

Berezin-Toeplitz quantization for compact Kähler manifolds. An introduction

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Abstract

The Berezin-Toeplitz operator and Berezin-Toeplitz deformation quantization schemes give quantization methods adapted to a Kähler structure on a manifold to be quantized. Here we present an introduction both to the definitions of its basic objects and to the results.

1 Introduction

For quantizable Kähler manifolds the Berezin - Toeplitz (BT) quantization scheme, both the operator quantization and the deformation quantization, supplies canonically defined quantizations. What makes the Berezin-Toeplitz quantization scheme so attractive is that it does not depend on further choices and that it does not only produce a formal deformation quantization, but one which is deeply related to some operator calculus.

Some time ago, in joint work with Martin Bordemann and Eckhard Meinrenken, the author showed that for compact Kähler manifolds it is a well-defined quantization scheme with correct semi-classical limit [14]. From the point of view of classical mechanics compact Kähler manifolds appear as phase space manifolds of restricted systems or of reduced systems. A typical example of its appearance is given by the spherical pendulum which after reduction has as phase-space the complex projective space.

Very recently, inspired by fruitful applications of the basic techniques of the Berezin-Toeplitz scheme beyond the quantization of classical systems, the interest in it revived considerably. For example these techniques show up in non-commutative geometry. More precisely, they appear in the approach to non-commutative geometry using fuzzy manifolds. The quantum spaces of the Berezin-Toeplitz quantization of level m , defined in Section 3 further down, are finite-dimensional in the compact case and the quantum operator of level m constitute finite-dimensional non-commutative matrix algebras. This is the arena of non-commutative fuzzy manifolds and gauge theories over them. The classical limit, the commutative manifold, is obtained as limit $m \rightarrow \infty$.

Another appearance of Berezin-Toeplitz quantization techniques as basic ingredients is in the pioneering work of Jørgen Andersen on the mapping class group (MCG) of surfaces in the context of Topological Quantum Field Theory (TQFT). Andersen gave also a lecture course at the school on his achievements. Beside other results, he was able to prove the asymptotic faithfulness of the mapping class group action on the space of covariantly constant sections of the Verlinde bundle with respect to the Axelrod-Witten-de la Pietra and Witten connection [3, 4], see also [51]. Furthermore, he showed that the MCG does not have Kazhdan's property T . Roughly speaking, a group has *property T* says that the identity representation is isolated in the space of all unitary representations of the group [5]. In these applications the manifolds to be quantized are the moduli spaces of certain flat connections on Riemann surfaces or, equivalently, the moduli space of stable algebraic vector bundles over smooth projective curves. Here further exciting research is going on, in particular, in the realm of TQFT and the construction of modular functors [6], [7, 8].

In general quite often moduli spaces come with a natural quantizable Kähler structure. Hence, it is not surprising that the Berezin-Toeplitz quantization scheme is of importance in moduli space problems. Non-commutative deformations, and a quantization is a non-commutative deformation, yield also informations about the commutative situation. These aspects clearly need further investigations.

It was the goal of the lecture course and it is the goal of this write-up to present a short introduction to the basic definitions and results on Berezin-Toeplitz quantization (both operator and deformation quantization) without proofs and too many details. The language presented was used in other lectures at the school and talks at the conference. The author hopes that it will be equally useful to the reader who aims to get a quick introduction to this exciting field. For a more detailed review, see [53]. There an extended list of references to the original literature and to reviews of other people concentrating on different aspects of the theory can be found, e.g. see [2], [54].

2 The geometric set-up

2.1 Quantizable Kähler manifolds

We will only consider phase-space manifolds which carry the structure of a Kähler manifold (M, ω) . Recall that in this case M is a complex manifold (let us say of complex dimension n) and ω , the Kähler form, is a non-degenerate closed positive $(1, 1)$ -form. This means that the Kähler form ω can be written with respect to

local holomorphic coordinates $\{z_i\}_{i=1,\dots,n}$ as

$$(2.1) \quad \omega = i \sum_{i,j=1}^n g_{ij}(z) dz_i \wedge d\bar{z}_j,$$

with local functions $g_{ij}(z)$ such that the matrix $(g_{ij}(z))_{i,j=1,\dots,n}$ is hermitian and positive definite.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with point-wise multiplication as associative product. A symplectic form on a differentiable manifold is a closed non-degenerate 2-form. In particular, we can consider our Kähler form ω as a symplectic form.

For a symplectic manifold M we can introduce on $C^\infty(M)$ a Lie algebra structure, the *Poisson bracket* $\{.,.\}$, in the following way. First we assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f , and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$(2.2) \quad \omega(X_f, \cdot) = df(\cdot), \quad \{f, g\} := \omega(X_f, X_g) .$$

This defines a Lie algebra structure in $C^\infty(M)$. Moreover, we obtain the Leibniz rule

$$\{fg, h\} = f\{g, h\} + \{f, h\}g, \quad \forall f, g, h \in C^\infty(M).$$

Such a compatible structure is called a *Poisson algebra*.

The next step in the geometric set-up is the choice of a quantum line bundle. A *quantum line bundle* for a given symplectic manifold (M, ω) is a triple (L, h, ∇) , where L is a complex line bundle, h a Hermitian metric on L , and ∇ a connection compatible with the metric h such that the (pre)quantum condition

$$(2.3) \quad \begin{aligned} \text{curv}_{L, \nabla}(X, Y) &:= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]} = -i\omega(X, Y), \\ \text{resp. } \text{curv}_{L, \nabla} &= -i\omega \end{aligned}$$

is fulfilled. A symplectic manifold is called *quantizable* if there exists a quantum line bundle.

In the situation of Kähler manifolds we require for a quantum line bundle that it is holomorphic and that the connection is compatible both with the metric h and the complex structure of the bundle. In fact, by this requirement ∇ will be uniquely fixed. If we choose local holomorphic coordinates on the manifold and a local holomorphic frame of the bundle the metric h will be represented by a function \hat{h} . In this case the curvature of the bundle can be given by $\bar{\partial}\partial \log \hat{h}$ and the quantum condition reads as

$$(2.4) \quad i \bar{\partial}\partial \log \hat{h} = \omega .$$

2.2 Examples

(a) Of course, \mathbb{C}^n is a Kähler manifold with Kähler form

$$(2.5) \quad \omega = i \sum_{k=1}^n dz_k \wedge d\bar{z}_k .$$

The Poisson bracket writes as

$$(2.6) \quad \{f, g\} = i \sum_{k=1}^n \left(\frac{\partial f}{\partial \bar{z}_k} \cdot \frac{\partial g}{\partial z_k} - \frac{\partial f}{\partial z_k} \frac{\partial g}{\partial \bar{z}_k} \right) .$$

The quantum line bundle is the trivial line bundle with hermitian metric fixed by the function $\hat{h}(z) = \exp(-\sum_{-k} \bar{z}_k z_k)$.

