \begin{vmatrix} x_{1,1} & x_{1,2} \\ x_{1,2} & x_{2,2} \end{vmatrix}, \quad \delta_3 = \begin{vmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{1,2} & x_{2,2} & x_{2,3} \\ x_{1,3} & x_{2,3} & x_{3,3} \end{vmatrix}, \quad \text{etc.}$$

are highest weight vectors for certain  $U(\ell)$ -representations. For  $1 \leq s \leq r$  and  $k \geq 1$ , the  $U(\ell)$ -representation  $\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k))$  is the sum of the irreducible representations having as highest weight vectors the monomials

$$\delta_1^\alpha \delta_2^\beta \dots \delta_s^\gamma, \quad \alpha + 2\beta + \dots + s\gamma = k,$$

and the restriction morphism

$$\Gamma^{\text{hol}}(\mathcal{O}_{Q_s}(k)) \longrightarrow \Gamma^{\text{hol}}(\mathcal{O}_{Q_{s-1}}(k))$$

has the span of the representations involving  $\delta_s$  explicitly as its kernel and, restricted to the span of those irreducible representations which do *not* involve  $\delta_s$ , this morphism is an isomorphism.

This situation may be interpreted in the following fashion: The composite

$$\mu_{2k}: \overline{\mathcal{O}}_s \subseteq \mathfrak{sp}(\ell, \mathbb{R}) \cong \mathfrak{sp}(\ell, \mathbb{R})^* \longrightarrow \mathfrak{u}(\ell)^*$$

is a singular momentum mapping for the  $U(\ell)$ -action on  $\overline{\mathcal{O}}_s$ ; actually, the adjoint  $\mathfrak{u}(\ell) \rightarrow C^\infty(\overline{\mathcal{O}}_s)$  of  $\mu_{2k}$  amounts to the composite of (7.4) with the inclusion of  $\mathfrak{u}(\ell)$  into  $\mathfrak{sp}(\ell, \mathbb{R})$ . The *irreducible*  $U(\ell)$ -representations which correspond to the *coadjoint orbits in the image*

$$\mu_{2k}(O_{s'} \setminus O_{s'-1}) \subseteq \mathfrak{u}(\ell)^*$$

of the stratum  $O_{s'} \setminus O_{s'-1}$  ( $1 \leq s' \leq s$ ) are precisely the irreducible representations having as highest weight vectors the monomials

$$\delta_1^\alpha \delta_2^\beta \dots \delta_{s'}^\gamma \quad (\alpha + 2\beta + \dots + s'\gamma = k)$$

involving  $\delta_{s'}$  explicitly, i. e. with  $\gamma \geq 1$ .

## 9 Holomorphic half-form quantization on the complexification of a compact Lie group

Recall that, given a general compact Lie group  $K$ , via the diffeomorphism (2.1), the complex structure on  $K^{\mathbb{C}}$  and the cotangent bundle symplectic structure on  $T^*K$  combine to  $K$ -bi-invariant Kähler structure. A global Kähler potential is given by the function  $\kappa$  defined by

$$\kappa(x e^{iY}) = |Y|^2, \quad x \in K, \quad Y \in \mathfrak{k}.$$

The function  $\kappa$  being a Kähler potential signifies that the symplectic structure on  $T^*K \cong K^{\mathbb{C}}$  is given by  $i\partial\bar{\partial}\kappa$ . Let  $\varepsilon$  denote the symplectic (or Liouville) volume form on  $T^*K \cong K^{\mathbb{C}}$ , and let  $\eta$  be the real  $K$ -bi-invariant (analytic) function on  $K^{\mathbb{C}}$  given by

$$\eta(x e^{iY}) = \sqrt{\left| \frac{\sin(\text{ad}(Y))}{\text{ad}(Y)} \right|}, \quad x \in K, \quad Y \in \mathfrak{k},$$

cf. [12] (2.10). Thus  $\eta^2$  is the density of Haar measure on  $K^{\mathbb{C}}$  relative to Liouville measure  $\varepsilon$ .

Half-form Kähler quantization on  $K^{\mathbb{C}}$  leads to the Hilbert space

$$\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)$$

of holomorphic functions on  $K^{\mathbb{C}}$  that are square-integrable relative to  $e^{-\kappa/\hbar}\eta\varepsilon$  [12]. Thus the scalar product in this Hilbert space is given by

$$\langle \psi_1, \psi_2 \rangle = \frac{1}{\text{vol}(K)} \int_{K^{\mathbb{C}}} \overline{\psi_1} \psi_2 e^{-\kappa/\hbar} \eta \varepsilon.$$

Relative to left and right translation,  $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)$  is a unitary  $(K \times K)$ -representation, and the Hilbert space associated with  $\mathcal{P}$  by reduction after quantization is the subspace

$$\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)^K$$

of  $K$ -invariants relative to conjugation.

