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Problems on asymptotic analysis over convex polytopes

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Abstract

In this paper, a survey of results in two topics on asymptotic analysis over convex polytopes, obtained in the papers [22, 25], one of which is related to representation theory of compact Lie groups and another is asymptotic formulas of sections of a line bundle over a toric Kähler manifold, is given.

1 Introduction

Convex polytopes often appear in many areas of mathematics. In particular, they play essential roles in representation theory of compact Lie groups and the theory of toric varieties. Combinatorial aspects of polytopes describe some algebraic structures in representation theory and geometrical structures in the theory of toric varieties. In representation theory, multiplicities of weights or irreducible summands are important quantities. But, many of the well-known formulas on multiplicities are given as alternating sums, and it would not be so easy to find effective estimates for these quantities. Then, as in [12], it would be reasonable to find asymptotic formulas for these quantities. Problems on asymptotic behavior of sections of line bundles over compact Kähler manifolds are intensively investigated. They are interesting problems in themselves, and also they often provide important information for complex geometrical problems. There is an enormous literature in this direction. We just refer to [6] for this direction.

In this paper, we give a survey of results on asymptotic analysis in these two topics, obtained in the papers [22, 25]. Let us give a brief account on the materials discussed here. First, we give an asymptotic formula of a quantity called a *lattice path counting function*. This quantity is defined as the number of lattice paths on a lattice in a vector space starting from the origin each step of which is in a fixed finite subset of the lattice. This quantity is a natural generalization of the binomial coefficient, and it goes well with the probability theory. Main result for this quantity is regarded as a result on large deviation, but we also give other asymptotic formulas, for example, corresponding to the local central limit theorem.

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Second topic is on an asymptotic behavior of distribution functions of sections of line bundles over a compact toric Kähler manifold, which we call *toric monomials*. In general, problems on asymptotic behavior of eigenfunctions of elliptic operators with discrete spectrum is very difficult. Indeed, one of simplest problems in this direction would be to find weak limits of modulus square of eigenfunctions. But, this problem is already hard. Indeed, it is known as quantum ergodicity problems when the classical counterpart is chaotic and there are many open problems. Even if the classical dynamical system is completely integrable, this problem is still difficult to resolve completely. (To our knowledge, one can find complete answer to this problem only in the case of the standard sphere. See [15].) So then, it would be useful to find reasonable and simple 'toy model' where one can settle almost all problems in this topic, such as weak limits, estimation of supremum norm, asymptotics of L^p-norm, pointwise asymptotics and asymptotic behavior of distribution functions. The toric varieties often provide a simple model for difficult problems, and this is the case with us. Namely, the projective toric Kähler manifolds are regarded as compactified phase spaces with completely integrable systems (torus actions on toric manifolds) whose joint eigenfunctions are toric monomials. So, our toric monomials are regarded as a model of (micro-local lifts of) joint eigenfunctions for completely integrable system.

Throughout this paper, the parameter which is made to tend to infinity is denoted by N. We note here that in each topic we are going to address the parameter N has a physical meaning. For the lattice path counting functions and the multiplicities of weights, the parameter N can be regarded as 'number of particles', because it is the parameter for tensor powers of a fixed representation and the 'classical phase space' of it would be N-fold product of a coadjoint orbit. Hence the limit $N \to \infty$ would be regarded as a thermodynamic limit. For the asymptotics of distribution functions of toric monomials, the limit $N \to \infty$ represents a semiclassical limit, because it is the parameter for the tensor power of a fixed line bundle over a toric Kähler manifold.

The organization of this paper is as follows. In Section 2, we define the lattice path counting functions and investigate its properties. In particular, we give an asymptotic formula (2.14) for the lattice path counting function. The formula (2.14) is a general formula, and we then use the formula (2.14) to give various asymptotic properties of the lattice path counting functions. These asymptotic formulas are used, in Section 3, to find asymptotic formulas for multiplicities of weights and irreducibles in the high tensor powers of a fixed irreducible representation of a compact Lie group. Section 4 is devoted to the study of toric monomials of a projective smooth toric variety. In particular, we give a sketch of proof of an asymptotic formula for the rescaled distribution functions of toric monomials.

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2 Asymptotic behavior of lattice path counting functions

In this section, we consider a problem on asymptotic behavior of lattice paths. In particular, we give various asymptotic results for the lattice path counting functions, which are explained their naturality along with their probabilistic background.

2.1 Lattice path counting functions

To begin with, let us prepare notation. Let X be a real vector space of dimension m, and let I be a lattice in X, that is, I is a co-compact discrete subgroup of X. Let X^* be the dual space of X, and let I^* be the dual lattice of I, that is, I^* is the lattice in X^* defined by $I^* = \{\gamma \in X^* \,;\, \langle \gamma, x \rangle \in \mathbb{Z},\, x \in I\}$. Let $S \subset I^*$ be a finite subset which is assumed to satisfy the following non-degeneracy condition;

$$\text{span}_{\mathbb{R}}\{\alpha - \beta \,;\, \alpha, \beta \in S\} = X^*. \tag{2.1}$$

For each positive integer N, we define the set $S(N)$ of lattice paths of length N with steps in S by

$$S(N) = \{\gamma \in I^* \,;\, \gamma = \beta_1 + \cdots + \beta_N \ \text{ for some } \ \beta_1, \ldots, \beta_N \in S\}. \tag{2.2}$$

Fix a positive function $c : S \to \mathbb{R}_{>0}$ on S which we call a weight function. Then, the main object in this section is the *weighted lattice path counting function* $\mathcal{P}_N^c : I^* \to \mathbb{R}$ with weight c defined by

$$\mathcal{P}_N^c(\gamma) = \begin{cases} \displaystyle\sum_{\substack{\beta_1,\ldots,\beta_N \in S \\ \gamma = \beta_1 + \cdots + \beta_N}} c(\beta_1) \cdots c(\beta_N) & \text{if } \gamma \in S(N), \\ 0 & \text{if } \gamma \notin S(N). \end{cases} \tag{2.3}$$

The function $\mathcal{P}_N^c$ often appears especially in probability theory and representation theory. We will explain a representation theoretical aspect of the function $\mathcal{P}_N^c$ in the next section. In this section, we discuss its probabilistic aspect and derive various asymptotic formulas for $\mathcal{P}_N^c$ as $N \to \infty$. Here is a typical example.

Example 2.1. Set $X = \mathbb{R}^m$ and use the standard Euclidean inner product to identify X^* with $\mathbb{R}^m$. We take the standard lattice $\mathbb{Z}^m$ for $I = I^*$. Let $\{e_1, \dots, e_m\}$ be the standard basis of $\mathbb{Z}^m$ and let $\Sigma = \mathrm{ch}\,(0, e_1, \dots, e_m)$. Here, for a subset $A \subset X^*$, $\mathrm{ch}\,(A)$ denotes the convex hull of A. Fix a positive integer p and set $S = (p\Sigma) \cap \mathbb{Z}^m$, the lattice points in the dilated polytope $p\Sigma$. Define the weight function $c : S \to \mathbb{R}_{>0}$ by

$$c(\beta) = \binom{p}{\beta} = \frac{p!}{\beta_1! \cdots \beta_m!(p - |\beta|)!},$$

where $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{R}^m$ and $|\beta| = \sum_{j=1}^m \beta_j$. Then, it is easy to show that $S(N) = (Np\Sigma) \cap \mathbb{Z}^m$ and

$$\mathcal{P}_N^c(\gamma) = \binom{Np}{\gamma}, \quad \gamma \in S(N).$$

Remark 2.2. For our finite set S in X^*, set $P = \mathrm{ch}\,(S)$. By definition, P is a convex polytope in X^*. Clearly the set $S(N)$ of lattice paths of length N with each step in S is contained in $NP \cap I^*$. However, in general, it is not necessary to have $S(N) = (NP) \cap I^*$. Indeed, let $X = X^* = \mathbb{R}^3$, $I = I^* = \mathbb{Z}^3$ and $S = \{(0,0,0), (1,0,0), (0,1,0), (1,1,2)\}$. The set $S \setminus \{(0,0,0)\}$ forms a basis of $\mathbb{R}^3$ (but not of $\mathbb{Z}^3$), and hence P is a simplex. Then, the point $(1,1,1)$ is in $(2P) \cap \mathbb{Z}^3$ but not in $S(2)$. It is a bit subtle problem whether or not we have $(NP) \cap \mathbb{Z}^m = S(N)$ for general S. It is related to the (projective) normality of the toric variety defined by the finite set S. See Section 4.

2.2 Asymptotic behavior of the binomial coefficients

In Example 2.1 we see that the lattice path counting functions $\mathcal{P}_N^c$ are regarded as a generalization of the binomial (multinomial) coefficients. In elementary probability theory, asymptotic properties of the binomial coefficients $\binom{N}{k_N}$ as $N \to \infty$ are related to the (local) central limit theorem or de Moivre-Laplace theorem. (In probability theory, the parameter N is the number of Bernoulli trials.) We just remind to the readers the following asymptotic properties of the binomial coefficients. In the following, we set $d_N(k) = k - N/2$. (For the proof, one just use Stirling's formula.)

$$\binom{N}{k} \sim \begin{cases} \text{(CL)} \quad 2^N \sqrt{\frac{2}{\pi N}} e^{-\frac{2d_N(k)^2}{N}} & \text{if } d_N(k) = o(N^{2/3}), \\ \text{(MD)} \quad 2^N \sqrt{\frac{2}{\pi N}} e^{-\frac{2d_N(k)^2}{N} - \frac{Nf(2d_N(k)/N)}{2}} & \text{if } d_N(k) = o(N), \\ \text{(SD)} \quad \frac{e^{-N(a\log a + (1-a)\log(1-a))}}{\sqrt{2\pi N a(1-a)}} & \text{if } k \sim aN,\ 0 < a < 1, \\ \text{(RE)} \quad \frac{N^{k_o}}{k_o!} & \text{if } k = k_o,\ N - k_o. \end{cases} \tag{2.4}$$

Here, in the second line, the function $f(x)$ is given by

$$f(x) + x^2 = g(x) := (1 + x)\log(1 + x) + (1 - x)\log(1 - x). \tag{2.5}$$

Let us consider the above asymptotic behavior of the binomial coefficients. In the case of (CL) and (MD), the exponent in the exponential is given by

$$-\frac{N}{2}g(2d_N(k)/2) = -\frac{2d_N(k)^2}{N} - \frac{N}{2}f(2d_N(k)/N).$$

But in the case of (CL), the first term x^2 of the function $g(x)$ dominates the decay rate because $Nf(2d_N(k)/N) = o(N^{-1/3})$. We call the case (CL) central limit region for k since the asymptotic form of the binomial coefficients in this region is Gaussian. (Note that, in the central limit region, if $d_N(k) = o(N^{1/2})$, the exponent $d_N(k)^2/N$ is bounded.) In the next case (MD), called moderate deviations, the second term is of order $o(N)$ which is the same as that of the first term. Thus, one can not ignore the second term in this region. Both cases of (CL) and (MD), the growth is governed by the exponent $\log 2$. But, in the case of strong deviations (SD), the growth is governed by the positive number $a\log(1/a) + (1 - a)\log(1/(1 - a))$ which is strictly less than $\log 2$ if $a \neq 1/2$. Finally, in the case of (RE), which we call the region of rare events, the binomial coefficients have polynomial growth rate rather than the exponential one.

2.3 Probabilistic aspects

In the previous subsection, we described the asymptotic behavior of the binomial coefficients, which is related to the central limit theorem and other limit theorems in probability theory. In this subsection, we give an account on probabilistic aspects of the lattice path counting function $\mathcal{P}_N^c(\gamma)$. Consider the function k_S^c on X defined by

$$k_S^c(\tau) = \sum_{\alpha \in S} c(\alpha) e^{\langle \alpha, \tau \rangle}, \qquad \tau \in X. \tag{2.6}$$

Then, it is easy to show that

$$k_S^c(\tau)^N = \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) e^{\langle \gamma, \tau \rangle}. \tag{2.7}$$

Thus, if we set $V(S, c) = \sum_{\alpha \in S} c(\alpha)$, then $V(S, c) = k_S^c(0)$ and $V(S, c)^N = \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma)$. Therefore, for any positive integer N, the measures

$$\begin{aligned} dm_N &= \frac{1}{V(S, c)^N} \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) \delta_{\gamma/N}, \\ d\mu_N &= \frac{1}{V(S, c)^N} \sum_{\gamma \in I^*} \mathcal{P}_N^c(\gamma) \delta_{\frac{1}{\sqrt{N}}(\gamma - N m_{S,c}^*)}, \end{aligned} \tag{2.8}$$

are probability measures on X^*. Here $m^*_{S,c} \in \mathrm{Int}\,(P)$ denotes the center of mass given by

$$m^*_{S,c} = \frac{1}{V(S,c)} \sum_{\alpha \in S} c(\alpha)\alpha, \tag{2.9}$$

and δ_x denotes the Dirac delta measure at x. The measure dm_N is supported on the polytope $P = \mathrm{ch}\,(S)$ while the support of the measure $d\mu_N$ is larger than P; it is supported on $\sqrt{N}P$ if the center of mass $m^*_{S,c}$ is the origin. The limit theorem we would like to mention first is the following.