(b) The **Riemann sphere** is the complex projective line $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\} \cong S^2$. With respect to the quasi-global coordinate z the form can be given as

$$(2.7) \quad \omega = \frac{i}{(1+z\bar{z})^2} dz \wedge d\bar{z} .$$

For the Poisson bracket one obtains

$$(2.8) \quad \{f, g\} = i(1+z\bar{z})^2 \left(\frac{\partial f}{\partial \bar{z}} \cdot \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} \right) .$$

Recall that the points in $\mathbb{P}^1(\mathbb{C})$ correspond to lines in \mathbb{C}^2 passing through the origin. If we assign to every point in $\mathbb{P}^1(\mathbb{C})$ the line it represents we obtain a holomorphic line bundle, called the tautological line bundle. The hyper plane section bundle is dual to the tautological bundle. It turns out that it is a quantum line bundle. Hence $\mathbb{P}^1(\mathbb{C})$ is quantizable.

(c) The above example generalizes to the n -dimensional **complex projective space** $\mathbb{P}^n(\mathbb{C})$. The Kähler form is given by the Fubini-Study form

$$(2.9) \quad \omega_{FS} := i \frac{(1+|w|^2) \sum_{i=1}^n dw_i \wedge d\bar{w}_i - \sum_{i,j=1}^n \bar{w}_i w_j dw_i \wedge d\bar{w}_j}{(1+|w|^2)^2} .$$

The coordinates w_j , $j = 1, \dots, n$ are affine coordinates $w_j = z_j/z_0$ on the affine chart $U_0 := \{(z_0 : z_1 : \dots : z_n) \mid z_0 \neq 0\}$. Again, $\mathbb{P}^n(\mathbb{C})$ is quantizable with the hyper plane section bundle as quantum line bundle.

(d) The (complex-) **one-dimensional torus** can be given as $M = \mathbb{C}/\Gamma_\tau$ where $\Gamma_\tau := \{n + m\tau \mid n, m \in \mathbb{Z}\}$ is a lattice with $\tau \in \mathbb{C}$, $\text{Im } \tau > 0$. As Kähler form we take

$$(2.10) \quad \omega = \frac{i\pi}{\text{Im } \tau} dz \wedge d\bar{z} ,$$

with respect to the coordinate z on the covering space \mathbb{C} . Clearly, this form is invariant under the lattice Γ_τ and hence well-defined on M . For the Poisson bracket one obtains

$$(2.11) \quad \{f, g\} = i \frac{\operatorname{Im} \tau}{\pi} \left(\frac{\partial f}{\partial \bar{z}} \cdot \frac{\partial g}{\partial z} - \frac{\partial f}{\partial z} \frac{\partial g}{\partial \bar{z}} \right) .$$

The corresponding quantum line bundle is the theta line bundle of degree 1, i.e. the bundle whose global sections are scalar multiples of the Riemann theta function.

(e) The **unit disc**

$$(2.12) \quad \mathcal{D} := \{z \in \mathbb{C} \mid |z| < 1\}$$

is a (non-compact) Kähler manifold. The Kähler form is given by

$$(2.13) \quad \omega = \frac{2i}{(1 - z\bar{z})^2} dz \wedge d\bar{z} .$$

For every **compact Riemann surface** M of genus $g \geq 2$ the unit disc \mathcal{D} is the universal covering space and M can be given as a quotient of \mathcal{D} by a Fuchsian subgroup of $SU(1, 1)$, whose elements act by fractional linear transformations.

Recall for $R = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$ with $|a|^2 - |b|^2 = 1$ (as an element of $SU(1, 1)$) the fractional linear transformation is given as

$$(2.14) \quad z \mapsto R(z) := \frac{az + b}{\bar{b}z + \bar{a}} .$$

The Kähler form (2.13) is invariant under fractional linear transformations. Hence, it defines a Kähler form on M . The quantum bundle is the canonical bundle, i.e. the bundle whose local sections are the holomorphic differentials. Its global sections can be identified with automorphic forms of weight 2 with respect to the Fuchsian group.

2.3 Conditions for being quantizable

The above examples might create the wrong impression that every Kähler manifold is quantizable. This is not the case. Above we introduced one-dimension tori. Higher dimensional tori can be given as Kähler manifold in a completely analogous manner as quotients \mathbb{C}^n/L where L is a $2n$ -dimensional lattice. But only those higher dimensional complex tori are quantizable which are abelian varieties, i.e. which admit enough theta functions. It is well-known that for $n \geq 2$ a generic torus will not be an abelian variety.

In the language of differential geometry a line bundle is called a positive line bundle if its curvature form (up to a factor of $1/i$) is a positive form. As the Kähler form is positive the quantum condition (2.3) yields that a quantum line bundle L is a positive line bundle.

Now let M is a quantizable **compact** Kähler manifold with quantum line bundle L . Kodaira's embedding theorem says that L is ample, i.e. that there exists a certain tensor power L^{m_0} of L such that the global holomorphic sections of L^{m_0} can be used to embed the phase space manifold M into a projective space of suitable dimension.

The embedding is defined as follows. Let $\Gamma_{hol}(M, L^{m_0})$ be the vector space of global holomorphic sections of the bundle L^{m_0} . Fix a basis s_0, s_1, \dots, s_N . We choose local holomorphic coordinates z for M and a local holomorphic frame $e(z)$ for the bundle L . After these choices the basis elements can be uniquely described by local holomorphic functions $\hat{s}_0, \hat{s}_1, \dots, \hat{s}_N$ defined via $s_j(z) = \hat{s}_j(z)e(z)$. The embedding is given by the map

$$(2.15) \quad M \hookrightarrow \mathbb{P}^N(\mathbb{C}), \quad z \mapsto \phi(z) = (\hat{s}_0(z) : \hat{s}_1(z) : \dots : \hat{s}_N(z)) .$$

Note that the point $\phi(z)$ in projective space neither depends on the choice of local coordinates nor on the choice of the local frame for the bundle L . Furthermore, a different choice of basis correspond to a $\text{PGL}(N, \mathbb{C})$ action on the embedding space and hence the embeddings are projectively equivalent. The “map” (2.15) could be given for every line bundle having nontrivial global sections. But it might happen that all sections have common zeros. For those points the map will not be defined. Furthermore, to be an embedding it should separate points and tangent directions. A line bundles whose global holomorphic sections will define an embedding into projective space, is called a *very ample line bundle*.

By this embedding quantizable compact Kähler manifolds are complex submanifolds of projective spaces. By Chow's theorem [52] they can be given as zero sets of homogenous polynomials, i.e. they are smooth projective varieties. The converse is also true. Given a smooth subvariety M of $\mathbb{P}^n(\mathbb{C})$ it will become a Kähler manifold by restricting the Fubini-Study form. The restriction of the hyper plane section bundle will be an associated quantum line bundle.

At this place a warning is necessary. the embedding is only an embedding as complex manifolds, not an isometric embedding as Kähler manifolds. This means that in general $\phi^{-1}(\omega_{FS}) \neq \omega$.