Let  $\varepsilon_T$  denote the Liouville volume form of  $T^*T \cong T^{\mathbb{C}}$ . There is a function  $\gamma$  on this space, made explicit in [28], such that the restriction mapping induces an isomorphism

$$(9.1) \quad \mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)^K \longrightarrow \mathcal{H}L^2(T^{\mathbb{C}}, e^{-\kappa/\hbar}\gamma\varepsilon_T)^W$$

of Hilbert spaces where the scalar product in  $\mathcal{H}L^2(T^{\mathbb{C}}, e^{-\kappa/\hbar}\gamma\varepsilon_T)^W$  is given by

$$(9.2) \quad \frac{1}{\text{vol}(K)} \int_{T^{\mathbb{C}}} \overline{\psi_1} \psi_2 e^{-\kappa/\hbar} \gamma \varepsilon_T.$$

## 10 Singular quantum structure: costratified Hilbert space

Let  $N$  be a stratified space. Thus  $N$  is a disjoint union  $N = \cup N_\lambda$  of locally closed subspaces  $N_\lambda$ , called *strata*, each stratum being an ordinary smooth manifold, and the mutual positions of the strata are made precise in a way not spelled out here. Let  $\mathcal{C}_N$  be the category whose objects are the strata of  $N$  and whose morphisms are the inclusions  $Y' \subseteq \overline{Y}$  where  $Y$  and  $Y'$  range over strata. We define a *costratified Hilbert space* relative to  $N$  or associated with the stratification of  $N$  to be a system which assigns a Hilbert space  $\mathcal{C}_Y$  to each stratum  $Y$ , together with a bounded linear map  $\mathcal{C}_{Y_2} \rightarrow \mathcal{C}_{Y_1}$  for each inclusion  $Y_1 \subseteq \overline{Y_2}$  such that, whenever  $Y_1 \subseteq \overline{Y_2}$  and  $Y_2 \subseteq \overline{Y_3}$ , the composite of  $\mathcal{C}_{Y_3} \rightarrow \mathcal{C}_{Y_2}$  with  $\mathcal{C}_{Y_2} \rightarrow \mathcal{C}_{Y_1}$  coincides with the bounded linear map  $\mathcal{C}_{Y_3} \rightarrow \mathcal{C}_{Y_1}$  associated with the inclusion  $Y_1 \subseteq \overline{Y_3}$ .

We now explain the construction of the costratified Hilbert space associated with the reduced phase space  $\mathcal{P}$ . This costratified structure is a *quantum analogue* of the *orbit type stratification*.

In the Hilbert space

$$\mathcal{H} = \mathcal{H}L^2(K^\mathbb{C}, e^{-\kappa/\hbar}\eta\varepsilon)^K \cong \mathcal{H}L^2(T^\mathbb{C}, e^{-\kappa/\hbar}\gamma\varepsilon_T)^W,$$

we single out subspaces associated with the strata in an obvious manner. For the special case

$$K = \mathrm{SU}(2), \quad \mathcal{P} = T^*K//K \cong \mathbb{C},$$

this comes down to the following procedure:

The elements of  $\mathcal{H}$  are ordinary holomorphic functions on  $K^\mathbb{C}$ . Being  $K$ -invariant, they are determined by their restrictions to  $T^\mathbb{C}$ ; these are  $W$ -invariant holomorphic functions on  $T^\mathbb{C}$ , and these  $W$ -invariant holomorphic functions, in turn, are determined by the holomorphic functions on

$$\mathcal{P} = K^\mathbb{C}//K^\mathbb{C} \cong T^\mathbb{C}/W \cong \mathbb{C}$$

which they induce on that space. In terms of the realization of  $\mathcal{P}$  as the complex line  $\mathbb{C}$ , the stratification of  $\mathcal{P}$  reproduced in Subsection 2.3 above is given by the decomposition  $\mathbb{C} = \mathcal{P}_+ \cup \mathcal{P}_- \cup \mathcal{P}_1$  of  $\mathbb{C}$  into

$$\mathcal{P}_+ = \{2\} \subseteq \mathbb{C}, \quad \mathcal{P}_- = \{-2\} \subseteq \mathbb{C}, \quad \mathcal{P}_1 = \mathbb{C} \setminus \mathcal{P}_0 = \mathbb{C} \setminus \{2, -2\}.$$

The closed subspaces

$$\begin{aligned} \mathcal{V}_+ &= \{f \in \mathcal{H}; f|_{\mathcal{P}_+} = 0\} \subseteq \mathcal{H} \\ \mathcal{V}_- &= \{f \in \mathcal{H}; f|_{\mathcal{P}_-} = 0\} \subseteq \mathcal{H} \end{aligned}$$

are Hilbert spaces, and we *define* the Hilbert spaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  to be the orthogonal complements in  $\mathcal{H}$  so that

$$\mathcal{H} = \mathcal{V}_+ \oplus \mathcal{H}_+ = \mathcal{V}_- \oplus \mathcal{H}_-;$$

moreover, we take  $\mathcal{H}_1$  to be the entire space  $\mathcal{H}$ . The resulting system

$$\{\mathcal{H}; \mathcal{H}_1, \mathcal{H}_+, \mathcal{H}_-\},$$

together with the corresponding orthogonal projections, is the *costratified Hilbert space* associated with the stratification of  $\mathcal{P}$ . By construction, this costratified Hilbert space structure is a *quantum analogue* of the *orbit type stratification* of  $\mathcal{P}$ .