Theorem 2.3. *The measures dm_N tend weakly to $\delta_{m^*_{S,c}}$ as $N \to \infty$.*

The above theorem, which is known as the *law of large numbers*, suggests that the normalized lattice path counting function $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ decreases when γ is far from $Nm^*_{S,c}$. The measure $d\mu_N$ measures its decay when $\gamma - Nm^*_{S,c} = O(N^{1/2})$. The precise statement is given as a *central limit theorem*.

Theorem 2.4. *The measures $d\mu_N$ tend weakly to the Gaussian measure*

$$\frac{e^{-\langle A^{-1}x,x\rangle/2}}{(2\pi)^{m/2}\sqrt{\det A}}\,dx$$

as $N \to \infty$, where the positive definite symmetric matrix A is given by

$$A = \frac{1}{V(S,c)} \sum_{\alpha \in S} \alpha \otimes \alpha - m^*_{S,c} \otimes m^*_{S,c}. \tag{2.10}$$

For simplicity, we explain in the case where $m^*_{S,c}$ is the origin. If F is a subset in X^*, according to the central limit theorem, $\mu_N(F) = m_N(N^{-1/2}F)$ tends to $C\int_F e^{-\langle A^{-1}x,x\rangle/2}\,dx = CN^{-m/2}\int_{N^{1/2}F} e^{-\langle A^{-1}y,y\rangle/2N}\,dy$. Thus, when $\gamma \in N^{1/2}F$, that is $\gamma = O(N^{1/2})$, the central limit theorem suggests that, on average, the behavior of the quantity $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ would be expressed as $CN^{-m/2}e^{-\langle A^{-1}\gamma,\gamma\rangle/2N}$. When, $\gamma - Nm^*_S = O(N)$, which means that γ is in the region of strong deviations as explained for the case of binomial coefficients, the averaged behavior of $V(S,c)^{-N}\mathcal{P}^c_N(\gamma)$ is described by the following theorem, which is known as the *large deviation principle.*

Theorem 2.5. *Set $I^c_S(x) = \sup_{\tau \in X}\{\langle x,\tau\rangle - \log(k^c_S(\tau)/V(S,c))\}$, $x \in X^*$. Then the function I_S is lower semi-continuous, and, for any closed set F and open set U of the polytope P, the measures dm_N satisfies the following.*

$$\limsup_{N\to\infty} \frac{1}{N}\log m_N(F) \le -\inf_{x\in F} I^c_S(x), \quad \liminf_{N\to\infty} \frac{1}{N}\log m_N(U) \ge -\inf_{x\in U} I^c_S(x).$$

The function I_S^c in Theorem 2.5 is called the *rate function* in the theory of large deviations. Theorem 2.5 says that when $\gamma - Nm_S^* = O(N)$ (that is, $\gamma/N \in F$ for fixed subset F), $V(S,c)^{-N}\mathcal{P}_N^c(\gamma)$ behaves, on average, like $e^{-NI_S^c(\gamma)}$. Theorems 2.3, 2.4 and 2.5 are proved by a standard method, although complete proofs can be found in [25]. In the next subsection, we derive much more precise asymptotic formulas for $V(S,c)^{-N}\mathcal{P}_N^c(\gamma)$, which support the above discussion.

2.4 Asymptotic behavior of lattice paths counting functions

The various aspects of asymptotic behavior of the binomial coefficients as explained above suggest that our weighted lattice path counting function $\mathcal{P}_N^c(\gamma)$ would also has similar asymptotic behavior. Indeed this is true. In the rest of this section, we give such results. To introduce such results, let us prepare some more notation. In the following, the setting up described in Subsection 2.1 is used. Since the differences $\alpha - \beta$ $(\alpha, \beta \in S)$ spans the whole space X^* as in the assumption (2.1), these spans over $\mathbb{Z}$ a sublattice of I^*, which is denoted by $L(S)^*$. Then, its dual lattice $L(S)$ in X contains the original lattice I. We set $Z(S) = I^*/L(S)^*$, which is a finite abelian group. It is easy to see that the Hessian of the function $\log k_S^c$,

$$A_S^c(\tau) := \nabla^2 \log k_S^c(\tau), \qquad \tau \in X, \tag{2.11}$$

is positive definite, and hence $\log k_S^c(\tau)$ is a convex function on X. By using this fact, one can show that the gradient,

$$\mu_S^c(\tau) := \nabla \log k_S^c(\tau), \qquad \tau \in X, \tag{2.12}$$

defines a *diffeomorphism* $\mu_S^c : X \to \mathrm{Int}\,(P)$, where $\mathrm{Int}\,(P)$ is the interior of the polytope $P = \mathrm{ch}\,(S)$. (Note that by the assumption (2.1), the polytope P is of dimension m.) See [7] or [8] for the proof of this fact. Denote the inverse map of $\mu_S^c : X \to \mathrm{Int}\,(P)$ by $\tau_S^c : \mathrm{Int}\,(P) \to X$. Define the smooth function δ_S^c on $\mathrm{Int}\,(P)$ by

$$\delta_S(x) = \log k_S^c(\tau_S^c(x)) - \langle\, x, \tau_S^c(x)\,\rangle, \qquad x \in \mathrm{Int}\,(P). \tag{2.13}$$

Theorem 2.6. *Take $x_o \in \mathrm{Int}\,(P)$ and $\gamma_N \in (NP) \cap I^*$ such that $\gamma_N = Nx_o + o(N)$. Then, we have*

$$\mathcal{P}_N^c(\gamma_N) = (2\pi N)^{-m/2} \frac{|Z(S)| e^{N\delta_S^c(\gamma_N/N)}}{\sqrt{\det A_S^c(\tau_S^c(\gamma_N/N))}} (1 + O(N^{-1})). \tag{2.14}$$

In particular, we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathcal{P}_N^c(\gamma_N) = \delta_S^c(x_o). \tag{2.15}$$

Remark 2.7. In [25], the assumption that, the lattice points γ_N is in $S(N)$ for every sufficiently large N, is imposed. However, in the proof of Theorem 2.6, we only use the integral formula (2.20) below, and this comes from the fact that $\mathcal{P}_N^c(\gamma)$ is the coefficients of $e^{i\langle \gamma, \varphi \rangle}$ in the Fourier series $k_S^c(i\varphi)^N$. Thus, we do not need to impose such an assumption.

Remark 2.8. It is easy to show that the rate function I_S^c in the large deviation principle (Theorem 2.5) is given by

$$I_S^c(x) = -\delta_S^c(x) + \log V(S,c), \quad x \in \mathrm{Int}\,(P).$$

Example 2.9. Let us examine the formula (2.15) for the binomial coefficients. As in Example 2.1 with $m=1$, let p be a positive integer, and let $S = \Sigma \cap \mathbb{Z} = \{0, 1, \dots, p\}$ with $\Sigma = \mathrm{ch}\,(0,p) = [0,p]$. Define $c : S \to \mathbb{R}_{>0}$ by $c(\beta) = \binom{p}{\beta}$. Then, as in Example 2.1, we have $\mathcal{P}_N^c(\gamma) = \binom{Np}{\gamma}$. In this case, the finite abelian group $Z(S)$ is trivial. The function k_S^c on $X^* = \mathbb{R}$ is given by $k_S^c(\tau) = (1+e^\tau)^p$, and hence

$$\mu_S^c(\tau) = \frac{pe^\tau}{1+e^\tau}, \quad \tau_S^c(x) = \log\left(\frac{x}{p-x}\right), \quad x \in \mathrm{Int}\,(P),\ \tau \in X = \mathbb{R}.$$

This shows that the function δ_S^c in this case is given by

$$\delta_S^c(x) = \log\left(\frac{p^p}{x^x(p-x)^{p-x}}\right), \qquad x \in \mathrm{Int}\,(P) = (0,p).$$

Then, the formula (2.15) can be deduced easily from Stirling's formula.

In the previous sections, we saw that $\mathcal{P}_N^c(\gamma_N)$ behaves differently when γ_N have different behavior as $N \to \infty$. In turn, Theorem 2.6 contains only one asymptotic formula (2.14). However, the asymptotic formula (2.14) in Theorem 2.6 is rather general. Indeed one can prove various asymptotic results from this formula similar to what was described for binomial coefficients. Let us explain how one can deduce them from just one formula (2.14). First, we note that since the weight function c is positive everywhere on the finite set S, the center of mass $m_{S,c}^*$ defined in (2.9) is in $\mathrm{Int}\,(P)$. Hence we can take $\gamma_N = Nm_S^* + d_N$ with $d_N = o(N^s)$, $0 \le s \le 2/3$ for the sequence γ_N in Theorem 2.6. A direct computation using the Taylor expansions around $x = m_{S,c}^*$ of the functions $\sqrt{\det A_S^c(\tau_S^c(x))}$ and $\delta_S^c(x)$ show that

$$\sqrt{\det A_S^c(\tau_S^c(\gamma_N/N))} = \sqrt{\det A}(1 + O(N^{-(1-s)})),$$
$$N\delta_S^c(\gamma_N/N) = N \log V(S,c) - \langle A^{-1}d_N, d_N \rangle/(2N) + o(N^{3s-2}),$$

where the symmetric matrix A is defined in (2.10). From these combined with the formula (2.14), we obtain the following local central limit theorem.

Theorem 2.10. *Let $0 \le s \le 2/3$ and $\gamma_N = Nm^*_{S,c} + d_N$ with $d_N = o(N^s)$. Then, we have*

$$\mathcal{P}^c_N(\gamma_N) = (2\pi N)^{-m/2} \frac{|Z(S)| V(S,c)^N e^{-\langle A^{-1} d_N, d_N \rangle/(2N)}}{\sqrt{\det A}} (1 + o(N^{3s-2})).$$

Next, we take $\gamma_N = N\alpha + f$ for some $f \in L(S)^*$ and $\alpha \in S \cap \mathrm{Int}\,(P)$. Then, one can apply directly Theorem 2.6. But in this case, one can also take $\gamma_N = N\alpha$ in the sketch of proof of Theorem 2.6 explained in the next subsection, and one has the following.

Theorem 2.11. *Let $f \in L(S)^*$ and let $\alpha \in S \cap \mathrm{Int}\,(P)$. Then we have*

$$\mathcal{P}^c_N(N\alpha + f) = (2\pi N)^{-m/2} \frac{|Z(S)| e^{-\langle f, \tau^c_S(\alpha) \rangle + N\delta^c_S(\alpha)}}{\sqrt{\det A^c_S(\tau^c_S(\alpha))}} (1 + O(N^{-1})).$$

Remark 2.12. Theorem 2.11 is a result on large deviations. For results on large deviations in a more general setting and from a geometrical point of view, see [16].

2.5 Method of stationary phase and sketch of proof

In this subsection, we give a sketch of proof of Theorem 2.6. To prove Theorem 2.6, we use a theorem on the method of stationary phase. First of all, let us give some account on this method.

Let $U \subset \mathbb{R}^m$ be an open set. Let $u \in C^\infty_0(U)$ and $\Phi \in C^\infty(U)$ with $\mathrm{Re}\,\Phi \ge 0$. Consider the following integral

$$I_N(u) = \int_U e^{-N\Phi(x)} u(x)\, dx. \tag{2.16}$$

We call the function Φ the phase function for the integral $I_N(u)$. The method of stationary phase is a method for studying asymptotic behavior of the integral of the form $I_N(u)$ as $N \to \infty$. To explain this method, suppose first that $\nabla\Phi \ne 0$ near $\mathrm{supp}\,(u)$. In this case, the first order differential operator,

$$L = -\frac{1}{|\nabla\Phi(x)|^2} \sum_{j=1}^m \overline{\frac{\partial\Phi}{\partial x_j}} \frac{\partial}{\partial x_j},$$

is well-defined near $\mathrm{supp}\,(u)$, where $|\nabla\Phi(x)|^2 = \sum_{j=1}^m \left|\frac{\partial\Phi}{\partial x_j}\right|^2$. Then it is straightforward to see that $L(e^{-N\Phi}) = Ne^{-N\Phi}$. Substituting this into the definition (2.16) of the integral $I_N(u)$ and integrating by parts show

$$I_N(u) = \frac{1}{N} \int_U e^{-N\Phi} ({}^tLu)\, dx,$$

where ${}^tL$ is the adjoint operator of L given by ${}^tLu = \sum_{j=1}^{m} \frac{\partial}{\partial x_j}\left(\frac{1}{|\nabla\Phi|^2}\overline{\frac{\partial\Phi}{\partial x_j}}u\right)$. Repeating this procedure, one has $I_N(u) = O(N^{-\infty})$, namely, for any positive integer k, one has $|I_N(u)| \leq C_k N^{-k}$ with a positive constant C_k.