3 Berezin-Toeplitz operators

In this section we will consider an operator quantization. This says that we will assign to each differentiable (differentiable to every order) function f on our Kähler manifold M (i.e. on our “phase space”) the Berezin-Toeplitz (BT) quantum operator T_f . More precisely, we will consider a whole family of operators $T_f^{(m)}$. These

operators are defined in a canonical way. As we know from the Groenewold-van Howe theorem we cannot expect that the Poisson bracket on M can be represented by the Lie algebra of operators if we require certain desirable conditions, see [1] for further details. The best we can expect is that we obtain it at least “asymptotically”. In fact, this is true.

3.1 Definition of the operators

Let (M, ω) be a quantizable Kähler manifold and (L, h, ∇) a quantum line bundle. We assume that L is already very ample. We consider all its tensor powers

$$(3.1) \quad (L^m, h^{(m)}, \nabla^{(m)}).$$

Here $L^m := L^{\otimes m}$. If \hat{h} corresponds to the metric h with respect to a local holomorphic frame e of the bundle L then \hat{h}^m corresponds to the metric $h^{(m)}$ with respect to the frame $e^{\otimes m}$ for the bundle L^m . The connection $\nabla^{(m)}$ will be the induced connection.

We introduce a scalar product on the space of sections. We adopt the convention that a hermitian metric (and a scalar product) is anti-linear in the first argument and linear in the second argument. First we take the Liouville form $\Omega = \frac{1}{n!} \omega^{\wedge n}$ as volume form on M and then set for the scalar product and the norm

$$(3.2) \quad \langle \varphi, \psi \rangle := \int_M h^{(m)}(\varphi, \psi) \Omega, \quad \|\varphi\| := \sqrt{\langle \varphi, \varphi \rangle},$$

for the space $\Gamma_\infty(M, L^m)$ of global C^∞ -sections. Let $L^2(M, L^m)$ be the L^2 -completed space of bounded sections with respect to this norm. Furthermore, let $\Gamma_{hol}^{(b)}(M, L^m)$ be the closed subspace consisting of those global holomorphic sections which are bounded. These spaces are the quantum spaces of the theory. Note that in case that M is compact $\Gamma_{hol}(M, L^m) = \Gamma_{hol}^{(b)}(M, L^m)$ and the spaces are finite-dimensional. Let

$$(3.3) \quad \Pi^{(m)} : L^2(M, L^m) \rightarrow \Gamma_{hol}^{(b)}(M, L^m)$$

be the projection.

Definition 3.1. For $f \in C^\infty(M)$ the Toeplitz operator $T_f^{(m)}$ (of level m) is defined by

$$(3.4) \quad T_f^{(m)} := \Pi^{(m)}(f \cdot) : \Gamma_{hol}^{(b)}(M, L^m) \rightarrow \Gamma_{hol}^{(b)}(M, L^m).$$

In words: One takes a holomorphic section s and multiplies it with the differentiable function f . The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

The linear map

$$(3.5) \quad T^{(m)} : C^\infty(M) \rightarrow \text{End}(\Gamma_{hol}^{(b)}(M, L^m)), \quad f \rightarrow T_f^{(m)} = \Pi^{(m)}(f \cdot), \quad m \in \mathbb{N}_0.$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. It will neither be a Lie algebra homomorphism nor an associative algebra homomorphism as in general

$$T_f^{(m)} T_g^{(m)} = \Pi^{(m)}(f \cdot) \Pi^{(m)}(g \cdot) \Pi^{(m)} \neq \Pi^{(m)}(fg \cdot) \Pi = T_{fg}^{(m)}.$$

Definition 3.2. The Berezin-Toeplitz (BT) quantization is the map

$$(3.6) \quad C^\infty(M) \rightarrow \prod_{m \in \mathbb{N}_0} \text{End}(\Gamma_{hol}^{(b)}(M, L^m)), \quad f \rightarrow (T_f^{(m)})_{m \in \mathbb{N}_0}.$$

In case that M is a compact Kähler manifold on a fixed level m the BT quantization is a map from the infinite-dimensional commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. The finite-dimensionality is due to the compactness of M . A lot of classical information will get lost. To recover this information one has to consider not just a single level m but all levels together as done in the above definition. In this way a family of finite-dimensional(matrix) algebras and a family of maps is obtained, which in the classical limit “converges” to the algebra $C^\infty(M)$.

3.2 Approximation results for the compact Kähler case

In the following we will only deal with compact quantizable Kähler manifolds. We assume that the quantum line bundle L is already very ample (i.e. its sections give an embedding into projective space). This is not much of a restriction. If L is not very ample we choose $m_0 \in \mathbb{N}$ such that the bundle L^{m_0} is very ample and take this bundle as quantum line bundle with respect to the rescaled Kähler form $m_0 \omega$ on M . The underlying complex manifold structure will not change.

Recall that in the compact case we have $\Gamma_{hol}(M, L^m) = \Gamma_{hol}^{(b)}(M, L^m)$. Denote for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and by

$$(3.7) \quad \|T_f^{(m)}\| := \sup_{\substack{s \in \Gamma_{hol}(M, L^m) \\ s \neq 0}} \frac{\|T_f^{(m)} s\|}{\|s\|}$$

the operator norm with respect to the norm (3.2) on $\Gamma_{hol}(M, L^m)$. The following theorem was shown in 1994.

Theorem 3.3. [Bordemann, Meinrenken, Schlichenmaier] [14]

(a) For every $f \in C^\infty(M)$ there exists a $C > 0$ such that

$$(3.8) \quad |f|_\infty - \frac{C}{m} \leq \|T_f^{(m)}\| \leq |f|_\infty.$$

In particular, $\lim_{m \rightarrow \infty} \|T_f^{(m)}\| = \|f\|_\infty$.

(b) For every $f, g \in C^\infty(M)$

$$(3.9) \quad \|\text{mi}[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}\| = O\left(\frac{1}{m}\right).$$

(c) For every $f, g \in C^\infty(M)$

$$(3.10) \quad \|T_f^{(m)} T_g^{(m)} - T_{f \cdot g}^{(m)}\| = O\left(\frac{1}{m}\right).$$

These results are contained in Theorem 4.1, 4.2, and in Section 5 in [14]. The proofs make reference to the symbol calculus of generalised Toeplitz operators as developed by Boutet-de-Monvel and Guillemin [17]. See [53] for a sketch. Only on the basis of this theorem we are allowed to call our scheme a quantizing scheme. The properties in the theorem might be rephrased as *the BT operator quantization has the correct semiclassical limit*. In other words it is a strict quantization in the sense of Rieffel [44] as formulated in the book by Landsman [36]. This notion is closely related to the notion of continuous field of C^* -algebras.