## 11 The holomorphic Peter-Weyl theorem

Choose a dominant Weyl chamber in the maximal torus  $\mathfrak{t}$ . Given the highest weight  $\lambda$  (relative to the chosen dominant Weyl chamber), we will denote by  $\chi_\lambda^{\mathbb{C}}$  the irreducible character of  $K^{\mathbb{C}}$  associated with  $\lambda$ .

**Theorem 11.1** (Holomorphic Peter-Weyl theorem). *The Hilbert space*

$$\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)$$

*contains the vector space  $\mathbb{C}[K^{\mathbb{C}}]$  of representative functions on  $K^{\mathbb{C}}$  as a dense subspace and, as a unitary  $(K \times K)$ -representation, this Hilbert space decomposes as the direct sum*

$$\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon) \cong \widehat{\bigoplus}_{\lambda \in \widehat{K^{\mathbb{C}}}} V_\lambda^* \otimes V_\lambda$$

*of  $(K \times K)$ -isotypical summands, each such summand being written here as  $V_\lambda^* \otimes V_\lambda$  where  $V_\lambda$  refers to the irreducible  $K$ -representation associated with the highest weight  $\lambda$ .*

A proof of this theorem and relevant references can be found in [26]. The holomorphic Peter-Weyl theorem entails that the irreducible characters  $\chi_\lambda^{\mathbb{C}}$  of  $K^{\mathbb{C}}$  constitute a Hilbert space basis of

$$\mathcal{H} = \mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)^K.$$

Given the highest weight  $\lambda$ , we will denote by  $\chi_\lambda$  the corresponding irreducible character of  $K$ ; plainly,  $\chi_\lambda$  is the restriction to  $K$  of the character  $\chi_\lambda^{\mathbb{C}}$ . As usual, let  $\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$ , the half sum of the positive roots and, for a highest weight  $\lambda$ , let

$$(11.1) \quad C_\lambda := (\hbar\pi)^{\dim(K)/2} e^{\hbar|\lambda+\rho|^2},$$

where  $|\lambda + \rho|$  refers to the norm of  $\lambda + \rho$  relative to the inner product on  $\mathfrak{k}$ . In view of the ordinary Peter-Weyl theorem, the  $\{\chi_\lambda\}$ 's constitute an *orthonormal* basis of the Hilbert space  $L^2(K, dx)^K$ .

**Theorem 11.2.** *The assignment to  $\chi_\lambda$  of  $C_\lambda^{-1/2}\chi_\lambda^{\mathbb{C}}$ , as  $\lambda$  ranges over the highest weights, yields a unitary isomorphism*

$$(11.2) \quad L^2(K, dx)^K \longrightarrow \mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar\eta\varepsilon})^K$$

of Hilbert spaces.

By means of this isomorphism, the costratified Hilbert space structure arising from *stratified Kähler* quantization as explained earlier carries over to the Schrödinger quantization.

## 12 Quantum Hamiltonian and Peter-Weyl decomposition

In the Kähler quantization, only the constants are quantizable while in the Schrödinger quantization, functions that are at most quadratic in generalized momenta are quantizable. In particular, the classical Hamiltonian (2.2) of our model is quantizable in the Schrödinger quantization, having as associated quantum Hamiltonian the operator

$$(12.1) \quad H = -\frac{\hbar^2}{2}\Delta_K + \frac{\nu}{2}(3 - \chi_1)$$

on  $L^2(K, dx)^K$ . The operator  $\Delta_K$ , in turn, arises from the non-positive Laplace-Beltrami operator  $\tilde{\Delta}_K$  associated with the bi-invariant Riemannian metric on  $K$  as follows: The operator  $\tilde{\Delta}_K$  is essentially self-adjoint on  $C^\infty(K)$  and has a unique extension  $\Delta_K$  to an (unbounded) self-adjoint operator on  $L^2(K, dx)$ . The spectrum being discrete, the domain of this extensions is the space of functions of the form  $f = \sum_n \alpha_n \varphi_n$  such that  $\sum_n |\alpha_n|^2 \lambda_n^2 < \infty$  where the  $\varphi_n$ 's range over the eigenfunctions and the  $\lambda_n$ 's over the eigenvalues of  $\tilde{\Delta}_K$ .