Next, suppose that the phase function Φ satisfies $\mathrm{Re}\,(\Phi) > 0$ near $\mathrm{supp}\,(u)$. Then, since $\mathrm{supp}\,(u)$ is assumed to be compact, one can find a positive constant α such that $\mathrm{Re}\,(\Phi) \geq \alpha$ near $\mathrm{supp}\,(u)$. This shows that $I_N(u) = O(e^{-N\alpha})$. Therefore, one find that the contribution to $I_N(u)$ as $N \to \infty$ comes from neighborhoods of points $x \in U$ where $\mathrm{Re}\,(\Phi)(x) = 0$ and $\nabla\Phi(x) = 0$. Traditional method of stationary phase considers the case where the phase function Φ is pure imaginary, namely $\mathrm{Re}\,(\Phi) \equiv 0$. In this case, we set $\Phi = i\phi$ with a real-valued function ϕ. Suppose also that there exists a point $x_o \in U$ such that $\nabla\phi(x_o) = 0$, the Hessian $\nabla^2\phi(x_o)$ is non-degenerate and $\nabla\phi(x) \neq 0$ for points x different from x_o. Then, by the Morse lemma, there exists a neighborhood V of x_o and a diffeomorphism $\kappa : V \to \kappa(V) \subset \mathbb{R}^m$ such that $\kappa(x_o) = 0$, $\nabla\kappa(x_o) = \mathrm{Id}$ and

$$\phi\circ\kappa^{-1}(y) = \phi(x_o) + \langle\, Ay, y\,\rangle/2, \quad A := \nabla^2\phi(x_o).$$

Changing a variable $x = \kappa(y)$ will show that $I_N(u) = \tilde{I}_N(\tilde{u}) + O(N^{-\infty})$, where $\tilde{u}$ equals $|\det\nabla\kappa^{-1}(y)|u(\kappa^{-1}y)$ times a cut-off function near the origin and

$$\tilde{I}_N(\tilde{u}) = e^{-iN\phi(x_o)}\int_{\kappa(V)} e^{-iN\langle\, Ay,y\,\rangle/2}\tilde{u}(y)\,dy.$$

Using Plancherel formula and the well-known formula,

$$\mathcal{F}^{-1}[e^{-iN\langle\, Ay,y\,\rangle/2}](\xi) = \frac{e^{-i\pi\,\mathrm{sgn}(A)/4}}{(2\pi N)^{m/2}|\det A|^{1/2}}e^{i\langle\, A^{-1}\xi,\xi\,\rangle/(2N)},$$

where $\mathcal{F}^{-1}$ is the inverse of the Fourier transform $\mathcal{F}$, shows that the integral $I_N(u)$ equals

$$\frac{e^{-iN\phi(x_o)-i\pi\,\mathrm{sgn}(A)/4}}{(2\pi N)^{m/2}|\det A|^{1/2}}\int_{\mathbb{R}^m} e^{i\langle\, A^{-1}\xi,\xi\,\rangle/(2N)}\widehat{\tilde{u}}(\xi)\,d\xi$$

modulo terms of order $O(N^{-\infty})$. Then, a Taylor expansion of the exponential function shows that the integral $I_N(u)$ has the following asymptotic expansion

$$I_N(u) \sim \left(\frac{2\pi}{N}\right)^{m/2}\frac{e^{-iN\phi(x_o)-i\pi\,\mathrm{sgn}(\nabla^2\phi(x_o))/4}}{|\det\nabla^2\phi(x_o)|^{1/2}}\sum_{k\geq 0}(A_k u)(x_o)N^{-k}, \tag{2.17}$$

where A_k is a differential operator of order $2k$ with $A_0 = I$. This is a usual method of stationary phase. However, in the above lines, we used the Morse lemma, and the differential operators A_k contains derivatives of the diffeomorphism κ. Hence it is not suitable to compute lower order terms explicitly. Furthermore, as is seen below, in our case, the phase function Φ itself depends on the parameter N. So, one can not apply, at least directly, the above method. Fortunately, there is a version of the method of stationary phase which is quite useful.

Theorem 2.13. *Let U be an open set in $\mathbb{R}^m$ and let K be a compact set in U. Let $u \in C_0^\infty(K)$ and let $\Phi \in C^\infty(U)$ such that $\mathrm{Re}\,(\Phi) \geq 0$. Suppose that there exists a point $x_o \in K$ such that $\mathrm{Re}\,(\Phi(x_o)) = 0$, $\nabla\Phi(x_o) = 0$, $\det \nabla^2\Phi(x_o) \neq 0$ and that $\nabla\Phi(x) \neq 0$ for $x \in K$ different from x_o. Then, for each positive integer k,*

$$I_N(u) = \left(\frac{2\pi}{N}\right)^{m/2} \frac{e^{-N\Phi(x_o)}}{\sqrt{\det \nabla^2\Phi(x_o)}} \sum_{j=0}^{k-1} (L_j u)(x_o) N^{-j} + R_k(N), \tag{2.18}$$

with the error estimate

$$|R_k(N)| \leq C_k(\Phi) \|u\|_{C^{2k}(U)} N^{-k}, \tag{2.19}$$

where $C_k(\Phi)$ is a positive constant. The differential operator L_j at x_o is given by

$$(L_j u)(x_o) = (-1)^j \sum_{\substack{\nu,\mu \geq 0 \\ \nu-\mu=j, 2\nu \geq 3\mu}} \frac{1}{2^\nu \mu! \nu!} [\langle \nabla^2\Phi(x_o)^{-1} D, D \rangle^\nu (g_\Phi^\mu u)](x_o),$$

$$g_\Phi(x) = \Phi(x) - \Phi(x_o) - \frac{1}{2}\langle \nabla^2\Phi(x_o)(x - x_o), x - x_o \rangle.$$

Furthermore, suppose that B is a subset of $C^\infty(U)$ such that;

- *every $\Phi \in B$ satisfies $\mathrm{Re}\,(\Phi) \geq 0$, $\mathrm{Re}\,(\Phi(x_o)) = 0$, $\nabla\Phi(x_o) = 0$, $\det \nabla^2\Phi(x_o) \neq 0$ and that $\nabla\Phi(x) \neq 0$ for $x \in K$ different from x_o, where x_o is fixed;*
- *$\|\Phi\|_{C^{3k+1}(U)}$ is bounded from above uniformly in $\Phi \in B$;*
- *$|x - x_o|/|\nabla\Phi(x)|$ is bounded from above uniformly in $x \in U$ and $\Phi \in B$.*

Then, the constant $C_k(\Phi)$ in (2.19) can be taken to be independent of $\Phi \in B$.

See [13, Section 7] for the proof of the above theorem.

Remark 2.14. It is easy to see that the third condition for $B \subset C^\infty(U)$ in Theorem 2.13 can be replaced by $\sup_{\Phi \in B} \|\nabla^2\Phi(x_o)^{-1}\| < \infty$. More precisely, if $\|\Phi\|_{C^3} \leq \alpha$ and $\|\nabla^2\Phi(x_o)^{-1}\| \leq \beta$ for each $\Phi \in B$, then, one can show that, when $|x - x_o| \leq 1/2\alpha\beta$, we have $\dfrac{|x - x_o|}{|\nabla\Phi(x)|} \leq 2\beta$. In particular, in the case where we can shrink the domain of integration suitably, the assumption that $\nabla\Phi(x) \neq 0$ for x different from x_o is satisfied if the Hessian at x_o is non-degenerate and its inverse is bounded from above.

We now give a sketch of proof of Theorem 2.6. We extend elements in X^* to the complex linear form on $X \otimes \mathbb{C}$. We write elements in $X \otimes \mathbb{C}$ as $w = \tau + i\varphi$, $\tau, \varphi \in X$. Then, the function k_S^c on X is naturally extended to $X \otimes \mathbb{C}$, and by (2.7), the lattice path counting function $\mathcal{P}_N^c(\gamma)$ has the following integral representation,

$$\mathcal{P}_N^c(\gamma) = \frac{1}{(2\pi)^m} \int_{T^m} e^{-i\langle \gamma, \varphi \rangle} k(i\varphi)^N \, d\varphi, \tag{2.20}$$

where $T^m = X/2\pi I$ is an m-dimensional torus, and the Lebesgue measure $d\varphi$ is normalized so that the volume of T^m is $(2\pi)^m$. Since the function $k(z) = k_S^c(\tau+i\varphi)$ ($z = e^{\tau+i\varphi}$, $\tau, \varphi \in X$) is holomorphic on the complex torus $\exp(X \otimes \mathbb{C}) \cong (\mathbb{C}^*)^m$, we can change the contour in the integral to obtain

(2.21)

$$\mathcal{P}_N^c(\gamma_N) = \frac{1}{(2\pi)^m}[k_S^c(\tau)e^{-\langle \gamma_N/N, \tau \rangle}]^N \int_{T^m} e^{-iN\langle \gamma_N/N, \varphi \rangle} \left(\frac{k_S^c(\tau+i\varphi)}{k_S^c(\tau)} \right)^N d\varphi,$$

which is valid for arbitrary $\tau \in X$. It is easy to show that $|k_S^c(\tau+i\varphi)| \leq k_S^c(\tau)$, and the equality holds if and only if $\varphi \in 2\pi L(S)$. Since $I \subset L(S)$, there is a natural surjective homomorphism $\pi_S : T^m \to T(S) := X/2\pi L(S)$. Then, the above equality condition is equivalent to say that $|k_S^c(\tau+i\varphi)| = k_S^c(\tau)$ if and only if $\varphi\,(\bmod\ 2\pi I)$ is in the kernel $\ker(\pi_S)$ of π_S, and which is naturally isomorphic to the finite abelian group $Z(S)$. Since (2.21) holds for any $\tau \in X$, we choose τ as $\tau_N = \tau_S^c(\gamma_N/N)$. Then, we have

$$e^{\delta_S^c(\gamma_N/N)} = k_S^c(\tau_N)e^{-\langle \gamma_N/N, \tau_N \rangle}.$$

We take a neighborhood U of the identity $0 \in \ker(\pi_S) \cong Z(S)$ so that $U \cap \ker(\pi_S) = \{0\}$ and take a cut-off function $\rho \in C_0^\infty(U)$ which is 1 near $\varphi = 0$. For any $g \in \ker(\pi_S)$, we set $U_g = U + g$ and $\rho_g(\varphi) = \rho(\varphi - g)$. Then, if we take U so small, there exists a constant $a > 0$ such that

$$\mathcal{P}_N^c(\gamma_N) = \frac{e^{N\delta_S^c(\gamma_N/N)}}{(2\pi)^m} \sum_{g \in \ker(\pi_S)} I_{N,g} \tag{2.22}$$

modulo terms of order $O(e^{-aN})$, where $I_{N,g}$ is given by

(2.23)

$$I_{N,g} = \int_{U_g} e^{-N\Phi_{N,g}(\varphi)} \rho_g(\varphi)\, d\varphi, \quad \Phi_{N,g} = i\langle \gamma_N/N, \varphi \rangle - \log \left(\frac{k_S^c(\tau_N+i\varphi)}{k_S^c(\tau_N)} \right).$$

Note that, if we introduce the function

$$\Phi(\tau,\varphi) := i\langle \mu_S^c(\tau), \varphi \rangle - \log \left(\frac{k_S^c(\tau+i\varphi)}{k_S^c(\tau)} \right)$$

on $B \times U_g$, where B is a closed ball with center $\tau_S^c(x_o)$, then we have $\Phi_{N,g}(\varphi) = \Phi(\tau_N, \varphi)$. From this expression, one can show that $\|\Phi_{N,g}\|_{C^k(U_g)}$ is bounded from above independently of N, where k is any integer greater than $[m/2]+1$. We take a representative $\varphi_g \in X$ of $g \in \ker(\pi_S)$ and identify U_g with a neighborhood of φ_g. Note that $\operatorname{Re}\Phi(\tau,\varphi) \geq 0$ and the equality holds for $(\tau,\varphi) \in B \times \overline{U_g}$ if and only if $\varphi = \varphi_g$. Then, we see that $\nabla\Phi_{N,g}(\varphi_g) = 0$. Furthermore, $\operatorname{Re}\Phi_{N,g}(\varphi) = 0$ on U_g if and only if $\varphi = \varphi_g$. A direct computation shows $e^{N\Phi_{N,g}(\varphi_g)} = 1$ and $\nabla^2\Phi_{N,g}(\varphi_g) = A_S^c(\tau_N)$. Since τ_N tends to $\tau_S^c(x_o)$ and since $x_o \in \operatorname{Int}(P)$, $A_S^c(\tau_N)$ has inverse whose norm is bounded uniformly in N. Therefore, Theorem 2.13 is applied and a direct computation shows Theorem 2.6.