Let us summarize further properties in the following

Proposition 3.4. *Let $f, g \in C^\infty(M)$, $n = \dim_{\mathbb{C}} M$ then*

(a)

$$(3.11) \quad \lim_{m \rightarrow \infty} \|[T_f^{(m)}, T_g^{(m)}]\| = 0.$$

(b) *The Toeplitz map*

$$C^\infty(M) \rightarrow \text{End}(\Gamma_{hol}(M, L^m)), \quad f \rightarrow T_f^{(m)},$$

is surjective.

(c)

$$(3.12) \quad T_f^{(m)*} = T_{\bar{f}}^{(m)}.$$

In particular, for real valued functions f the associated Toeplitz operator T_f is selfadjoint.

(d) *Let $A \in \text{End}(\Gamma_{hol}(M, L^m))$ be a selfadjoint operator then there exists a real valued function f , such that $A = T_f^{(m)}$.*

(e) *Denote the trace on $\text{End}(\Gamma_{hol}(M, L^m))$ by $\text{Tr}^{(m)}$ then*

$$(3.13) \quad \text{Tr}^{(m)}(T_f^{(m)}) = m^n \left(\frac{1}{\text{vol}(\mathbb{P}^n(\mathbb{C}))} \int_M f \Omega + O(m^{-1}) \right).$$

For the proofs, resp. references to the proofs, I refer to [53]. I like to stress the fact that the Toeplitz map is never injective on a fixed level m . But from $\|T_{f-g}^{(m)}\| \rightarrow 0$ for $m \rightarrow 0$ we can conclude that $f = g$.

There exists another quantum operator in the geometric setting, the operator of geometric quantization introduced by Kostant and Souriau. In a first step the prequantum operator associated to the bundle L^m for the function $f \in C^\infty(M)$ is defined as

$$(3.14) \quad P_f^{(m)} := \nabla_{X_f^{(m)}}^{(m)} + i f \cdot id.$$

Here $\nabla^{(m)}$ is the connection in L^m , and $X_f^{(m)}$ the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$, i.e. $m\omega(X_f^{(m)}, \cdot) = df(\cdot)$. Next one has to choose a polarization. In general it will not be unique. But in our complex situation there is canonical one by taking the projection to the space of holomorphic sections. This polarization is called *Kähler polarization*. The operator of geometric quantization is then defined by

$$(3.15) \quad Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}.$$

The Toeplitz operator and the operator of geometric quantization (with respect to the Kähler polarization) are related by

Proposition 3.5. (*Tuynman Lemma*) *Let M be a compact quantizable Kähler manifold then*

$$(3.16) \quad Q_f^{(m)} = i \cdot T_{f - \frac{1}{2m} \Delta f}^{(m)},$$

where Δ is the Laplacian with respect to the Kähler metric given by ω .

For the proof see [56], and [13] for a coordinate independent proof.

In particular the operators $Q_f^{(m)}$ and the $T_f^{(m)}$ have the same asymptotic behaviour. It should be noted that for (3.16) the compactness of M is essential.

Remark 3.6. Above we introduced Berezin-Toeplitz operators also for non-compact Kähler manifolds. Unfortunately, in this context the proofs of Theorem 3.3 do not work. One has to study examples or classes of examples case by case and to check whether the corresponding properties are correct. See [53] for list of references in this context.

4 Berezin-Toeplitz deformation quantization

4.1 What is a star product?

There is another approach to quantization. Instead of assigning noncommutative operators to commuting functions one might think about “deforming” the pointwise commutative multiplication of functions into a non-commutative product. It

is required to remain associative, the commutator of two elements should relate to the Poisson bracket of the elements, and it should reduce in the “classical limit” to the commutative situation.

It turns out that such a deformation which is valid for all differentiable functions cannot exist. A way out is to enlarge the algebra of functions by considering formal power series over them and to deform the product inside this bigger algebra. A first systematic treatment and applications in physics of this idea were given 1978 by Bayen, Flato, Fronsdal, Lichnerowicz, and Sternheimer [9]. There the notion of *deformation quantization* and *star products* were introduced. Earlier versions of these concepts were around due to Berezin [10], Moyal [39], and Weyl [58]. For a presentation of the history see [54]. We will show that for compact Kähler manifolds M , there is a natural star product.

We start even more general, with a Poisson manifold $(M, \{.,.\})$, i.e. a differentiable manifold with a Poisson bracket for the function such that $(C^\infty(M), \cdot, \{.,.\})$ is a Poisson algebra. Let $\mathcal{A} = C^\infty(M)[[\nu]]$ be the algebra of formal power series in the variable ν over the algebra $C^\infty(M)$.

Definition 4.1. A product \star on \mathcal{A} is called a (formal) star product for M (or for $C^\infty(M)$) if it is an associative $\mathbb{C}[[\nu]]$ -linear product which is ν -adically continuous such that

1. $\mathcal{A}/\nu\mathcal{A} \cong C^\infty(M)$, i.e. $f \star g \bmod \nu = f \cdot g$,
2. $\frac{1}{\nu}(f \star g - g \star f) \bmod \nu = -i\{f, g\}$,

where $f, g \in C^\infty(M)$.

Alternatively we can write

$$(4.1) \quad f \star g = \sum_{j=0}^{\infty} C_j(f, g)\nu^j ,$$

with $C_j(f, g) \in C^\infty(M)$ such that the C_j are bilinear in the entries f and g . The conditions (1) and (2) can be reformulated as

$$(4.2) \quad C_0(f, g) = f \cdot g, \quad \text{and} \quad C_1(f, g) - C_1(g, f) = -i\{f, g\} .$$

By the ν -adic continuity (4.1) fixes \star on \mathcal{A} . A (formal) deformation quantization is given by a (formal) star product. I will use both terms interchangeable.

There are certain additional conditions for a star product which are sometimes useful.

1. We call it “null on constants”, if $1 \star f = f \star 1 = f$, which is equivalent to the fact that the constant function 1 will remain the unit in \mathcal{A} . In terms of the coefficients it can be formulated as $C_k(f, 1) = C_k(1, f) = 0$ for $k \geq 1$. Here we always assume this to be the case for star products.

2. We call it selfadjoint if $\overline{f \star g} = \bar{g} \star \bar{f}$, where we assume $\bar{\nu} = \nu$.
3. We call it local if

$$\text{supp } C_j(f, g) \subseteq \text{supp } f \cap \text{supp } g, \quad \forall f, g \in C^\infty(M).$$

From the locality property it follows that the C_j are bidifferential operators and that the global star product defines for every open subset U of M a star product for the Poisson algebra $C^\infty(U)$. Such local star products are also called *differential star products*.

In the usual setting of deformation theory there always exists a trivial deformation. This is not the case here, as the trivial deformation of $C^\infty(M)$ to \mathcal{A} , which is nothing else as extending the point-wise product to the power series, is not allowed as it does not fulfill Condition (2) in Definition 4.1 (at least not if the Poisson bracket is non-trivial). In fact the existence problem is highly non-trivial. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [22], Omori-Maeda-Yoshioka [41], and Fedosov [29]. The general Poisson case was settled by Kontsevich [35].