Since the metric is bi-invariant, so is  $\Delta_K$ , whence  $\Delta_K$  restricts to a self-adjoint operator on  $L^2(K, dx)^K$ , which we still write as  $\Delta_K$ . By means of the isomorphism (11.2), we then transfer the Hamiltonian, in particular, the operator  $\Delta_K$ , to a self-adjoint operators on  $\mathcal{H}$ . Schur's lemma then tells us the following:

- (1) Each isotypical  $(K \times K)$ -summand  $L^2(K, dx)_\lambda$  of  $L^2(K, dx)$  in the Peter-Weyl decomposition is an eigenspace for  $\Delta_K$ ;
- (2) the representative functions are eigenfunctions for  $\Delta_K$ ;
- (3) the eigenvalue  $-\varepsilon_\lambda$  of  $\Delta_K$  corresponding to the highest weight  $\lambda$  is given by

$$\varepsilon_\lambda = (|\lambda + \rho|^2 - |\rho|^2).$$

Thus, in the holomorphic quantization on  $T^*K \cong K^{\mathbb{C}}$ , the free energy operator (i. e. without potential energy term) arises as the unique extension of the operator  $-\frac{1}{2}\Delta_K$  on  $\mathcal{H}$  to an unbounded self-adjoint operator, and the spectral decomposition thereof refines to the holomorphic Peter-Weyl decomposition of  $\mathcal{H}$ .

### 13 The lattice gauge theory model arising from $SU(2)$

In the rest of the paper we will discuss somewhat informally, for the special case where the underlying compact group is  $K = SU(2)$ , some of the implications for the physical interpretation ; see [29] for a leisurely somewhat more complete introduction and [28] for a systematic description.

To begin with, we write out the requisite data for the special case under consideration. We denote the roots of  $K = SU(2)$  relative to the dominant Weyl chamber chosen earlier by  $\alpha$  and  $-\alpha$ , so that  $\varrho = \frac{1}{2}\alpha$ . The invariant inner product on the Lie algebra  $\mathfrak{k}$  of  $K$  is of the form

$$(13.1) \quad -\frac{1}{2\beta^2} \text{tr}(Y_1 Y_2), \quad Y_1, Y_2 \in \mathfrak{k},$$

with a scaling factor  $\beta > 0$  which we will leave unspecified (e.g.,  $\beta = \frac{1}{\sqrt{8}}$  for the Killing form). Then

$$|\alpha|^2 = 4\beta^2, \quad |\varrho|^2 = \beta^2.$$

The highest weights are  $\lambda_n = \frac{n}{2}\alpha$ , where  $n = 0, 1, 2, \dots$  (twice the spin). Then

$$(13.2) \quad \varepsilon_n \equiv \varepsilon_{\lambda_n} = \beta^2 n(n+2), \quad C_n \equiv C_{\lambda_n} = (\hbar\pi)^{3/2} e^{\hbar\beta^2(n+1)^2},$$

cf. (11.1) for the significance of the notation  $C_{\lambda_n}$ . We will now write the complex characters  $\chi_{\lambda_n}^{\mathbb{C}}$  as  $\chi_n^{\mathbb{C}}$  ( $n \geq 0$ ). On  $T^{\mathbb{C}}$ , these complex characters are given by

$$(13.3) \quad \chi_n^{\mathbb{C}}(\text{diag}(z, z^{-1})) = z^n + z^{n-2} + \dots + z^{-n}, \quad z \in \mathbb{C} \setminus \{0\},$$

whereas, on  $T$ , the corresponding real characters take the form

$$(13.4) \quad \chi_n(\text{diag}(e^{ix}, e^{-ix})) = \frac{\sin((n+1)x)}{\sin(x)}, \quad x \in \mathbb{R}, \quad n \geq 0.$$

The Weyl group  $W$  permutes the two entries of the elements in  $T$ . Hence, the reduced configuration space  $\mathcal{X} = T/W$  can be parametrized by  $x \in [0, \pi]$  through  $x \mapsto \text{diag}(e^{ix}, e^{-ix})$ . In this parametrization, the measure  $\nu$  on  $T$  is given by

$$\nu dt = \frac{\text{vol}(K)}{\pi} \sin^2(x) dx.$$

It follows that the assignment to  $\psi \in C^\infty(T)^W$  of the function

$$x \mapsto \sqrt{2} \sin x \psi(\text{diag}(e^{ix}, e^{-ix})), \quad x \in [0, \pi],$$

defines a Hilbert space isomorphism from  $L^2(K, dx)^K$ , realized as a Hilbert space of  $W$ -invariant  $L^2$ -functions on  $T$ , onto the ordinary  $L^2[0, \pi]$ , where the inner

product in  $L^2[0, \pi]$  is normalized so that the constant function with value 1 has norm 1. In particular, given  $n \geq 0$ , the character  $\chi_n$  is mapped to the function given by the expression