3 Asymptotics of multiplicities in high tensor powers

In Section 2, we derived asymptotic formula for the lattice path counting function $\mathcal{P}_N^c(\gamma)$ for general weight function c on the set S of steps. In this section, we give an application of the formula in Theorem 2.11 to representation theory of compact connected Lie groups.

3.1 Quick review of representation theory of compact Lie groups

The representations we are going to consider is them for compact connected Lie groups. For structure theory and representation theory of compact Lie groups, we refer the readers to [3]. In the following we prepare and review some terminology for representation theory of compact Lie groups. Let G be a compact connected Lie group. For simplicity, we assume that G is semi-simple, that is, assume that the center $Z(G)$ of G is finite. Let T be a maximal torus in G and let $\mathfrak{g}$ and $\mathfrak{t}$ be the Lie algebra of G and T, respectively. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the dual space of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. Since T is abelian and compact, the exponential map $\exp : \mathfrak{t} \to T$ is a surjective homomorphism, and its kernel $I = \ker(\exp)$ is a lattice in $\mathfrak{t}$ so that $T = \mathfrak{t}/I$. The dual lattice I^* of I is called the weight lattice or the lattice of integral forms. The Weyl group W is the quotient group $N(T)/T$ of the normalizer $N(T)$ of T by T, which is known to be a finite group. The maximal torus T acts on the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ by the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}^{\mathbb{C}})$, and then $\mathfrak{g}^{\mathbb{C}}$ is decomposed into irreducible components of T-action as

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha,$$

where, for each $0 \neq \alpha \in I^*$, we set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}^{\mathbb{C}}\,;\, \mathrm{Ad}\,(\exp(\varphi))x = e^{2\pi i\langle \alpha, \varphi \rangle} x, \varphi \in \mathfrak{t}\}.$$

The set R, called the root system, is defined by $R = \{\alpha \in \mathfrak{t}^* \setminus \{0\}\,;\, \mathfrak{g}_\alpha \neq 0\}$. The elements in R are called the roots. Since we have assumed that the group G is semi-simple, we have $\mathrm{span}_{\mathbb{R}}(R) = \mathfrak{t}^*$. Furthermore, there exists a basis $\{\alpha_1, \dots, \alpha_m\} \subset R$ of $\mathfrak{t}^*$ ($m = \dim T$) such that each $\alpha \in R$ can be represented as a $\mathbb{Z}$-linear combination of $\{\alpha_1, \dots, \alpha_m\}$,

$$\alpha = \sum_{j=1}^m n_j(\alpha)\alpha_j,$$

where $n_j(\alpha) \geq 0$ for all j or $n_j(\alpha) \leq 0$ for all j. Such a basis $\{\alpha_1, \dots, \alpha_m\} \subset R$ is called a system of simple roots. Let $R_+ = \{\alpha \in R\,;\, n_j(\alpha) \geq 0,\, j = 1, \dots, m\}$ and set $R_- = -R_+$. Then, it is well-known that $R_- \subset R$ and $R = R_+ \cup R_-$ (disjoint union). The elements in R_+ are called the positive roots. The Weyl group W is finite and acts on $\mathfrak{t}$ and $\mathfrak{t}^*$. We choose a W-invariant inner product on $\mathfrak{t}$ which is denoted $\langle \cdot, \cdot \rangle$, and this inner product naturally induces a W-invariant inner product on $\mathfrak{t}^*$ which we continue to write as $\langle \cdot, \cdot \rangle$. A positive Weyl chamber, denoted by C, is a cone in $\mathfrak{t}^*$ defined by

$$C = \{\varphi \in \mathfrak{t}^* \,;\, \langle \varphi, \alpha \rangle > 0,\ \alpha \in R_+\}.$$

Then, the famous Weyl character formula states that there exists a bijection between $\overline{C} \cap I^*$ and the set of characters (restricted to the maximal torus T) of irreducible representations of G. Furthermore, for each $\lambda \in \overline{C} \cap I^*$, the corresponding irreducible representation, denoted by V_λ, has the character χ_λ on T given by

$$\chi_\lambda(t) = \frac{A(\lambda+\rho)(\varphi)}{\Delta(\varphi)}, \quad t = \exp(\varphi) \in T,\ \varphi \in \mathfrak{t}, \tag{3.1}$$

where ρ is half the sum of the positive roots, $\rho = \frac{1}{2}\sum_{\alpha \in R_+} \alpha$, and for $\alpha \in \mathfrak{t}^*$, the alternating sum $A(\alpha)$ is defined by

$$A(\alpha)(\varphi) = \sum_{w \in W} \operatorname{sgn}(w) e^{2\pi i \langle w\alpha, \varphi \rangle}, \quad \varphi \in \mathfrak{t}, \tag{3.2}$$

where $\operatorname{sgn}(w)$ is the determinant of the transformation $w : \mathfrak{t} \to \mathfrak{t}$. The function $\Delta(\varphi)$ is defined by $\Delta(\varphi) = A(\rho)(\varphi)$, which is called the Weyl denominator. Note that, since W preserves the inner product on $\mathfrak{t}$, $\operatorname{sgn}(w) = \pm 1$. The integral form $\lambda \in \overline{C} \cap I^*$ is called the dominant weight of the irreducible representation V_λ.

3.2 Multiplicities in high tensor powers

Let V_λ be the irreducible representation of G with the dominant weight $\lambda \in \overline{C} \cap I^*$, and let N be a positive integer. Then, the tensor product $V_\lambda^{\otimes N}$ is a representation space of T, and hence one has a weight space decomposition

$$V_\lambda^{\otimes N} = \bigoplus_{\mu \in I^*} V_{\lambda,N}(\mu), \quad V_{\lambda,N}(\mu) = \{v \in V_\lambda^{\otimes N} \,;\, \exp(\varphi)v = e^{2\pi i \langle \mu, \varphi \rangle} v,\ \varphi \in \mathfrak{t}\}. \tag{3.3}$$

We set

$$m_N(\lambda;\mu) = \dim_{\mathbb{C}} V_{\lambda,N}(\mu), \quad \mu \in I^*,$$

and call $m_N(\lambda;\mu)$ the multiplicity of the weight μ in $V_\lambda^{\otimes N}$. The space $V_\lambda^{\otimes N}$ is also a representation space of the compact Lie group G, and hence it can be decomposed into irreducible summands,

$$V_\lambda^{\otimes N} = \bigoplus_{\mu \in \overline{C} \cap I^*} a_N(\lambda;\mu) V_\mu,$$

where $a_N(\lambda;\mu) \in \mathbb{Z}_+$ is the number of times V_μ appears in $V_\lambda^{\otimes N}$. We call $a_N(\lambda;\mu)$ the multiplicity of the irreducible representation V_μ in $V_\lambda^{\otimes N}$. A natural problem in this setting is whether or not one can find an effective formula for the multiplicities $m_N(\lambda;\mu)$ or $a_N(\lambda;\mu)$ in terms of N, λ and μ. For example, when $N = 2$, Steinberg's formula states that $a_2(\lambda;\mu)$ can be written as

$$a_2(\lambda;\mu) = \sum_{v,w\in W} \operatorname{sgn}(vw)\mathfrak{p}(v(\lambda+\rho) + w(\lambda+\rho) - (\mu+2\rho)), \tag{3.4}$$

where $\mathfrak{p}$ is Kostant's partition function,

$$\mathfrak{p}(\lambda) = \sharp\left\{ (n_\alpha | \alpha \in R_+)\,;\, n_\alpha \in \mathbb{Z}_{\geq 0},\, \lambda = \sum_{\alpha\in R_+} n_\alpha \alpha \right\}. \tag{3.5}$$

One can also use Steinberg's formula repeatedly to represent $a_N(\lambda;\mu)$ in terms of Kostant's partition function. However, this formula is an alternating sum and it is easy to imagine that the result becomes quite complicated as N becomes large. Hence it would not be so easy to estimate how large the multiplicity $a_N(\lambda;\mu)$ is from this formula for large N.

In the rest of this section, we give results on the asymptotics of the multiplicities $m_N(\lambda;\mu)$ and $a_N(\lambda;\mu)$ which is an application of Theorem 2.11. For any $\lambda \in \overline{C} \cap I^*$, define $S_\lambda = \{\mu \in I^*\,;\, m_1(\lambda;\mu) \neq 0\}$. Namely, S_λ is the set of weights occurring in the irreducible representation V_λ. Let $P(\lambda)$ denote the convex hull of the orbit $W \cdot \lambda$ of the Weyl group through λ. Let Λ^* be the lattice in $\mathfrak{t}^*$ spanned by the root system R over $\mathbb{Z}$, which is often called the root lattice. Define the map $\mu_\lambda : \mathfrak{t} \to \mathfrak{t}^*$ by

$$\mu_\lambda(x) = \frac{1}{\sum_{\nu\in S_\lambda} m_1(\lambda;\nu)e^{\langle \nu,x\rangle}} \sum_{\mu\in S_\lambda} m_1(\lambda;\mu)e^{\langle \mu,x\rangle}\mu. \tag{3.6}$$

It is well-known that $W \cdot \lambda \subset S_\lambda \subset P(\lambda)$. Since each coefficient of $\mu \in S_\lambda$ in the definition the map μ_λ is positive, the image of the map μ_λ is contained in the (relative) interior $\operatorname{Int}(P(\lambda))$ of the polytope $P(\lambda)$. Furthermore, as for the case of the map μ_S^c defined by (2.12), it turns out that the map μ_λ is a diffeomorphism from $\mathfrak{t}$ onto $\operatorname{Int}(P(\lambda))$ if the dominant weight λ is in the interior C of the closed positive Weyl chamber $\overline{C}$. (See below for this point.) Denote by $\tau_\lambda : \operatorname{Int}(P(\lambda)) \to \mathfrak{t}$ the inverse of the map μ_λ.

Theorem 3.1. *Let $\lambda \in C \cap I^*$ and $\nu_o \in S_\lambda$. Suppose that ν_o is in the interior of the polytope $P(\lambda)$. Take $f \in \Lambda^*$. Then, we have the following formula.*

$$m_N(\lambda; N\nu_o + f) = (2\pi N)^{-m/2} \frac{|Z(G)| e^{N\delta_\lambda(\nu_o) - \langle f,\tau_\lambda(\nu_o)\rangle}}{\sqrt{\det A_\lambda(\nu_o)}}(1 + O(N^{-1})), \tag{3.7}$$

where $Z(G)$ is the center of G, and the function δ_λ on $\mathrm{Int}\,(P(\lambda))$ and the positive definite matrix $A_\lambda(\nu_o)$ is given by

$$\delta_\lambda(x) = \log\left(\sum_{\mu\in S_\lambda} m_1(\lambda;\mu)e^{\langle\,\mu,\tau_\lambda(x)\,\rangle}\right) - \langle\, x, \tau_\lambda(x)\,\rangle, \quad x \in \mathrm{Int}\,(P(\lambda)),$$

$$A_\lambda(\nu_o) = \sum_{\mu\in S_\lambda} \frac{m_1(\lambda;\mu)e^{\langle\,\mu,\tau_\lambda(\nu_o)\,\rangle}}{\sum_{\nu\in S_\lambda} m_1(\lambda;\nu)e^{\langle\,\nu,\tau_\lambda(\nu_o)\,\rangle}}\mu\otimes\mu - \nu_o\otimes\nu_o.$$

Note that we have some other asymptotic results for the multiplicities $m_N(\lambda;\mu)$ of weights in tensor power $V_\lambda^{\otimes N}$. See [25]. By using Theorem 3.1, one can find the following asymptotic result for the multiplicities of irreducible representations in $V_\lambda^{\otimes N}$.

Theorem 3.2. *Let $\lambda \in C \cap I^*$ and let $\nu_o \in \overline{C} \cap S_\lambda \cap \mathrm{Int}\,(P(\lambda))$. Then, we have the following formula.*

(3.8)

$$a_N(\lambda; N\nu_o) = (2\pi N)^{-m/2} e^{N\delta_\lambda(\nu_o)} \left(\frac{|Z(G)|\Delta(\tau_\lambda(\nu_o)/(2\pi i))e^{-\langle\,\rho,\tau_\lambda(\nu_o)\,\rangle}}{\sqrt{\det A_\lambda(\nu_o)}} + O(N^{-1}) \right),$$

where the Weyl denominator Δ is extended to the complexification $\mathfrak{t}\otimes\mathbb{C}$.

Remark 3.3. By using the Weyl denominator formula, we have

$$\Delta(\tau_\lambda(\nu_o)/(2\pi i)) = \prod_{\alpha\in R_+} \left(e^{\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle/2} - e^{-\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle/2}\right).$$

So, for example, when $\tau_\lambda(\nu_o)$ is in a wall of a Weyl chamber, that is, there is a $\alpha \in R_+$ such that $\langle\,\alpha,\tau_\lambda(\nu_o)\,\rangle = 0$, we have $\Delta(\tau_\lambda(\nu_o)/(2\pi i)) = 0$, and hence the leading term in the asymptotic formula (3.8) vanishes. In this case, the formula (3.8) is not relevant to estimate the multiplicity $a_N(\lambda; N\nu_o)$.