The next question is the classification of star products.

Definition 4.2. Given a Poisson manifold $(M, \{.,.\})$. Two star products \star and \star' associated to the Poisson structure $\{.,.\}$ are called equivalent if and only if there exists a formal series of linear operators

$$(4.3) \quad B = \sum_{i=0}^{\infty} B_i \nu^i, \quad B_i : C^\infty(M) \rightarrow C^\infty(M),$$

with $B_0 = id$ such that

$$(4.4) \quad B(f) \star' B(g) = B(f \star g).$$

For local star products in the general Poisson setting there are complete classification results. Here I will only consider the symplectic case. To each local star product \star its *Fedosov-Deligne class*

$$(4.5) \quad cl(\star) \in \frac{1}{i\nu}[\omega] + H_{dR}^2(M)[[\nu]]$$

can be assigned. Here $H_{dR}^2(M)$ denotes the 2nd deRham cohomology class of closed 2-forms modulo exact forms and $H_{dR}^2(M)[[\nu]]$ the formal power series with such classes as coefficients. Such formal power series are called *formal deRham classes*. In general we will use $[\alpha]$ for the cohomology class of a form α . This assignment gives a 1:1 correspondence between the formal deRham classes and the equivalence classes of star products.

For contractible manifolds we have $H_{dR}^2(M) = 0$ and hence there is up to equivalence exactly one local star product. This yields that locally all local star products of a manifold are equivalent to a certain fixed one, which is called the Moyal product. For these and related classification results see [23], [31], [12], [40].

For our compact Kähler manifolds we will have many different and even non-equivalent star products. The question is: is there a star product which is given in a natural way? The answer will be yes: the Berezin-Toeplitz star product to be introduced below. First we consider star products respecting the complex structure in a certain sense.

Definition 4.3. (Karabegov [32]) A star product is called *star product with separation of variables* if and only if

$$(4.6) \quad f \star h = f \cdot h, \quad \text{and} \quad h \star g = h \cdot g,$$

for every locally defined holomorphic function g , antiholomorphic function f , and arbitrary function h .

Recall that a local star product \star for M defines a star product for every open subset U of M . We have just to take the bidifferential operators defining \star . Hence it makes sense to talk about \star -multiplying with local functions.

Proposition 4.4. *A local \star product has the separation of variables property if and only if in the bidifferential operators $C_k(\cdot, \cdot)$ for $k \geq 1$ in the first argument only derivatives in holomorphic and in the second argument only derivatives in antiholomorphic directions appear.*

In Karabegov's original notation the rôles of the holomorphic and antiholomorphic functions is switched. Bordemann and Waldmann [15] called such star products *star products of Wick type*. Both Karabegov and Bordemann-Waldmann proved that there exists for every Kähler manifold star products of separation of variables type. See also Reshetikhin and Takhtajan [43] for yet another construction. But I like to point out that in all these constructions the result is only a formal star product without any relation to an operator calculus, which will be given by the Berezin-Toeplitz star product introduced in the next section.

Another warning is in order. The property of being a star product of separation of variables type will not be kept by equivalence transformations.

4.2 Berezin-Toeplitz star product

Again we restrict the situation to the compact quantizable Kähler case.

Theorem 4.5. *There exists a unique (formal) star product \star_{BT} for M*

$$(4.7) \quad f \star_{BT} g := \sum_{j=0}^{\infty} \nu^j C_j(f, g), \quad C_j(f, g) \in C^\infty(M),$$

in such a way that for $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f, g)$ for all m

$$(4.8) \quad \|T_f^{(m)} T_g^{(m)} - \sum_{0 \leq j < N} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)}\| \leq K_N(f, g) \left(\frac{1}{m}\right)^N.$$

The star product is null on constants and selfadjoint.

This theorem has been proven immediately after [14] was finished. It has been announced in [46],[47] and the proof was written up in German in [45]. A complete proof published in English can be found in [49].

For simplicity we write

$$(4.9) \quad T_f^{(m)} \cdot T_g^{(m)} \sim \sum_{j=0}^{\infty} \left(\frac{1}{m}\right)^j T_{C_j(f,g)}^{(m)} \quad (m \rightarrow \infty),$$

but we will always assume the strong and precise statement of (4.8).

Next we want to identify this star product. Let K_M be the canonical line bundle of M , i.e. the n^{th} exterior power of the holomorphic 1-differentials. The canonical class δ is the first Chern class of this line bundle, i.e. $\delta := c_1(K_M)$. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1, 1)$ -form ω_{can} . In our sign conventions we have $\delta = [\omega_{can}]$.

Together with Karabegov the author showed

Theorem 4.6. [34] (a) *The Berezin-Toeplitz star product is a local star product which is of separation of variable type.*

(b) *Its classifying Deligne-Fedosov class is*

$$(4.10) \quad cl(\star_{BT}) = \frac{1}{i} \left(\frac{1}{\nu} [\omega] - \frac{\delta}{2} \right)$$

for the characteristic class of the star product \star_{BT} .

(c) *The classifying Karabegov form associated to the Berezin-Toeplitz star product is*

$$(4.11) \quad -\frac{1}{\nu} \omega + \omega_{can}.$$

Remark 4.7. The Karabegov form

$$(4.12) \quad \widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots$$

is a formal series, where ω_{-1} is the Kähler form ω of the manifold and the forms ω_r , $r \geq 0$, are closed but not necessarily nondegenerate $(1,1)$ -forms on M . It was shown in [32] that all deformation quantizations with separation of variables on

the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by such formal forms (4.12). They might be considered as formal deformations of $(1/\nu)\omega_{-1}$. The reason that we have $-\frac{1}{\nu}\omega$ in (4.11) is that in Karabegov's terminology the role of the holomorphic and anti-holomorphic variables are switched. For a description of Karabegov's construction, see [53]. There details about the identification of the Berezin-Toeplitz star product in his classification can be found, see also [34].

Remark 4.8. By Tuynman's lemma (3.16) the operators of geometric quantization with Kähler polarization have the same asymptotic behaviour as the BT operators. As the latter defines a star product \star_{BT} it can be used to give also a star product \star_{GQ} associated to geometric quantization. Details can be found in [49]. This star product will be equivalent to the BT star product, but it is not of separation of variables type. The equivalence is given by the $\mathbb{C}[[\nu]]$ -linear map induced by

$$(4.13) \quad B(f) := f - \nu \frac{\Delta}{2} f = (id - \nu \frac{\Delta}{2}) f.$$

We obtain $B(f) \star_{BT} B(g) = B(f \star_{GQ} g)$.