$$(13.5) \quad \chi_n(x) = \sqrt{2} \sin((n+1)x).$$

In view of the isomorphism between  $L^2(K, dx)^K$  and  $L^2[0, \pi]$  and the isomorphism (11.2), we can work in an abstract Hilbert space  $\mathcal{H}$  with a distinguished orthonormal basis  $\{|n\rangle : n = 0, 1, 2, \dots\}$ . We achieve the passage to the holomorphic realization  $\mathcal{H}L^2(K^{\mathbb{C}}, e^{-\kappa/\hbar}\eta\varepsilon)^K$ , to the Schrödinger realization  $L^2(K, dx)^K$ , and to the ordinary  $L^2$ -realization  $L^2[0, \pi]$  by substitution of, respectively,  $C_n^{-1/2}\chi_n^{\mathbb{C}}$ ,  $\chi_n$ , and  $\sqrt{2}\sin(n+1)x$ , for  $|n\rangle$ . We remark that plotting wave functions in the realization of  $\mathcal{H}$  by  $L^2[0, \pi]$  has the advantage that, directly from the graph, one can read off the corresponding probability densities with respect to Lebesgue measure on the parameter space  $[0, \pi]$ .

We determine the subspaces  $\mathcal{H}_\tau$  for the special case  $K = \text{SU}(2)$ . The orbit type strata are  $\mathcal{P}_+$ ,  $\mathcal{P}_-$  and  $\mathcal{P}_1$ , where  $\mathcal{P}_\pm$  consists of the class of  $\pm\mathbf{1}$  and  $\mathcal{P}_1 = \mathcal{P} \setminus (\mathcal{P}_+ \cup \mathcal{P}_-)$ . (Recall that via the complex analytic isomorphism (2.3),  $\mathcal{P}_\pm$  is identified with the subset  $\{\pm 2\}$  of  $\mathbb{C}$ .) Since  $\mathcal{P}_1$  is dense in  $\mathcal{P}$ , the space  $\mathcal{V}_1$  reduces to zero and so  $\mathcal{H}_1 = \mathcal{H}$ . By definition, the subspaces  $\mathcal{V}_+$  and  $\mathcal{V}_-$  consist of the functions  $\psi \in \mathcal{H}$  that satisfy the constraints

$$(13.6) \quad \psi(\mathbf{1}) = 0, \quad \psi(-\mathbf{1}) = 0,$$

respectively. One can check that the system  $\{\chi_n^{\mathbb{C}} - (n+1)\chi_0^{\mathbb{C}} : n = 1, 2, 3, \dots\}$  forms a basis in  $\mathcal{V}_+$  and that the system  $\{\chi_n^{\mathbb{C}} + (-1)^n \frac{n+1}{2}\chi_1^{\mathbb{C}} : n = 0, 2, 3, \dots\}$  forms a basis in  $\mathcal{V}_-$ . Taking the orthogonal complements, we arrive at the following.

**Theorem 13.1.** *The subspaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  have dimension 1. They are spanned by the normalized vectors*

$$(13.7) \quad \psi_+ := \frac{1}{N} \sum_{n=0}^{\infty} (n+1) e^{-\hbar\beta^2(n+1)^2/2} |n\rangle,$$

$$(13.8) \quad \psi_- := \frac{1}{N} \sum_{n=0}^{\infty} (-1)^n (n+1) e^{-\hbar\beta^2(n+1)^2/2} |n\rangle,$$

respectively. The normalization factor  $N$  is determined by the identity

$$N^2 = \sum_{n=1}^{\infty} n^2 e^{-\hbar\beta^2 n^2}.$$

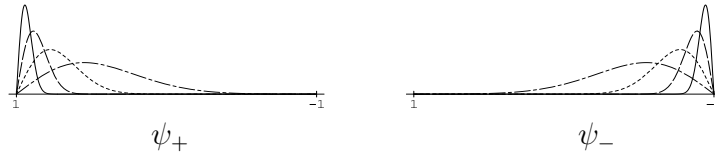
Hence, in Dirac notation, the orthogonal projections  $\Pi_{\pm}: \mathcal{H} \rightarrow \mathcal{H}_{\pm}$  are given by the expressions

$$(13.9) \quad \Pi_{\pm} = |\psi_{\pm}\rangle \langle\psi_{\pm}|.$$

In terms of the  $\theta$ -constant  $\theta_3(Q) = \sum_{k=-\infty}^{\infty} Q^{k^2}$ , the normalization factor  $N$  is determined by the identity

$$(13.10) \quad N^2 = \frac{1}{2} e^{-\hbar\beta^2} \theta'_3(e^{-\hbar\beta^2}).$$

The following figure shows plots of  $\psi_{\pm}$  in the realization of  $\mathcal{H}$  via  $L^2[0, \pi]$  for  $\hbar\beta^2 = 1/128$  (continuous line),  $1/32$  (long dash),  $1/8$  (short dash),  $1/2$  (alternating short-long dash).