Remark 3.4. Theorems 3.1 and 3.2 are formulas in large deviation. The local central limit theorems for the multiplicities $a_N(\lambda;\mu)$, $m_N(\lambda;\mu)$ also hold true. See [2], [25] for these formulas.

3.3 Multiplicities versus lattice path counting functions

In this subsection, we give a sketch of proof of Theorems 3.1 and 3.2. Indeed, these are proved by using the asymptotic formula in Theorem 2.11 of lattice path counting function $\mathcal{P}_N^c(\gamma)$. Let us explain how the lattice path counting function comes into the discussion. To use the lattice path model, we need to specify the vector space X, the lattice I, the finite set S in the dual lattice I^* satisfying the non-degeneracy condition (2.1) and the weight function $c : S \to \mathbb{R}_{>0}$.

In the representation theoretical setting, we take $\mathfrak{t}$ for the vector space X and the integer lattice ker(exp) for the lattice I. The finite set S is the set S_λ of weights occurring in the fixed irreducible representation V_λ. We define the weight function c_λ on S_λ by setting $c_\lambda(\mu) = m_1(\lambda;\mu)$. Then, we can consider the lattice path counting function $\mathcal{P}_N^\lambda = \mathcal{P}_N^{c_\lambda}$ on I^*. To be precise, we need to check that S_λ satisfies the condition (2.1) and to specify the lattice $L(S_\lambda)^*$. Let $\{\alpha_1, \ldots, \alpha_m\}$ be the system of simple roots which defines the fixed positive Weyl chamber C. We identify $\mathfrak{t}$ and $\mathfrak{t}^*$ by using the fixed W-invariant inner product. For any root $\alpha \in R$, define $\alpha^\vee = \frac{2}{\langle \alpha,\alpha \rangle}\alpha \in \mathfrak{t}$. The vector $\alpha^\vee$ is called an inverse root (or co-root). It is well-known that the set $R^\vee$ of all $\alpha^\vee$, $\alpha \in R$, is again a root system, but what we need is the fact that $R^\vee$ is contained in the integer lattice I. From this and the assumption that $\lambda \in C \cap I^*$, that is λ is not in the wall of the chamber C, the pairing $\langle \lambda, \alpha^\vee \rangle$ is a positive integer for each $\alpha \in R_+$. The reflection $s_\alpha : \mathfrak{t}^* \to \mathfrak{t}^*$ with respect to α is given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha, \quad x \in \mathfrak{t}^*.$$

The reflection with respect to the simple roots α_j $(j = 1, \ldots, m)$ is denoted by s_j. Then, it is well-known that $s_j \in W$ (actually, s_j's generate W) and the set of weights S_λ in V_λ is invariant under the action of the Weyl group W. Since, $\lambda \in S_\lambda$ with $m_1(\lambda;\lambda) = 1$, we see

$$\alpha_j = \frac{1}{\langle \lambda, \alpha_j^\vee \rangle}(\lambda - s_j(\lambda)) \in \operatorname{span}_{\mathbb{R}}\{\mu - \nu\,;\, \mu, \nu \in S_\lambda\}.$$

This shows that S_λ satisfies the condition (2.1). Indeed, one can say more. It is well-known that $\lambda - j\alpha$ is contained in S_λ for any positive root α and any integer j with $0 \le j \le \langle \lambda, \alpha^\vee \rangle$ (see [14]). This shows that the lattice $L(S_\lambda)^*$ coincides with the root lattice Λ^*. Then, the finite group $Z(S_\lambda)$, which is defined as the quotient $I^*/L(S_\lambda)^* \cong L(S_\lambda)/I$, is isomorphic to the quotient Λ/I, where Λ is the lattice in $\mathfrak{t}$ dual to Λ^*. The latter group Λ/I is known to be isomorphic to the center $Z(G)$. Thus, the lattice path counting function $\mathcal{P}_N^\lambda(\gamma)$ satisfies the assumptions made for Theorem 2.11, and hence we can apply it. But then the crucial fact is that we have

$$\mathcal{P}_N^\lambda(\mu) = m_N(\lambda;\mu), \quad \mu \in I^*. \tag{3.9}$$

Indeed, in this case the function $k_\lambda := k_{S_\lambda}^{c_\lambda}$ defined in (2.6) is given by

$$k_\lambda(\tau) = \sum_{\mu \in S_\lambda} m_1(\lambda;\mu) e^{\langle \mu, \tau \rangle}, \quad \tau \in \mathfrak{t},$$

which coincides with the character $\chi_\lambda(\tau/(2\pi i))$. Since χ_λ^N is the character of the representation $V_\lambda^{\otimes N}$ of G, (3.9) follows from (2.7) and (3.3). Hence, Theorem 3.1

follows directly from Theorem 2.11. To prove Theorem 3.2, we just need to use Theorem 3.1 with $f = \rho - w\rho$, $w \in W$ (these are indeed elements in $L(S)^* = \Lambda^*$), and the following identity;

$$a_N(\lambda;\mu) = \sum_{w\in W} \mathrm{sgn}(w) m_N(\lambda;\mu+\rho-w\rho). \tag{3.10}$$

This formula is obtained in [10]. To prove this, we observe that the character χ_λ^N of $V_\lambda^{\otimes N}$ can be written as

$$\chi_\lambda^N = \sum_{\mu\in\overline{C}\cap I^*} a_N(\lambda;\mu)\chi_\mu. \tag{3.11}$$

Then, multiplying this identity by the Weyl denominator Δ and using the Weyl character formula (3.1), we see

$$\Delta\chi_\lambda^N = \sum_{\mu\in\overline{C}\cap I^*,\, w\in W} \mathrm{sgn}(w) a_N(\lambda;\mu) e^{2\pi i w(\mu+\rho)}. \tag{3.12}$$

But, the weight decomposition (3.3) tells us that

$$\Delta\chi_\lambda^N = \sum_{\gamma\in I^*,\, w\in W} \mathrm{sgn}(w) m_N(\lambda;\gamma) e^{2\pi i(\gamma+w\rho)}. \tag{3.13}$$

In (3.12), note that, when $\mu \in \overline{C}$ we have $\mu+\rho \in C$, and hence $w(\mu+\rho) = \mu+\rho$ if and only if $w = 1$. Thus, the coefficient of $e^{2\pi i(\mu+\rho)}$ in (3.12) is $a_N(\lambda;\mu)$ while that in (3.13) is the right hand side of (3.10). From this, we conclude (3.10) and hence Theorem 3.2.

4 Distribution laws for toric monomials

In the previous sections, we consider the lattice path counting function or multiplicities of group representations. In these topics, the limit $N \to \infty$ can be regarded as a kind of thermodynamic limit because N can be regarded as a 'number of particles'. In turn, the problem we are going to address in this section is in the semiclassical limit. Namely, we consider asymptotic behavior of sections of a line bundle over a projective toric varieties.

4.1 Toric varieties from monomial embeddings

In this section, for simplicity, we set $X = X^* = \mathbb{R}^m$ and $I = I^* = \mathbb{Z}^m$. Let $S \subset \mathbb{Z}^m$ be a finite set and put $s = \sharp S$. As in the previous sections, we fix a positive function c on S. Assume that the set S satisfies the following stronger assumption than (2.1):

$$\mathrm{span}_{\mathbb{Z}}\{\alpha-\beta\,;\,\alpha,\beta\in S\} = \mathbb{Z}^m. \tag{4.1}$$

We denote the standard coordinates on $\mathbb{C}^s$ by $\zeta = (\zeta_\alpha)_{\alpha\in S}$ and the homogeneous coordinates of points in the complex projective space $\mathbb{C}P^{s-1}$ of dimension $s-1$ by $[\zeta] = [\zeta_\alpha]_{\alpha\in S}$, $\zeta \in \mathbb{C}^s \setminus \{0\}$. Denote by $T^m_{\mathbb{C}} = (\mathbb{C}^*)^m$ a complex torus of dimension m and consider the map

$$\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}, \quad \Phi_S(t) = [c(\alpha)^{1/2} t^\alpha]_{\alpha\in S}, \tag{4.2}$$

where, for $t = (t_1, \dots, t_m) \in T^m_{\mathbb{C}}$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m$, we set $t^\alpha = t_1^{\alpha_1} \cdots t_m^{\alpha_m}$. The condition (4.1) assures that the map Φ_S is injective and is an embedding, which we call a *monomial embedding*. Define

$$\mathcal{O}_S = \Phi_S(T^m_{\mathbb{C}}), \quad M_S = \overline{\mathcal{O}_S}^{\text{Zariski}}, \tag{4.3}$$

where $\overline{\mathcal{O}_S}^{\text{Zariski}}$ denotes the Zariski closure of $\mathcal{O}_S$, which means that M_S is the smallest algebraic variety containing $\mathcal{O}_S$. We call the projective variety M_S a *toric variety*. Usually, toric varieties are, by definition, algebraic varieties which is irreducible, normal, and on which $T^m_{\mathbb{C}}$ acts algebraically with an open dense orbit. Our varieties of the form M_S admit these properties except the *normality*. The structures and properties of the varieties of the form M_S are described in [9] and in the article by A. Cannas da Silva in [1]. In this section, we give a brief account on these varieties. First, we give just one example. For other examples, see [1] where one can find many examples and exercises.

Example 4.1. Let $m = 1$. Take a positive integer p. Set $S = \{0, 1, \dots, p\}$. Take $c \equiv 1$. Then, the monomial embedding $\Phi_S : \mathbb{C}^* \to \mathbb{C}P^p$ is given by $\Phi_S(t) = [1 : t : t^2 : \cdots : t^p]$. Hence the variety M_S coincides with the image of the Veronese embedding $V : \mathbb{C}P^1 \to \mathbb{C}P^p$ given by

$$V([z_1 : z_2]) = [z_2^p : z_1 z_2^{p-1} : \cdots : z_1^{p-1} z_2 : z_1^p].$$

This shows that M_S is isomorphic to $\mathbb{C}P^1$.

We introduced the variety M_S by using the positive function c on S. But, the structure of M_S does not depend on the choice of the function c. Indeed, let X be the variety of the form M_S obtained by letting $c \equiv 1$. Let $C \in \mathrm{GL}(s, \mathbb{C})$ be the diagonal matrix whose components are given by $c(\alpha)^{1/2}$, $\alpha \in S$. Then, we have $M_S = CX$. However, when the variety M_S is smooth, the Kähler structure on M_S induced by the Fubini-Study form on $\mathbb{C}P^{s-1}$ depends on the choice of the weight function c.

Thus, for simplicity, we set $c \equiv 1$ in the rest of this subsection. Let $\mathbb{Z}^s_+$ denotes the set of lattice points in $\mathbb{R}^s$ whose components are all non-negative integers. Then it is not hard to show that, the homogeneous ideal $I_S \subset \mathbb{C}[\mathbb{Z}^s_+]$ defining the variety M_S, where $\mathbb{C}[\mathbb{Z}^s_+]$ denotes the algebra of polynomials in s-variables over $\mathbb{C}$, is generated by

$$\left\{ \zeta^\nu - \zeta^{\nu'} \;\middle|\; \nu, \nu' \in \mathbb{Z}^s_+,\ \sum_{\alpha\in S} \nu_\alpha \alpha = \sum_{\alpha\in S} \nu'_\alpha \alpha,\ \sum_\alpha \nu_\alpha = \sum_\alpha \nu'_\alpha \right\}.$$

To look closer at the ideal I_S, we set $A(S) := \{(\alpha, 1) \in \mathbb{Z}^{m+1}\,;\, \alpha \in S\}$. Let $S_{A(S)}$ denote the additive semigroup generated by $A(S)$ and let $\mathbb{C}[S_{A(S)}]$ the semigroup algebra. As an algebra, $\mathbb{C}[S_{A(S)}]$ has generators $(z, w)^{(\alpha,1)} = z^\alpha w$ $(\alpha \in S)$, where z and w are a complex m-variables and a complex variable, respectively. Let $\pi : \mathbb{R}^s \to \mathbb{R}^{m+1}$ be the linear map defined by $\pi(x) = \sum_\alpha x_\alpha(\alpha, 1)$, $x = (x_\alpha)_{\alpha\in S}$, and let $\hat{\pi} : \mathbb{C}[\mathbb{Z}^s_+] \to \mathbb{C}[S_{A(S)}]$ be a surjective homomorphism defined by

$$\hat{\pi}(\zeta^\nu) = z^{\sum_\alpha \nu_\alpha \alpha} w^{\sum_\alpha \nu_\alpha} = (z, w)^{\pi(\nu)}.$$

The following lemma is easy to prove and hence we omit the proof.

Lemma 4.2. *We have* $\ker(\hat{\pi}) = I_S$. *In particular,* I_S *is a prime homogeneous ideal and* M_S *is irreducible. The homogeneous coordinate ring of* M_S *is isomorphic to the semigroup ring* $\mathbb{C}[S_{A(S)}]$.