Remark 4.9. From (3.13) the following complete asymptotic expansion for $m \rightarrow \infty$ can be deduced [49], [16]):

$$(4.14) \quad \text{Tr}^{(m)}(T_f^{(m)}) \sim m^n \left(\sum_{j=0}^{\infty} \left(\frac{1}{m} \right)^j \tau_j(f) \right), \quad \text{with } \tau_j(f) \in \mathbb{C}.$$

We define the $\mathbb{C}[[\nu]]$ -linear map

$$(4.15) \quad \text{Tr} : C^\infty(M)[[\nu]] \rightarrow \nu^{-n} \mathbb{C}[[\nu]], \quad \text{Tr } f := \nu^{-n} \sum_{j=0}^{\infty} \nu^j \tau_j(f),$$

where the $\tau_j(f)$ are given by the asymptotic expansion (4.14) for $f \in C^\infty(M)$ and for arbitrary elements by $\mathbb{C}[[\nu]]$ -linear extension.

Proposition 4.10. [49] *The map Tr is a trace, i.e., we have*

$$(4.16) \quad \text{Tr}(f \star g) = \text{Tr}(g \star f).$$

5 Berezin's coherent states, symbols, and transform

5.1 The disc bundle

We will assume that M is a compact quantizable Kähler manifold with very ample quantum line bundle L , i.e. L has enough global holomorphic sections

to embed M into projective space. From the bundle¹ (L, h) we pass to its dual $(U, k) := (L^*, h^{-1})$ with dual metric k . Inside of the total space U we consider the circle bundle

$$(5.1) \quad Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

the (open) disc bundle, and (closed) disc bundle respectively

$$(5.2) \quad D := \{\lambda \in U \mid k(\lambda, \lambda) < 1\}, \quad \bar{D} := \{\lambda \in U \mid k(\lambda, \lambda) \leq 1\}.$$

Let $\tau : U \rightarrow M$ the projection (maybe restricted to the subbundles).

For the projective space $\mathbb{P}^N(\mathbb{C})$ with the hyperplane section bundle H as quantum line bundle the bundle U is just the tautological bundle. Its fiber over the point $z \in \mathbb{P}^N(\mathbb{C})$ consists of the line in \mathbb{C}^{N+1} which is represented by z . In particular, for the projective space the total space of U with the zero section removed can be identified with $\mathbb{C}^{N+1} \setminus \{0\}$. The same picture remains true for the, via the very ample quantum line bundle in projective space embedded, manifold M . The quantum line bundle will be the pull-back of H (i.e. its restriction to the embedded manifold) and its dual is the pull-back of the tautological bundle.

In the following we use $E \setminus 0$ to denote the total space of a vector bundle E with the image of the zero section removed. Starting from the real valued function $\hat{k}(\lambda) := k(\lambda, \lambda)$ on U we define $\tilde{a} := \frac{1}{2i}(\partial - \bar{\partial}) \log \hat{k}$ on $U \setminus 0$ (the derivation are taken with respect to the complex structure on U) and denote by α its restriction to Q . With the help of the quantization condition (2.3) we obtain $d\alpha = \tau^*\omega$ (with the deRham differential $d = d_Q$) and that in fact $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \alpha$ is a volume form on Q . Indeed α is a contact form for the contact manifold Q . As far as the integration is concerned we get

$$(5.3) \quad \int_Q (\tau^*f)\mu = \int_M f \Omega, \quad \forall f \in C^\infty(M).$$

Recall that Ω is the Liouville volume form on M .

With respect to μ we take the L^2 -completion $L^2(Q, \mu)$ of the space of functions on Q . By the natural circle action the bundle Q is an S^1 -bundle and the tensor powers of U can be viewed as associated line bundles. Sections of $L^m = U^{-m}$ can be identified with functions ψ on Q which satisfy the equivariance condition $\psi(c\lambda) = c^m\psi(\lambda)$, i.e. which are homogeneous of degree m . This identification is given via the map

$$(5.4) \quad \gamma_m : L^2(M, L^m) \rightarrow L^2(Q, \mu), \quad s \mapsto \psi_s \quad \text{where} \quad \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

which turns out to be an isometry onto its image, where on $L^2(M, L^m)$ we take the scalar product (3.2).

¹As the connection ∇ will not be needed anymore, I will drop it in the notation.

The generalized *Hardy space* \mathcal{H} is the closure of the space of those functions in $L^2(Q, \mu)$ which can be extended to holomorphic functions on the whole disc bundle \overline{D} . The generalized *Szegö projector* is the projection

$$(5.5) \quad \Pi : L^2(Q, \mu) \rightarrow \mathcal{H} .$$

The space \mathcal{H} is preserved by the S^1 -action. It can be decomposed into eigenspaces $\mathcal{H} = \prod_{m=0}^{\infty} \mathcal{H}^{(m)}$ where $c \in S^1$ acts on $\mathcal{H}^{(m)}$ as multiplication by c^m . The Szegö projector is S^1 invariant and can be decomposed into its components, the Bergman projectors

$$(5.6) \quad \hat{\Pi}^{(m)} : L^2(Q, \mu) \rightarrow \mathcal{H}^{(m)} .$$

If we restrict (5.4) on the holomorphic sections we obtain the isometry

$$(5.7) \quad \gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)} .$$

In the case of $\mathbb{P}^N(\mathbb{C})$ this correspondence is nothing else as the identification of the global sections of the m^{th} tensor powers of the hyper plane section bundle with the homogenous polynomial functions of degree m on \mathbb{C}^{N+1} .

Remark 5.1. In this set-up the notion of Toeplitz structure (Π, Σ) , as developed by Boutet de Monvel and Guillemin in [17, 30] can be applied. After some work this leads to a proof of most of the statements in Theorem 3.3. A sketch of these techniques and of the proof can be found in [53]

5.2 Coherent States

We recall the correspondence (5.4) $\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$ of m -homogeneous functions ψ_s on U with sections of L^m . To obtain this correspondence we fixed the section s and varied α .

Now we do the opposite. We fix $\alpha \in U \setminus 0$ and vary the sections s . Obviously this yields a linear form on $\Gamma_{hol}(M, L^m)$ and hence with the help of the scalar product (3.2) we make the following

Definition 5.2. (a) The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the unique element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ such that

$$(5.8) \quad \langle e_\alpha^{(m)}, s \rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha)))$$

for all $s \in \Gamma_{hol}(M, L^m)$.

(b) The *coherent state (of level m)* associated to $x \in M$ is the projective class

$$(5.9) \quad e_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \quad \alpha \in \tau^{-1}(x), \alpha \neq 0 .$$

Of course, we have to show that the object in (b) is well-defined. Recall that $\langle \cdot, \cdot \rangle$ denotes the scalar product on the space of global sections $\Gamma_\infty(M, L^m)$. In our convention it will be anti-linear in the first argument and linear in the second argument. The coherent vectors are antiholomorphic in α and fulfill

$$(5.10) \quad e_{\alpha\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}, \quad c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

Note that $e_\alpha^{(m)} \equiv 0$ would imply that all sections will vanish at the point $x = \tau(\alpha)$. Hence, the sections of L cannot be used to embed M into projective space, which is a contradiction to the very-ampleness of L . Hence, $e_\alpha^{(m)} \not\equiv 0$ and due to (5.10) the class

$$[e_\alpha^{(m)}] := \{s \in \Gamma_{hol}(M, L^m) \mid \exists c \in \mathbb{C}^* : s = c \cdot e_\alpha^{(m)}\}$$

is a well-defined element of the projective space $\mathbb{P}(\Gamma_{hol}(M, L^m))$, only depending on $x = \tau(\alpha) \in M$.