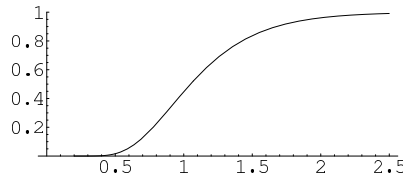


## 14 Tunneling between strata

Computing the inner product of  $\psi_+$  and  $\psi_-$ ,

$$\langle \psi_+, \psi_- \rangle = \frac{1}{N^2} \sum_{n=1}^{\infty} (-1)^{n+1} n^2 e^{-\hbar\beta^2 n^2} = - \frac{\theta'_3(-e^{-\hbar\beta^2})}{\theta'_3(e^{-\hbar\beta^2})},$$

we observe that the subspaces  $\mathcal{H}_+$  and  $\mathcal{H}_-$  are not orthogonal. They share a certain overlap which depends on the combined parameter  $\hbar\beta^2$ . The absolute square  $|\langle \psi_+, \psi_- \rangle|^2$  yields the tunneling probability between the strata  $\mathcal{P}_+$  and  $\mathcal{P}_-$ , i. e., the probability for a state prepared at  $\mathcal{P}_+$  to be measured at  $\mathcal{P}_-$  and vice versa. The following figure shows a plot of the tunneling probability against  $\hbar\beta^2$ . For large values, this probability tends to 1 whereas for  $\hbar\beta^2 \rightarrow 0$ , i.e., in the semiclassical limit, it vanishes.



## 15 Energy eigenvalues and eigenstates

Passing to the realization of  $\mathcal{H}$  via  $L^2[0, \pi]$  and applying the general formula for the radial part of the Laplacian on a compact group, see [13, §II.3.4], from the description (12.1) of the quantum Hamiltonian, viz.

$$H = -\frac{\hbar^2}{2} \Delta_K + \frac{\nu}{2} (3 - \chi_1),$$

we obtain the formal expression

$$-\frac{\hbar^2\beta^2}{2}\left(\frac{d^2}{dx^2}+1\right)+\frac{\nu}{2}(3-\chi_1)$$

for  $H$  on  $L^2[0, \pi]$ . Hence the stationary Schrödinger equation can be written as

$$(15.1) \quad \left(\frac{d^2}{dx^2}+2\tilde{\nu}\cos(x)+\left(\frac{2E}{\hbar^2\beta^2}+1-3\tilde{\nu}\right)\right)\psi(x)=0,$$

where  $\tilde{\nu} = \frac{\nu}{\hbar^2\beta^2} \equiv \frac{1}{\hbar^2\beta^2g^2}$ , and  $E$  refers to the eigenvalue. The change of variable  $y = (x - \pi)/2$  leads to the Mathieu equation

$$(15.2) \quad \frac{d^2}{dy^2}f(y) + (a - 2q\cos(2y))f(y) = 0,$$

where

$$(15.3) \quad a = \frac{8E}{\hbar^2\beta^2} + 4 - 12\tilde{\nu}, \quad q = 4\tilde{\nu};$$

here  $f$  refers to a Whitney smooth function on the interval  $[-\pi/2, 0]$  satisfying the boundary conditions

$$(15.4) \quad f(-\pi/2) = f(0) = 0.$$

For the theory of the Mathieu equation and its solutions, called *Mathieu functions*, see [1]. For certain characteristic values of the parameter  $a$  depending analytically on  $q$  and usually denoted by  $b_{2n+2}(q)$ ,  $n = 0, 1, 2, \dots$ , solutions satisfying (15.4) exist. Given  $a = b_{2n+2}(q)$ , the corresponding solution is unique up to a complex factor and can be chosen to be real-valued. It is usually denoted by  $\text{se}_{2n+2}(y; q)$ , where ‘se’ stands for *sine elliptic*.

Thus, in the realization of  $\mathcal{H}$  via  $L^2[0, \pi]$ , the stationary states are given by

$$(15.5) \quad \xi_n(x) = (-1)^{n+1}\sqrt{2}\left(\text{se}_{2n+2}\left(\frac{x-\pi}{2}; 4\tilde{\nu}\right)\right), \quad n = 0, 1, 2, \dots,$$

and the corresponding eigenvalues by

$$E_n = \frac{\hbar^2\beta^2}{2}\left(\frac{b_{2n+2}(4\tilde{\nu})}{4} + 3\tilde{\nu} - 1\right).$$

The factor  $(-1)^{n+1}$  ensures that, for  $\tilde{\nu} = 0$ , we get  $\xi_n = \chi_n$ . According to [1, §20.5], for any value of the parameter  $q$ , the functions

$$\sqrt{2}\text{se}_{2n+2}(y; q), \quad n = 0, 1, 2, \dots,$$

form an orthonormal basis in  $L^2[-\pi/2, 0]$  and the characteristic values satisfy  $b_2(q) < b_4(q) < b_6(q) < \dots$ . Hence, the  $\xi_n$ 's form an orthonormal basis in  $\mathcal{H}$  and the eigenvalues  $E_n$  are nondegenerate.