Let $\phi : T^m_{\mathbb{C}} \to T^s_{\mathbb{C}}$ be an injective homomorphism defined by $\phi(t) = (t^\alpha)_{\alpha\in S}$. Then, $T^m_{\mathbb{C}}$ acts on $\mathbb{C}P^{s-1}$ through the homomorphism ϕ and the monomial embedding $\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}$ is equivariant. Clearly, the image $\mathcal{O}_S$ of Φ_S is an orbit of $T^m_{\mathbb{C}}$-action on $\mathbb{C}P^{s-1}$. Furthermore, it is not hard to show that $\mathcal{O}_S = \{[\zeta] \in M_S\,;\, \zeta_\alpha \neq 0,\ \alpha \in S\}$, and hence $\mathcal{O}_S$ is open in M_S. Thus, up to the normality, the variety M_S is toric.

In general, a projective variety X in $\mathbb{C}P^{s-1}$ is said to be *normal* if the local ring o_p is integrally closed the function field of X for each $p \in X$ (see [11]). For each $\alpha \in S$, let $U_\alpha \subset \mathbb{C}P^{s-1}$ be the open set given by $\{\zeta_\alpha \neq 0\}$. We know that $U_\alpha \cong \mathbb{C}^{s-1}$ and $\{U_\alpha\}_{\alpha\in S}$ covers $\mathbb{C}P^{s-1}$. When $X \subset \mathbb{C}P^{s-1}$ is a projective variety, each $U_\alpha \cap X$ is an affine variety. Then, the normality of X is equivalent to the condition that the affine coordinate ring of $X \cap U_\alpha$ is integrally closed for each $\alpha \in S$. To describe the conditions for the normality of our variety M_S, let us prepare some more notation. We set $P = \mathrm{ch}\,(S)$, the convex hull of S. Then, the set $\mathcal{V}(P)$ of vertices of P is in S. For any $p \in \mathcal{V}(P)$, let $S_p \subset \mathbb{Z}^m$ be the semigroup generated by $\{\alpha - p\,;\, \alpha \in S\}$. Then, we have the following theorem.

Theorem 4.3. *Suppose that the finite set* S *in* $\mathbb{Z}^m$ *satisfies the condition* (4.1). *Then, the following conditions are equivalent.*

1. *The projective variety* M_S *defined by* (4.3) *is normal.*
2. *For all* $p \in \mathcal{V}(P)$, *we have* $S_p = K(S_p) \cap \mathbb{Z}^m$, *where* $K(S_p)$ *denotes the cone generated by* S_p.
3. *There exists a positive integer* N_o *such that for any integer* $N \geq N_o$, *we have* $S(N) = (NP) \cap \mathbb{Z}^m$, *where* $S(N)$ *is defined in* (2.2)

Note that the second condition in Theorem 4.3 comes from the fact that the family of open sets $\{U_p \cap M_S\}_{p\in\mathcal{V}(P)}$ is an open covering. See [21, Lemma 13.10, Theorem 13.11] for the proofs of this fact and Theorem 4.3.

Remark 4.4. A projective variety X is said to be projectively normal if its homogeneous coordinate ring is integrally closed. For the toric variety M_S constructed above (with S satisfying (4.1)), the projective normality is equivalent to that we have $S_{A(S)} = K(S_{A(S)}) \cap \mathbb{Z}^{m+1}$. See [9], [21]. Furthermore, under the assumption (4.1), one can show that this condition holds if and only if $S(N) = (NP) \cap \mathbb{Z}^m$ for *any* positive integer N. As we will see in the next section, if $S = P \cap \mathbb{Z}^m$ with the Delzant lattice polytope P, the corresponding toric variety M_S is smooth and normal. However, even in this case, it is not clear whether M_S is projectively normal or not. See [4], [19] for this issue.

4.2 Smooth projective toric varieties

We have constructed a toric variety M_S from a finite set $S \subset \mathbb{Z}^m$ satisfying the condition (4.1) through the monomial embedding $\Phi_S : T^m_{\mathbb{C}} \to \mathbb{C}P^{s-1}$, $s = \sharp S$. Our interest is in asymptotic analysis, and it would be reasonable to use smooth toric variety. In this section, we consider such a variety. From now on, we assume that our finite set S is of the form $S = P \cap \mathbb{Z}^m$ where P is a lattice polytope, which means that each vertex of the polytope P lies in the lattice $\mathbb{Z}^m$. In this case, we write Φ_P, $\mathcal{O}_P$, M_P instead of Φ_S, $\mathcal{O}_S$, M_S, respectively. Furthermore, we assume that the polytope P is Delzant. Recall that a polytope P in $\mathbb{R}^m$ is said to be *Delzant* if, for each vertex p of P, there exist exactly m edges emanating from p and there exists a lattice basis $\{w_1, \ldots, w_m\}$ of $\mathbb{Z}^m$ such that each edge emanating from p lies on the half line $\{p + tw_j\,;\, t \geq 0\}$ for some j. Then, the following fact is well-known.

Proposition 4.5. *The toric variety M_S constructed above is smooth if S is of the form $S = P \cap \mathbb{Z}^m$ with a Delzant lattice polytope P.*

In [9], the corresponding fact is in Corollary 3.2, Chapter 5. There, the conditions for M_S to be smooth is described in a different fashion. However, one can check that the Delzant condition implies these conditions. In the following we give a sketch of proof of Proposition 4.5, which is similar to the proof of the fact that the complex projective space is smooth.

Proof. Define the map $\mu_s : \mathbb{C}P^s \to \mathbb{R}^s$ by

$$\mu_s([\zeta]) = \sum_{\alpha \in S} \frac{|\zeta_\alpha|^2}{\sum_{\beta \in S} |\zeta_\beta|^2} e_\alpha,$$

where e_α ($\alpha \in S$) is the standard basis of $\mathbb{R}^s$. We then define the map $\mu_P^c : M_P \to \mathbb{R}^m$ by the composition

$$\mu_P^c : M_P \overset{\iota_P}{\hookrightarrow} \mathbb{C}P^s \overset{\mu_s}{\to} \mathbb{R}^s \overset{p}{\to} \mathbb{R}^m, \tag{4.4}$$

where the linear map $p : \mathbb{R}^s \to \mathbb{R}^m$ is defined as $p(e_\alpha) = \alpha$, $\alpha \in S = P \cap \mathbb{Z}^m$. Note that the map μ_P^c depends on the choice of the weight function c on $S = P \cap \mathbb{Z}^m$. The map μ_P^c is continuous in the usual topology on M_S and its image coincides with the Delzant polytope P. It is not so hard to show that $\mu_P^c(M_P \setminus \mathcal{O}_P) = \partial P$, where $\mathcal{O}_P = \Phi_S(T_{\mathbb{C}}^m)$ is the image of the monomial embedding $\Phi_S : T_{\mathbb{C}}^m \to \mathbb{C}P^{s-1}$. (To prove this, one will need to use the fact that $\mathcal{O}_P$ is dense in M_P in the usual topology. See [17] for this fact.) Furthermore, one can show that the following holds.

1. For each face f of P, we have $(\mu_P^c)^{-1}(f) = \{[\zeta] \in M_P\,;\, \zeta_\alpha = 0,\ \alpha \in S \setminus f\}$.
2. For each face f, we have $(\mu_P^c)^{-1}(\mathrm{ri} f) = \{[\zeta] \in (\mu_P^c)^{-1}(f)\,;\, \zeta_\alpha \neq 0,\ \alpha \in S \cap f\}$, where $\mathrm{ri} f$ is the relative interior of f in the affine hull of f.
3. For each face f, $(\mu_P^c)^{-1}(\mathrm{ri} f)$ is a $T_{\mathbb{C}}^m$-orbit.
4. For each vertex $p \in \mathcal{V}(P)$, the open set $U_p \cap M_P = \{[\zeta] \in M_P\,;\, \zeta_p \neq 0\}$ is given by

$$U_p \cap M_P = \bigcup_{f:\,\text{face of } P,\ p \in f} (\mu_P^c)^{-1}(\mathrm{ri} f). \tag{4.5}$$

(The correspondence between the open faces of P and the $T_{\mathbb{C}}^m$-orbit in M_P is proved in [9]. But, one can show the above facts in an elementary method similar to the proof of Lemma 3.10 in [24]. Note that the above facts hold for general finite set S satisfying (4.1).) From these facts, the decomposition $M_P = \bigcup_{f:\text{face of} P} (\mu_P^c)^{-1}(\mathrm{ri} f)$ gives the orbit decomposition of the $T_{\mathbb{C}}^m$-action on M_P.

Now, fix a vertex p of P. Since P is Delzant, there exists a lattice basis $\{v_j\}_{j=1}^m$ of $\mathbb{Z}^m$ such that each edge emanating from p lies in a half line $\{p + tv_j\,;\, t \geq 0\}$. Define a matrix Γ_p with integer components by the formula $\Gamma_p v_j = e_j$ ($j = 1, \ldots, m$) where $\{e_j\}_{j=1}^m$ is the standard basis of $\mathbb{Z}^m$. Since $\{v_j\}$ is a lattice basis, the determinant of Γ_p is ± 1. Then, we define a map

$$\phi_p : \mathbb{C}^m \to U_p \cap M_P, \quad \phi_p(w) = [w^{\Gamma_p(\alpha - p)}]_{\alpha \in S}. \tag{4.6}$$

Note that ϕ_p is well-defined because $\Gamma_p(\alpha - p) \in \mathbb{Z}_+^m$. (This follows from the fact that each $\alpha - p$ can be written as a linear combination of v_j with coefficients in $\mathbb{Z}_+$.) Define

$$\psi_p : U_p \cap M_P \to \mathbb{C}^m, \quad \psi_p([\zeta_\alpha]_{\alpha \in S}) = \left(\frac{\zeta_{\alpha_1}}{\zeta_p}, \ldots, \frac{\zeta_{\alpha_m}}{\zeta_p} \right),$$

where $\alpha_j \in S$ is characterized by $v_j = \alpha_j - p$. Let us show that $\phi_p^{-1} = \psi_p$. It is easy to show that $\psi_p \circ \phi_p(w) = w$ for $w \in \mathbb{C}^m$. To prove $\phi_p \circ \psi_p([\zeta]) = [\zeta]$ for

$[\zeta] \in U_p \cap M_P$, we need to use the structure of the orbit decomposition described above. We set $\alpha - p = \sum_{j=1}^m c_j(\alpha) v_j$ with $c_j(\alpha) \in \mathbb{Z}_+$. We fix $[\zeta_\alpha]_{\alpha \in S} \in U_p \cap M_P$ and put

$$\lambda_\alpha = \left(\frac{\zeta_{\alpha_1}}{\zeta_p}\right)^{c_1(\alpha)} \cdots \left(\frac{\zeta_{\alpha_m}}{\zeta_p}\right)^{c_m(\alpha)}, \quad \alpha \in S = P \cap \mathbb{Z}^m.$$

Then, we must show that $[\zeta_\alpha]_{\alpha \in S} = [\lambda_\alpha]_{\alpha \in S}$. There are m facets (faces of codimension 1) of P containing the vertex p, which we denote by $F_1, \ldots, F_m$, where F_j is characterized by $\alpha_j \notin F_j$ $(j = 1, \ldots, m)$. For any $I \subset \{1, \ldots, m\}$, we set

$$f_I = \bigcap_{j \in I} F_j,$$

which is a face of P containing p. All the faces containing p are of the form f_I for some $I \subset \{1, \ldots, m\}$. By (4.5), there is a unique $I \subset \{1, \ldots, m\}$ such that $[\zeta_\alpha]_{\alpha \in S} \in (\mu_P^c)^{-1}(\mathrm{ri} f_I)$. By the fact that $(\mu_P^c)^{-1}(\mathrm{ri} f_I)$ is a $T_{\mathbb{C}}^m$-orbit, there exist $c \in \mathbb{C}^*$ and $z \in T_{\mathbb{C}}^m$ such that $\zeta_\alpha = c z^\alpha$ $(\alpha \in S \cap f_I)$, $\zeta_\alpha = 0$ $(\alpha \in S \setminus f_I)$. Note that $\alpha_j \notin f_I$ if and only if $j \in I$. From this one can show that $\alpha \in S \setminus f_I$ if and only if $c_j(\alpha) \neq 0$ for some $j \in I$. Thus, we have $\lambda_\alpha = 0$ for $\alpha \in S \setminus f_I$. Since $\zeta_{\alpha_j} = c z^{\alpha_j}$ for $j \notin I$, we have, for $\alpha \in S \cap f_I$,

$$\lambda_\alpha = \prod_{j \notin I} \left(\frac{\zeta_{\alpha_j}}{\zeta_p}\right)^{c_j(\alpha)} = \prod_{j \notin I} (z^{\alpha_j - p})^{c_j(\alpha)} = z^\alpha = c^{-1}\zeta_\alpha,$$

which shows $[\lambda_\alpha]_\alpha = [\zeta_\alpha]_{\alpha \in S}$. Therefore, we have $\psi_p = \phi_p^{-1}$, and hence ϕ_p is a homeomorphism. Now, it is not so hard to show, by a direct computation with the orbit decomposition described above, that the coordinate change $\phi_q^{-1} \circ \phi_p$ $(p, q \in \mathcal{V}(P))$ is holomorphic. $\square$

4.3 Toric monomials

In the rest of this section, let P be a Delzant lattice polytope and let $S = P \cap \mathbb{Z}^m$. Then, we have a compact complex submanifold $M_P := M_S$ in $\mathbb{C}P^{s-1}$. Denote the inclusion of M_P into $\mathbb{C}P^{s-1}$ by $\iota_P : M_P \hookrightarrow \mathbb{C}P^{s-1}$. Let ω_{FS} be the Fubini-Study Kähler form on $\mathbb{C}P^{s-1}$. Then, the 2-form $\omega_P^c = \iota_P^* \omega_{\mathrm{FS}}$ is a Kähler form on M_P. The 2-form ω_P^c is integral in the sense that there exists a line bundle L_P^c over M_P such that $c_1(L_P^c) = [\omega_P^c]$ in (the image of) $H^2(M_P, \mathbb{Z})$. Indeed, let $\mathcal{O}(1) \to \mathbb{C}P^{s-1}$ denote the hyperplane section bundle. The bundle $\mathcal{O}(1)$ is the dual to the tautological line bundle over $\mathbb{C}P^{s-1}$. Then, the pull-back $L_P^c = \iota_P^* \mathcal{O}(1) \to M_P$ has this property.