This kind of coherent states go back to Berezin. A coordinate independent version and extensions to line bundles were given by Rawnsley [42]. They also exist in the non-compact setting, as the linear form given by the evaluation of the sections is continuous, see again [42].

The coherent states play an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [18, 19, 20, 21], via Berezin's covariant symbols. In these works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U . It has the advantage that one can consider all tensor powers of L together on an equal footing.

Remark 5.3. The *coherent state embedding* is the antiholomorphic embedding

$$(5.11) \quad M \rightarrow \mathbb{P}(\Gamma_{hol}(M, L^m)) \cong \mathbb{P}^N(\mathbb{C}), \quad x \mapsto [e_{\tau^{-1}(x)}^{(m)}].$$

Here $N = \dim \Gamma_{hol}(M, L^m) - 1$. Here we will understand under $\tau^{-1}(x)$ always a non-zero element of the fiber over x . The coherent state embedding is up to conjugation the embedding (2.15) with respect to an orthonormal basis of the sections.

5.3 Berezin symbols

In this subsection I will be rather short, but details and complete proofs can be found in [53]. We start with the

Definition 5.4. The *covariant Berezin symbol* $\sigma^{(m)}(A)$ (of level m) of an operator $A \in \text{End}(\Gamma_{hol}(M, L^m))$ is defined as

$$(5.12) \quad \sigma^{(m)}(A) : M \rightarrow \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)} \rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle}, \quad \alpha \in \tau^{-1}(x).$$

As the factors appearing in (5.10) will cancel, it is a well-defined function on M .

We introduce the the *coherent projectors* used by Rawnsley

$$(5.13) \quad P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \quad \alpha \in \tau^{-1}(x).$$

in the convenient bra-ket notation of physicists. With their help the covariant symbol can be expressed as

$$(5.14) \quad \sigma^{(m)}(A)(x) = \text{Tr}(AP_x^{(m)}).$$

From the definition of the symbol it follows that $\sigma^{(m)}(A)$ is real analytic and that $\sigma^{(m)}(A^*) = \overline{\sigma^{(m)}(A)}$.

Rawnsley [42] introduced a very helpful function on the manifold M relating the local metric in the bundle with the scalar product on coherent states.

In our dual description we define it in the following way.

Definition 5.5. *Rawnsley's epsilon function* is the function

$$(5.15) \quad M \rightarrow C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}(x), \quad \alpha \in \tau^{-1}(x).$$

Indeed it is an extremely interesting function encoding geometric information. In [53, Prop. 6.6] it is shown that for any orthonormal basis s_1, s_2, \dots, s_k of $\Gamma_{hol}(M, L^m)$ it calculates to

$$(5.16) \quad \epsilon^{(m)}(x) = \sum_{j=1}^k h^{(m)}(s_j, s_j)(x).$$

The function $\epsilon^{(m)}$ is strictly positive. Hence, we can define the modified measure

$$(5.17) \quad \Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$$

for the space of functions on M and obtain a modified scalar product $\langle \cdot, \cdot \rangle_\epsilon^{(m)}$ for $C^\infty(M)$.

In the case that the functions $\epsilon^{(m)}$ will be constant as function of the points of the manifold it calculates as

$$(5.18) \quad \epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{\text{vol } M}.$$

Here $\text{vol } M$ denotes the volume of the manifold with respect to the Liouville measure. Now the question arises when $\epsilon^{(m)}$ will be constant, resp. when the measure $\Omega_\epsilon^{(m)}$ will be the standard measure (up to a scalar). If there is a transitive

group action on the manifold and everything, e.g. Kähler form, bundle, metric, is homogenous with respect to the action this will be the case. An example is given by $M = \mathbb{P}^N(\mathbb{C})$. By a result of Rawnsley [42], resp. Cahen, Gutt and Rawnsley [18], $\epsilon^{(m)} \equiv \text{const}$ if and only if the quantization is projectively induced. This means that under the conjugate of the coherent state embedding (2.15), the Kähler form ω of M coincides with the pull-back of the Fubini-Study form. Note that in general situations this is not the case.

Definition 5.6. Given an operator $A \in \text{End}(\Gamma_{\text{hol}}(M, L^m))$ then a *contravariant Berezin symbol* $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of A is defined by the representation of the operator A as integral

$$(5.19) \quad A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)} \Omega_\epsilon^{(m)}(x),$$

if such a representation exists.

Very important is that we put “a” and not “the” in the definition, as in general the contravariant symbol will not be unique. But

Proposition 5.7. *The Toeplitz operator $T_f^{(m)}$ admits a representation (5.19) with*

$$(5.20) \quad \check{\sigma}^{(m)}(T_f^{(m)}) = f ,$$

i.e. the function f is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. Moreover, every operator $A \in \text{End}(\Gamma_{\text{hol}}(M, L^m))$ has a contravariant symbol.

As the Toeplitz map is surjective the last statement in the proposition is clear.

We introduce on $\text{End}(\Gamma_{\text{hol}}(M, L^m))$ the Hilbert-Schmidt norm

$$(5.21) \quad \langle A, C \rangle_{HS} = \text{Tr}(A^* \cdot C) .$$

Theorem 5.8. *The Toeplitz map $f \rightarrow T_f^{(m)}$ and the covariant symbol map $A \rightarrow \sigma^{(m)}(A)$ are adjoint:*

$$(5.22) \quad \langle A, T_f^{(m)} \rangle_{HS} = \langle \sigma^{(m)}(A), f \rangle_\epsilon^{(m)} .$$

Let us collect some related results

Proposition 5.9.

(a)

$$(5.23) \quad \langle A, B \rangle_{HS} = \langle \sigma^{(m)}(A), \check{\sigma}^{(m)}(B) \rangle_\epsilon^{(m)} .$$

(b) *The covariant symbol map $\sigma^{(m)}$ is injective.*

(c)

$$(5.24) \quad \text{Tr} A = \int_M \sigma^{(m)}(A) \Omega_\epsilon^{(m)} = \int_M \check{\sigma}^{(m)}(A) \Omega_\epsilon^{(m)} .$$

Remark 5.10. Under certain very restrictive conditions Berezin covariant symbols can be used to construct a star product, called the *Berezin star product*. As said above the symbol map

$$(5.25) \quad \sigma^{(m)} : \text{End}(\Gamma_{hol}(M, L^m)) \rightarrow C^\infty(M)$$

is injective. Its image is a subspace $\mathcal{A}^{(m)}$ of $C^\infty(M)$, called the subspace of covariant symbols of level m . If $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ are elements of this subspace the operators A and B will be uniquely fixed. Hence also $\sigma^{(m)}(A \cdot B)$. Now one takes

$$(5.26) \quad \sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B)$$

as definition for an associative and noncommutative product $\star_{(m)}$ on $\mathcal{A}^{(m)}$. The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m . In certain special situations like those studied by Berezin himself [11] and Cahen, Gutt, and Rawnsley [18] the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m \in \mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^\infty(M)$. Indeed, in the cases considered, the manifold is a homogenous manifold and the epsilon function $\epsilon^{(m)}$ is a constant. A detailed analysis shows that then a star product can be given.