Figure 2 shows the energy eigenvalues  $E_n$  and the level separation  $E_{n+1} - E_n$  for  $n = 0, \dots, 8$  as functions of  $\tilde{\nu}$ . Figure 3 displays the eigenfunctions  $\xi_n$ ,  $n = 0, \dots, 3$ , for  $\tilde{\nu} = 0, 3, 6, 12, 24$ . The plots have been generated by means of the built-in Mathematica functions `MathieuS` and `MathieuCharacteristicB`.

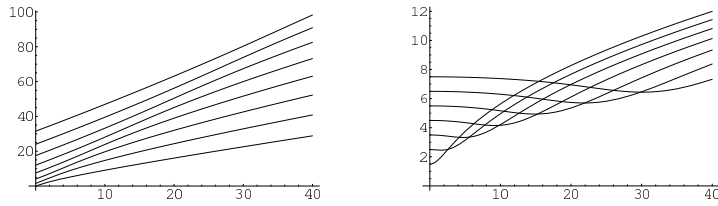


Figure 2: Energy eigenvalues  $E_n$  (left) and transition energy values  $E_{n+1} - E_n$  (right) for  $n = 0, \dots, 7$  in units of  $\hbar^2 \beta^2$  as functions of  $\tilde{\nu}$ .

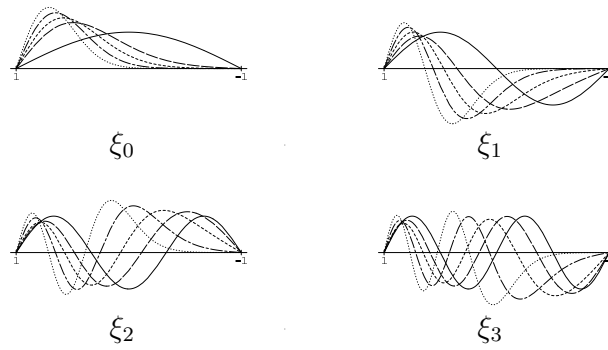


Figure 3: Energy eigenfunctions  $\xi_0, \dots, \xi_3$  for  $\tilde{\nu} = 0$  (continuous line), 3 (long dash), 6 (short dash), 12 (alternating short-long dash), 24 (dotted line).

## 16 Expectation values of the costratification orthoprojectors

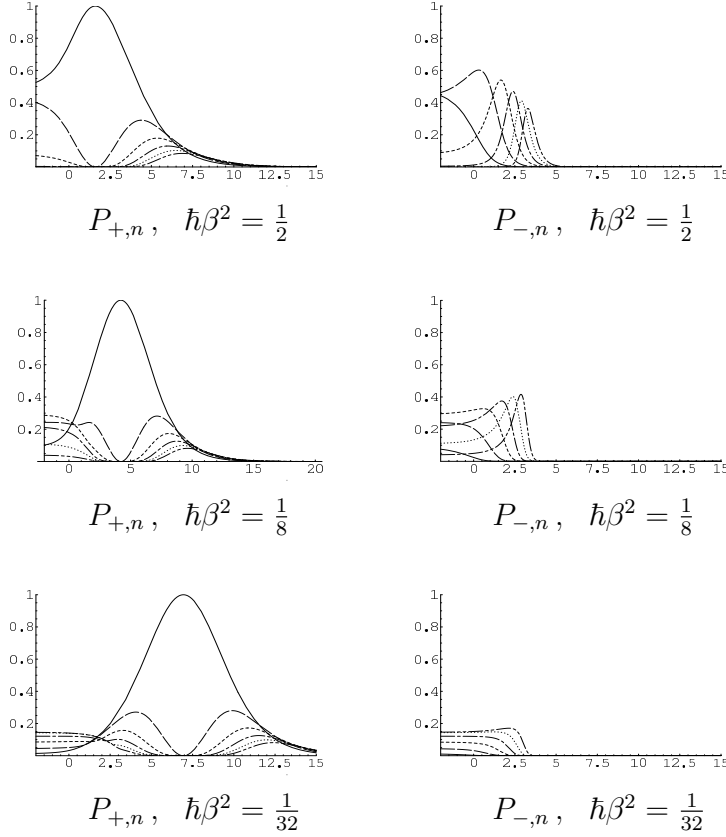


Figure 4: Expectation values  $P_{+,n}$  and  $P_{-,n}$  for  $n = 0$  (continuous line),  $n = 1$  (long dash),  $n = 2$  (short dash),  $n = 3$  (long-short dash),  $n = 4$  (dotted line) and  $n = 5$  (long-short-short dash), plotted over  $\log \tilde{\nu}$  for  $\hbar\beta^2 = \frac{1}{2}, \frac{1}{8}, \frac{1}{32}$ .