Our toric variety M_P is smooth and hence is normal as a projective variety. (Normality is checked by the second condition in Theorem 4.3 and the Delzant condition.) Thus, it is equivariantly equivalent to a toric variety constructed from a fan. The fan corresponding to M_P is the 'normal fan' of the Delzant polytope

P ([9]). We do not need to use the fan in this paper, and hence we omit the description of M_P in terms of the fan. However, we mention that we can use the theory of toric variety constructed from the fan, as described in [7], [20]. For example, the space of global holomorphic sections $H^0(M_P, (L_P^c)^{\otimes N})$ of the N-th tensor power of the line bundle L_P^c is decomposed into weight spaces for the $T_{\mathbb{C}}^m$-action as

$$H^0(M_P, (L_P^c)^{\otimes N}) = \bigoplus_{\alpha \in (NP) \cap \mathbb{Z}^m} \mathbb{C}\chi_\alpha^N \quad (N \geq 1),$$

where χ_α^N is a weight vector with weight α. The sections χ_α^N are just monomials on the open orbit $\mathcal{O}_P \cong T_{\mathbb{C}}^m$. We call these sections *toric monomials*. Our purpose is to investigate various asymptotic formulas for sections in $H^0(M_P, (L_P^c)^{\otimes N})$ as $N \to \infty$. So, it is useful to describe concretely the sections χ_α^N ($\alpha \in (NP) \cap \mathbb{Z}^m$) for every sufficiently large N. Since our variety M_P is normal, there exists a positive integer N_o such that we have

$$H^0(M_P, (L_P^c)^{\otimes N}) = \iota_P^* H^0(\mathbb{C}P^{s-1}, \mathcal{O}(N)) \tag{4.7}$$

for every $N \geq N_o$. (One can also use the third condition in Theorem 4.3 to prove (4.7).) Recall that the holomorphic sections of $\mathcal{O}(N) \to \mathbb{C}P^{s-1}$ are regarded as homogeneous polynomials in $\mathbb{C}^s$ of degree N. In particular, for $N = 1$, define $\lambda_\alpha \in (\mathbb{C}^s)^*$ ($\alpha \in S = P \cap \mathbb{Z}^m$) as the coordinate functions on $\mathbb{C}^s$. Then, the set $\{\lambda_\alpha\}_{\alpha \in S}$ gives a basis of $H^0(\mathbb{C}P^{s-1}, \mathcal{O}(1))$. Hence, the sections

$$\chi_\alpha = c(\alpha)^{-1/2} \iota_P^* \lambda_\alpha \in H^0(M_P, L_P^c), \quad \alpha \in S,$$

form a basis of $H^0(M_P, L_P^c)$. For $N \geq N_o$, we set

$$\chi_\alpha^N = \chi_{\beta_1} \otimes \cdots \otimes \chi_{\beta_N}, \quad \alpha = \beta_1 + \cdots + \beta_N \in (NP) \cap \mathbb{Z}^m, \ \beta_j \in S.$$

This does not depend on the choice of $\beta_1, \ldots, \beta_N$ for fixed $\alpha \in (NP) \cap \mathbb{Z}^m$. Then the toric monomials χ_α^N, $\alpha \in (NP) \cap \mathbb{Z}^m$ form a basis of $H^0(M_P, (L_P^c)^{\otimes N})$. It is proved by a direct computation that the basis $\{\chi_\alpha^N\,;\, \alpha \in (NP) \cap \mathbb{Z}^m\}$ forms an orthogonal basis with respect to the inner product

$$\langle\, s, t\, \rangle = \int_{M_P} h_P^N(s(z), t(z))\, (\omega_P^c)^m / m!,$$

where h_P^N is the Hermitian metric on $(L_P^c)^{\otimes N}$ induced from the Fubini-Study Hermitian metric on $\mathcal{O}(1)$. Thus, we normalize each χ_α^N as

$$\varphi_\alpha^N = \frac{1}{\|\chi_\alpha^N\|} \chi_\alpha^N, \quad \alpha \in (NP) \cap \mathbb{Z}^m$$

to form an orthonormal basis $\{\varphi_\alpha^N\,;\, \alpha \in (NP) \cap \mathbb{Z}^m\}$ of $H^0(M_P, (L_P^c)^{\otimes N})$. Then, our problem is to investigate asymptotic behavior of $|\varphi_\alpha^N(z)|_P^2 = h_P^N(\varphi_\alpha^N(z), \varphi_\alpha^N(z))$.

Remark that the map $\mu_P^c : M_P \to P \subset \mathbb{R}^m$ defined by (4.4) is the moment map for symplectic action of the real torus T^m on the symplectic manifold (M_P, ω_P^c). The moment map μ_P^c is (by definition) invariant under T^m-action and, when it is restricted to the open orbit $\mathcal{O}_P := \mathcal{O}_S$ $(S = P \cap \mathbb{Z}^m)$, the map $\mu_P^c : \mathcal{O}_P \to \mathrm{Int}\,(P)$ defines, by using the coordinate $z = e^{\tau/2+i\varphi}$ $(\tau, \varphi \in \mathbb{R}^m)$ on $\mathcal{O}_P$, the map $\mathbb{R}^m \to \mathrm{Int}\,(P)$ denoted also by μ_P^c. The map μ_P^c is given explicitly by

$$\mu_P^c(\tau) = \sum_{\alpha \in S} \frac{c(\alpha) e^{\langle \alpha, \tau \rangle}}{\sum_{\beta \in S} c(\beta) e^{\langle \beta, \tau \rangle}} \alpha, \quad \tau \in \mathbb{R}^m. \tag{4.8}$$

The map (4.8) is the same as that defined in (2.12). Then, we have its potential function k_P^c on $\mathbb{R}^m$ defined in (2.6). The function k_P^c defines a function on the open orbit $\mathcal{O}_P$, which is also denoted by k_P^c, by $k_P^c(\Phi_S(z)) = k_S^c(\tau)$, $z = e^{\tau/2+i\varphi} \in T_{\mathbb{C}}^m$, $\tau, \varphi \in \mathbb{R}^m$. An important fact is that the function k_P^c so defined is a Kähler potential of ω_P^c on $\mathcal{O}_P$. Indeed one can check directly that

$$\omega_P^c = \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \log k_P^c.$$

From this, the volume form $d\mathrm{vol} = (\omega_P^c)^m / m!$ is given, on the open orbit $\mathcal{O}_P$, by

$$d\mathrm{vol} = \frac{1}{(2\pi)^m} \det A_P^c(\tau)\, d\tau d\varphi, \tag{4.9}$$

where $A_P^c(\tau) = \nabla^2 \log k_P^c(\tau)$ is a positive definite symmetric matrix.

4.4 Asymptotic behavior of toric monomials

There are various aspects of asymptotic behavior of $|\varphi_\alpha^N(z)|_P^2$. In this subsection, we give some of asymptotic results for this functions. The results in this subsection can be found in [22]. Among results in [22], the most typical result is the following. To state the theorem, let us prepare some notation. As mentioned in Section 2, the map $\mu_P^c : \mathbb{R}^m \to \mathrm{Int}\,(P)$ defined in (4.4), (4.8) is a diffeomorphism. We denote its inverse map by $\tau_P^c : \mathrm{Int}\,(P) \to \mathbb{R}^m$. Denote by δ_P^c the function on $\mathrm{Int}\,(P)$ defined by the formula (2.13). Then, we define the function b_P^c on $\mathrm{Int}\,(P) \times \mathrm{Int}\,(P)$ by

$$b_P^c(x, y) = \delta_P^c(y) - \delta_P^c(x) + \langle\, y - x, \tau_P^c(y)\, \rangle. \tag{4.10}$$

For simplicity of notation, we set

$$c(P, x) := \frac{1}{\sqrt{\det A_P^c(\tau_P^c(x))}}, \quad x \in \mathrm{Int}\,(P). \tag{4.11}$$

Theorem 4.6. *Let $D_\alpha(t)$ denote the distribution function of the the push-forward measure $(|\varphi_\alpha^N(z)|_P^2)_* d\mathrm{vol}$ on the real line, which is defined explicitly by*

$$D_\alpha(t) := \mathrm{vol}\left(z \in M_P\,;\, |\varphi_\alpha^N(z)|_P^2 > t\right). \tag{4.12}$$

Suppose that the sequence of lattice points $\alpha_N \in (NP) \cap \mathbb{Z}^m$ satisfies $\alpha_N = N x_o + O(1)$ with a point x_o in $\mathrm{Int}\,(P)$.

1. For $t > 0$, we have

$$D_{\alpha_N}(t) \sim \frac{(\pi m)^{m/2}}{c(P, x_o)\Gamma(m/2+1)} \left(\frac{\log N}{N}\right)^{m/2}. \tag{4.13}$$

2. For $0 < t < c(P, x_o)$, we have

$$\begin{aligned} &\lim_{N\to\infty} \left(\frac{N}{2\pi}\right)^{m/2} D_{\alpha_N}\left(\left(\frac{N}{2\pi}\right)^{m/2} t\right) \\ &\qquad = \frac{1}{c(P, x_o)\Gamma(m/2+1)} \left(\log\left(\frac{c(P, x_o)}{t}\right)\right)^{m/2}. \end{aligned} \tag{4.14}$$

3. For $t > 0$, we have

$$\lim_{N\to\infty} D_{\alpha_N}(e^{-Nt}) = \mathrm{vol}(x \in \mathrm{Int}\,(P)\,;\, b_P^c(x_o, x) < t), \tag{4.15}$$

where vol *denotes the Euclidean volume.*

Remark 4.7. The formula (4.14) is also proved in [5]. The Kähler structure used in [5] is the one naturally induced from the standard Kähler form on $\mathbb{C}^d$ (d is the number of facets of P) through the GIT description of the toric manifold M_P.

Before giving a sketch of proof, we give an explanation on Theorem 4.6. For simplicity, consider the case where $\alpha_N = N\alpha$ with $\alpha \in \mathrm{Int}\,(P) \cap \mathbb{Z}^m$. The sections $\varphi_{N\alpha}^N$ are expected to concentrate on the fiber $(\mu_P^c)^{-1}(\alpha)$ of the moment map $\mu_P^c : M_P \to P$. (Indeed, one can show that the measure $|\varphi_{N\alpha}^N|_P^2\, d\mathrm{vol}$ tends weakly to the uniform measure on the fiber $(\mu_P^c)^{-1}(\alpha) \cong T^m$.) Since the function $|\varphi_{N\alpha}^N|_P^2$ is invariant under the action of the real torus T^m, and since M_P/T^m is homeomorphic to P, this function induces a function on the polytope P. So, suppose that the function $|\varphi_{N\alpha}^N|_P^2$ is like a Gaussian bump around α. Consider its sub-level sets, $L_N(t) = \{|\varphi_{N\alpha}^N|_P^2 \geq t\}$ in $\mathrm{Int}\,(P)$. When t is a fixed positive constant, which is the case of the formula (4.13), since the Gaussian bump becomes quite sharp around α as N tends to infinity, the volume of the sub-level set $L_N(t)$ becomes small as $N \to \infty$, and how small it becomes does not depend on the constant t because the corresponding measure finally converges to the Dirac delta at α. This is the formula (4.13). Thus, to find a correct limit of $D_{N\alpha}(t)$, we need to rescale the constant t so that $t = t_N$ depends on N. The formula (4.14) gives the correct rescaling when $t = t_N$ becomes large as $N \to \infty$. In this case, since the level sets become upper and upper as N tends to infinity, the rescaling in (4.14) gives the information around the center of the concentration. The formula (4.14) shows that the way of concentration is rather universal, because it does not depend much on the geometry of M_P. In turn, in the formula (4.15), the rescaling $t = t_N$ is made by e^{-Nt} which decays as N tends to infinity. This means that the level sets become lower and lower as N tends to the infinity. In this rescale, the distribution function can grasp the information about the tale of the bump, and the formula (4.15) shows that the tale of the bump contains much geometric information, and such information is contained in the function $b_P^c(\alpha, \cdot)$.