For further examples, for which this method works (not necessarily compact) see other articles by Cahen, Gutt, and Rawnsley [19, 20, 21]. For related results see also work of Moreno and Ortega-Navarro [38], [37]. In particular, also the work of Engliš [27, 26, 25, 24]. Reshetikhin and Takhtajan [43] gave a construction of a (formal) star product using formal integrals in the spirit of the Berezin's covariant symbol construction.

6 Berezin transform

Starting from $f \in C^\infty(M)$ we can assign to it its Toeplitz operator $T_f^{(m)} \in \text{End}(\Gamma_{hol}(M, L^m))$ and then assign to $T_f^{(m)}$ the covariant symbol $\sigma^{(m)}(T_f^{(m)})$. It is again an element of $C^\infty(M)$.

Definition 6.1. The map

$$(6.1) \quad C^\infty(M) \rightarrow C^\infty(M), \quad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)})$$

is called the *Berezin transform (of level m)*.

From the point of view of Berezin's approach the operator $T_f^{(m)}$ has as a contravariant symbol f . Hence $I^{(m)}$ gives a correspondence between contravariant symbols and covariant symbols of operators. The Berezin transform was introduced and studied by Berezin [11] for certain classical symmetric domains in \mathbb{C}^n .

These results were extended by Unterberger and Upmeyer [57], see also Engliš [25, 26, 27] and Engliš and Peetre [28]. Obviously, the Berezin transform makes perfect sense in the compact Kähler case which we consider here.

Theorem 6.2. [34] *Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ evaluated at the point x has a complete asymptotic expansion in powers of $1/m$ as $m \rightarrow \infty$*

$$(6.2) \quad I^{(m)}(f)(x) \sim \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i},$$

where $I_i : C^\infty(M) \rightarrow C^\infty(M)$ are maps with

$$(6.3) \quad I_0(f) = f, \quad I_1(f) = \Delta f.$$

Here the Δ is the usual Laplacian with respect to the metric given by the Kähler form ω .

Complete asymptotic expansion means the following. Given $f \in C^\infty(M)$, $x \in M$ and an $r \in \mathbb{N}$ then there exists a positive constant A such that

$$(6.4) \quad \left| I^{(m)}(f)(x) - \sum_{i=0}^{r-1} I_i(f)(x) \frac{1}{m^i} \right|_\infty \leq \frac{A}{m^r}.$$

Remark 6.3. The asymptotic of the Berezin transform is rather useful. It contains a lot of geometric information about the manifold. Moreover, the asymptotic expansion of the Berezin transform supplies a different proof of Theorem 3.3, part (a), using the relation

$$(6.5) \quad |I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \leq \|T_f^{(m)}\| \leq |f|_\infty.$$

(see [53].)

Remark 6.4. The Berezin transform can be expressed by the Bergman kernels. Recall from Section 5 the Szegő projectors $\Pi : L^2(Q, \mu) \rightarrow \mathcal{H}$ and its components $\hat{\Pi}^{(m)} : L^2(Q, \mu) \rightarrow \mathcal{H}^{(m)}$, the Bergman projectors. The Bergman projectors have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha, \beta)$ defined on $Q \times Q$, i.e.

$$(6.6) \quad \hat{\Pi}^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha, \beta) \psi(\beta) \mu(\beta).$$

The Bergman kernels can be expressed with the help of the coherent vectors.

$$(6.7) \quad \mathcal{B}_m(\alpha, \beta) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle, \quad \alpha, \beta \in Q.$$

Let $x, y \in M$ and choose $\alpha, \beta \in Q$ with $\tau(\alpha) = x$ and $\tau(\beta) = y$ then the functions

$$(6.8) \quad u_m(x) := \mathcal{B}_m(\alpha, \alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle,$$

$$(6.9) \quad v_m(x, y) := \mathcal{B}_m(\alpha, \beta) \cdot \mathcal{B}_m(\beta, \alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle$$

are well-defined on M and on $M \times M$ respectively. An integral representation of the Berezin transform is obtained by

$$(6.10) \quad \begin{aligned} (I^{(m)}(f))(x) &= \frac{1}{\mathcal{B}_m(\alpha, \alpha)} \int_Q \mathcal{B}_m(\alpha, \beta) \mathcal{B}_m(\beta, \alpha) \tau^* f(\beta) \mu(\beta) \\ &= \frac{1}{u_m(x)} \int_M v_m(x, y) f(y) \Omega(y) . \end{aligned}$$

In [34] an asymptotic expansion of the Bergman kernel is shown. The above formula is then the starting point in [34] for the proof of the existence of the asymptotic expansion of the Berezin transform. Again I refer to [53] for more details and more arguments.

Remark 6.5. As everything is ready now I like to close with a result of the pullback of the Fubini-Study form. Starting from the Kähler manifold (M, ω) and after choosing an orthonormal basis of the space $\Gamma_{hol}(M, L^m)$ we obtain an embedding

$$\phi^{(m)} : M \rightarrow \mathbb{P}^{N(m)}$$

of M into projective space of dimension $N(m)$. On $\mathbb{P}^{N(m)}$ we have the standard Kähler form, the Fubini-Study form ω_{FS} . The pull-back $(\phi^{(m)})^* \omega_{FS}$ will not depend on the orthogonal basis chosen for the embedding. But in general it will not coincide with a scalar multiple of the Kähler form ω we started with.

It was shown by Zelditch [59], by generalizing a result of Tian [55], that $(\phi^{(m)})^* \omega_{FS}$ admits a complete asymptotic expansion in powers of $\frac{1}{m}$ as $m \rightarrow \infty$. In fact it is related to the asymptotic expansion of the Bergman kernel (6.8) along the diagonal. The pull-back can be given as [59, Prop.9]

$$(6.11) \quad (\phi^{(m)})^* \omega_{FS} = m\omega + i \partial \bar{\partial} \log u_m(x) .$$

If we replace in the asymptotic expansion $1/m$ by the formal variable ν , and denote the resulting formal series by $\mathbb{F}(\cdot)$, we obtain the Karabegov form of the star product “dual” to the Berezin-Toeplitz star product:

$$(6.12) \quad \widehat{\omega} = \mathbb{F}((\phi^{(m)})^* \omega_{FS}).$$

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