On the level of the observables, the costratification is given by the orthoprojectors  $\Pi_{\pm}$  onto the subspaces  $\mathcal{H}_{\pm}$ . We discuss their expectation values in the energy eigenstates,

$$P_{\pm,n} := \langle \xi_n | \Pi_{\pm} \xi_n \rangle,$$

i.e., the probability that the system prepared in the stationary state  $\xi_n$  is measured in the subspace  $\mathcal{H}_{\pm}$ . According to (13.9),

$$(16.1) \quad P_{\pm,n} = |\langle \xi_n | \psi_{\pm} \rangle|^2.$$

As  $\text{se}_{2n+2}$  is odd and  $\pi$ -periodic, it can be expanded as

$$\text{se}_{2n+2}(y; q) = \sum_{k=0}^{\infty} B_{2k+2}^{2n+2}(q) \sin((2k+2)y),$$

with Fourier coefficients  $B_{2k+2}^{2n+2}(q)$  satisfying certain recurrence relations [1, §20.2]. Due to (13.5),

$$(16.2) \quad \langle \xi_n | k \rangle = (-1)^{n+k} B_{2k+2}^{2n+2}(4\tilde{\nu}),$$

whence (13.7) and (13.8) yield the expressions

$$(16.3) \quad \langle \xi_n | \psi_+ \rangle = \frac{(-1)^n}{N} \sum_{k=0}^{\infty} (-1)^k (k+1) e^{-\hbar\beta^2(k+1)^2/2} B_{2k+2}^{2n+2}(4\tilde{\nu}),$$

$$(16.4) \quad \langle \xi_n | \psi_- \rangle = \frac{(-1)^n}{N} \sum_{k=0}^{\infty} (k+1) e^{-\hbar\beta^2(k+1)^2/2} B_{2k+2}^{2n+2}(4\tilde{\nu}).$$

Together with (16.1), this procedure leads to formulas for  $P_{\pm,n}$ . The functions  $P_{\pm,n}$  depend on the parameters  $\hbar$ ,  $\beta^2$  and  $\nu$  only via the combinations  $\hbar\beta^2$  and  $\tilde{\nu} = \nu/(\hbar^2\beta^2)$ . Figure 4 displays  $P_{\pm,n}$  for  $n = 0, \dots, 5$  as functions of  $\tilde{\nu}$  for three specific values of  $\hbar\beta^2$ , thus treating  $\tilde{\nu}$  and  $\hbar\beta^2$  as independent parameters. This is appropriate for the discussion of the dependence of the functions  $P_{\pm,n}$  on the coupling parameter  $g$  for fixed values of  $\hbar$  and  $\beta^2$ . The plots have been generated by Mathematica through numerical integration.

For  $n = 0$ , the function  $P_{+,n}$  has a dominant peak which is enclosed by less prominent maxima of the other  $P_{+,n}$ 's and moves to higher  $\tilde{\nu}$  when  $\hbar\beta^2$  decreases. That is to say, for a certain value of the coupling constant, the state  $\psi_+$  which spans  $\mathcal{H}_+$  seems to coincide almost perfectly with the ground state. If the two states coincided exactly then (16.2) would imply that, for a certain value of  $q$ , the coefficients  $B_{2k+2}^{2n+2}(q)$  would be given by  $(-1)^{n+k} \frac{1}{N} (k+1) e^{-\hbar\beta^2(k+1)^2/2}$ . However, this is not true; the latter expressions do not satisfy the recurrence relations valid for the coefficients  $B_{2k+2}^{2n+2}(q)$  for any value of  $q$ .

## 17 Outlook

For  $K = \text{SU}(2)$  it remains to discuss the dynamics relative to the costratified structure and to explore the probability flow into and out of the subspaces  $\mathcal{H}_{\pm}$ . More generally, it would be worthwhile carrying out this program for  $K = \text{SU}(n)$ ,  $n \geq 3$ . For  $K = \text{SU}(3)$ , the orbit type stratification of the reduced phase space consists of a 4-dimensional stratum, a 2-dimensional stratum, and three isolated points. Thereafter the approach should be extended to arbitrary lattices.

The notion of costratified Hilbert space implements the stratification of the reduced classical phase space on the level of states. The significance of the stratification for the quantum observables remains to be clarified. Then the physical role of this stratification can be studied in more realistic models like the lattice QCD of [30, 32, 33].

A number of applications of the theory of stratified Kähler spaces have already been mentioned. Using the approach to lattice gauge theory in [19], we intend

to develop elsewhere a rigorous approach to the quantization of certain lattice gauge theories by means of the Kähler quantization scheme for stratified Kähler spaces explained in the present paper. We plan to apply this scheme in particular to situations of the kind explored in [34]–[36] and to compare it with the approach to quantization in these papers. Constrained quantum systems occur in molecular mechanics as well, see e. g. [45] and the references there. Perhaps the Kähler quantization scheme for stratified Kähler spaces will shed new light on these quantum systems.

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