Remark 4.8. We have explained Theorem 4.6 by supposing that the function $|\varphi_{N\alpha}^N|_P^2$ looks like Gaussian. Indeed this is seen by the pointwise asymptotics of this function (see Theorem 4.9 below). However, one could accept this exposition by the following fact. Suppose we are given a Gaussian function

$$g(u) = \frac{e^{-\langle Au, u\rangle/2}}{\sqrt{\det A}}, \quad u \in \mathbb{R}^m$$

on $\mathbb{R}^m$ with the measure

$$d\nu_A = \frac{\det A}{(2\pi)^{m/2}}\, du$$

so that $\int_{\mathbb{R}^m} g(u)\, d\nu_A(u) = 1$. Then, a direct computation tells us that the distribution function is given by

$$\nu_A(u \in \mathbb{R}^m\,;\, g(u) > t) = \frac{1}{c_A \Gamma(m/2+1)} \left(\log\left(\frac{c_A}{t}\right)\right)^{m/2}$$

for $0 < t \leq c_A$ with the constant $c_A = 1/\sqrt{\det A}$. Therefore, one can say that the rescaled distributions in (4.14) for the toric monomials have a universal Gaussian form around the center of the localization.

4.5 Inverting a moment problem

Among various asymptotic formulas in Theorem 4.6, we give a sketch of proof of the formula (4.14). The formula (4.14) is one of consequences of the following theorem about pointwise asymptotic behavior of $|\varphi_{\alpha_N}^N(z)|_P^2$.

Theorem 4.9. *Let $\alpha_N \in (NP) \cap \mathbb{Z}^m$ be a sequence of lattice points in NP such that $\alpha_N = N x_o + O(1)$ with a point $x_o \in \operatorname{Int}(P)$. Then, we have*

$$(4.16) \quad |\varphi_{\alpha_N}^N(z)|_P^2 = c(P, x_o)\left(\frac{N}{2\pi}\right)^{m/2} e^{-N[b_P^c(x_o, x) + \langle x_o - \alpha_N/N, \tau_P^c(x) - \tau_P^c(x_o)\rangle]}(1 + O(N^{-1})),$$

where we write $z = e^{\tau_P^c(x)/2 + i\varphi}$ with $x \in \operatorname{Int}(P)$. This holds uniformly in $z \in T_{\mathbb{C}}^m \cong \mathcal{O}_P$.

Remark 4.10. A special boundary case where $\alpha_N = N\alpha$ with $\alpha \in \partial P \cap \mathbb{Z}^m$ is also handled in [22] by using the local coordinates on $U_p \cap M_P$ ($p \in \mathcal{V}(P)$) described in the proof of Proposition 4.5. In [23], general boundary case is analyzed.

Indeed, taking k-th power of the formula (4.16) in Theorem 4.9, combined with a technical estimate, shows the following asymptotic formula for the L^{2k}-norms.

Theorem 4.11. *Suppose that the sequence $\alpha_N \in (NP) \cap \mathbb{Z}^m$ of lattice points and a point $x_o \in \mathrm{Int}\,(P)$ satisfy the condition in Theorem 4.6. Then, for the L^{2k}-norm $\|\varphi^N_{\alpha_N}\|_{2k}$ of the section $\varphi^N_{\alpha_N}$ has the following asymptotic behavior;*

$$\|\varphi^N_{\alpha_N}\|^{2k}_{2k} = \frac{c(P, x_o)^{k-1}}{k^{m/2}} \left(\frac{N}{2\pi}\right)^{(k-1)m/2} (1 + O_k(N^{-1})), \tag{4.17}$$

where O_k means that the estimate $O(N^{-1})$ depends on k.

Let us explain how one can deduce the formula (4.14) from Theorem 4.11. To take the rescaling in the formula (4.14) into account, let us introduce the measure $d\mathrm{v}_N$ and the function f_N on M_P defined by

$$d\mathrm{v}_N = \left(\frac{N}{2\pi}\right)^{m/2} d\mathrm{vol}_P, \quad f_N(z) = \left(\frac{N}{2\pi}\right)^{-m/4} |\varphi_{\alpha_N}(z)|_P,$$

so that $\|f_N\|_{L^2(d\mathrm{v}_N)} = 1$. According to the formula (4.17), we have

$$\|f_N\|^{2k}_{L^{2k}(d\mathrm{v}_N)} = \frac{c(P, x_o)^{k-1}}{k^{m/2}}(1 + O_k(N^{-1})). \tag{4.18}$$

Consider the push-forward measure $|f_N|^2_* d\mathrm{v}_N$. By using the pointwise asymptotic formula (4.16), one can show that

$$\lim_{N\to\infty} \|f_N\|^2_\infty = c(P, x_o), \tag{4.19}$$

and hence the support of the push-forward measures $|f_N|^2_* d\mathrm{v}_N$ are contained in a bounded set in $[0, +\infty)$ independent of N. The distribution function $F_N(t) := (|f_N|^2_* d\mathrm{v}_N)([t, +\infty))$ of the measure $|f_N|^2_* d\mathrm{v}_N$ is given by the rescaled distribution function, $F_N(t) = (\frac{N}{2\pi})^{m/2} D_{\alpha_N}((\frac{N}{2\pi})^{m/2} t)$, in the formula (4.14). The limit of the k-th moment as $N \to \infty$ of the measure $|f_N|^2_* d\mathrm{v}_N$ is given by

(4.20)
$$\int_{\mathbb{R}} x^k \, d(|f_N|^2_* d\mathrm{v}_N)(x) = \|f_N\|^{2k}_{L^{2k}(d\mathrm{v}_N)} = \frac{c(P, x)^{k-1}}{k^{m/2}}(1 + O_k(N^{-1})) \to \frac{c(P, x)^{k-1}}{k^{m/2}}.$$

Now, we note that the measure

$$d\rho_N(x) = x \, d(|f_N|^2_* d\mathrm{v}_N)(x)$$

on the real line is a probability measure supported in a bounded interval in $[0, +\infty)$ independent of N. Then, if the sequence of probability measures $d\rho_N$ tend weakly to a probability measure, say $d\rho$, the limit measure $d\rho$ would be supported on

$[0, c(P,x)]$ by (4.19), and the distribution function $F_N(t)$ would have a limit because

$$F_N(t) = \int \chi_{(t,+\infty)}(x)\, d(|f_N|_*^2 dv_N)(x) = \int \frac{1}{x}\chi_{(t,+\infty)}(x)\, d\rho_N(x)$$
$$\to \int \frac{1}{x}\chi_{(t,+\infty)}(x)\, d\rho(x),$$

where $\chi_{(t,+\infty)}$ is the characteristic function of the interval $(t, +\infty)$. Furthermore, by (4.20), we must have

$$\int x^k \, d\rho(x) = \lim_{N\to\infty} \int x^k d\rho_N(x) = \frac{c(P,x)^k}{(k+1)^{m/2}}.$$

So, we arrive at a *moment problem*, that is, to find a probability measure $d\rho$ whose k-th moment is given by $\frac{c(P,x)^k}{(k+1)^{m/2}}$. For this, we have the following lemma.

Lemma 4.12. *Let ρ be a compactly supported probability measure on $\mathbb{R}$. Suppose that there exists a positive integer h and a positive number c such that, for any non-negative integer k,*

$$\int x^k \, d\rho(x) = \frac{c^k}{(k+1)^{h/2}}.$$

Then, we have

$$d\rho(x) = \frac{1}{c\Gamma(h/2)}\chi_{(0,c)}(x)\left(\log\left(\frac{c}{x}\right)\right)^{h/2-1}.$$

See [22, Lemma 4.1] for the proof of Lemma 4.12. From (4.20) and Lemma 4.12, it is not hard to show the formula (4.14). Therefore, what we need is to prove Theorem 4.9.

4.6 Pointwise asymptotic formula

In this subsection, we give a sketch of proof of Theorem 4.9. Since the formula 4.16 is a local estimate on $\mathcal{O}_P \cong T^m_{\mathbb{C}}$, we use the local coordinates $z = e^{\tau/2+i\varphi}$, $\tau, \varphi \in \mathbb{R}^m$, on $T^m_{\mathbb{C}}$. By definition of the Fubini-Study Hermitian metric on $(L_P^c)^{\otimes N} = \iota_P^*\mathcal{O}(N)$, the modulus square of the monomial $\chi^N_{\alpha_N}$ can be written as

$$|\chi^N_{\alpha_N}(z)|_P^2 = e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N\rangle]}, \quad z = e^{\tau/2+i\varphi}.$$

Hence to consider the pointwise behavior of $\varphi^N_{\alpha_N} = \frac{1}{\|\chi^N_{\alpha_N}\|}\chi^N_{\alpha_N}$, it is enough to consider behavior of the L^2-norm $\|\chi^N_{\alpha_N}\|$. By (4.9), we have

$$\|\chi^N_{\alpha_N}\|^2 = \int_{\mathbb{R}^m} e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N\rangle]} \det A_P(\tau)\, d\tau. \tag{4.21}$$

The critical point of the function $\log k_P^c(\tau) - \langle \tau, \alpha_N/N \rangle$ is given by $\mu_P^c(\tau) = \alpha_N/N$, that is $\tau = \tau_P^c(\alpha_N/N)$, which depends on the parameter N. So, we discuss as follows. We note that, in (4.21), the function $\det A_P(\tau)$ is a positive integrable function on $\mathbb{R}^m$ by (4.9) and the fact that the open orbit $\mathcal{O}_P$ is dense in M_P. Since $\alpha_N/N = x_o + O(N^{-1})$, we can choose an open ball U in $\mathrm{Int}\,(P)$ around x_o such that $\overline{U} \subset \mathrm{Int}\,(P)$ and $\alpha_N/N \in U$ for every sufficiently large N. As in [22, Lemma 3.3], there exist positive constants R, c such that $\log k_P^c(\tau) - \langle \tau, x \rangle \geq c|\tau|$ for any $(x, \tau) \in \overline{U} \times \mathbb{R}^m$, $|\tau| \geq R$. We may choose $R > 0$ so that $|\tau_P^c(\alpha_N/N)| < R$ for every sufficiently large N. Thus, the integral in (4.21) equals

$$\int_{\mathbb{R}^m} e^{-N[\log k_P^c(\tau) - \langle \tau, \alpha_N/N \rangle]} g(\tau) \det A_P(\tau)\, d\tau \tag{4.22}$$

modulo a term of order $O(e^{-cRN})$, where we inserted a cut-off function $0 \leq g(\tau) \leq 1$ satisfying $g(\tau) = 1$ for $|\tau| \leq 2R$. Changing the integral variable $\tau = \tau_P^c(x)$, the integral in (4.22) is written in the form

$$e^{-N\delta_P^c(x_o)} \int_{\mathrm{Int}\,(P)} e^{-Nb_P^c(x_o,x)} R_N(x_o, x) g(\tau_P^c(x))\, dx, \tag{4.23}$$

where the function $b_P^c(x_o, x)$ is defined in (4.10) and the function $R_N(x_o, x)$ is given by $R_N(x_o, x) = e^{\langle \alpha_N - Nx_o, \tau_P^c(x) \rangle}$. Since $\alpha_N = Nx_o + O(1)$, derivatives of $R_N(x_o, x)$ of any order are bounded by constants on the support of $g(\tau_P^c(x))$. For fixed $x_o \in \mathrm{Int}\,(P)$, it is easy to show that the function $b_P^c(x_o, x)$ has unique critical point at $x = x_o$ with Hessian $A_P^c(\tau_P^c(x_o))$. Since $b_P^c(x_o, x_o) = 0$, a standard argument involving the Morse lemma and the Fourier transform of Gaussian functions as in Section 2 shows

$$\|\chi_{\alpha_N}^N\|^2 = \left(\frac{N}{2\pi}\right)^{-m/2} \sqrt{\det A_P^c(\tau_P^c(x_o))}\, e^{-N[\delta_P^c(x_o) + \langle x_o - \alpha_N/N, \tau_P^c(x_o) \rangle]} (1 + O(N^{-1})). \tag{4.24}$$

From this and a direct computation, one conclude (4.16).

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