LSF Research Working Paper Series

N°. 13-09

Date: October 2013

Title: Asymmetric contests with risky rents

Author(s)*: Jean-Daniel Guigou, Bruno Lovat, Marc Boissaux

Abstract: We study a rent-seeking proportional-prize contest between two heterogeneously risk-averse players and given prize amounts are normally distributed rather than known ex ante. We establish existence and unicity of a Nash equilibrium linked to this contest, and study the equilibrium efforts implied. We then obtain a similar equilibrium result for a winner-takes-all lottery contest within the same risky rent context, and compare optimal efforts as well as expected utilities between the two contest types.

***Corresponding Author's Address:** Tel: +352 46 66 44 6317; Fax : +352 46 66 44 6835
E-mail address: jean-daniel.guigou@uni.lu

The opinions and results mentioned in this paper do not reflect the position of the Institution.

The LSF Research Working Paper Series is available
online:
http://www.en.uni.lu/recherche/fdef/luxembourg_school_of_finance_research_in_finance/working_papers

For editorial correspondence, please contact:
martine.zenner@uni.lu

University of Luxembourg
Faculty of Law, Economics and
Finance
Luxembourg School of Finance
4 Rue Albert Borschette
L-1246 Luxembourg

Asymmetric contests with risky rents

Jean-Daniel Guigou ^{*1}, Bruno Lovat ^{†2} and Marc Boissaux ^{‡1}

¹Luxembourg School of Finance, University of Luxembourg, Luxembourg

²BETA, Université de Lorraine, Nancy, France

October 2, 2013

Abstract

We study a rent-seeking proportional-prize contest between two heterogeneously risk-averse players and given prize amounts are normally distributed rather than known ex ante. We establish existence and unicity of a Nash equilibrium linked to this contest, and study the equilibrium efforts implied. We then obtain a similar equilibrium result for a winner-takes-all lottery contest within the same risky rent context, and compare optimal efforts as well as expected utilities between the two contest types.

1 Introduction

Rent-seeking contests are activities entered by a number of contestants (or players), who invest an effort aimed to maximise either their chance of winning a given prize (or *rent*) or the proportion of a given rent they will win. This quite general definition can be specialised to model and study a large number of economic situations which can be described within this sort of framework. Examples covered in the literature include lobbying, litigation, sports contests, advertising campaigns, elections, R&D races or military conflict: see [7] for a recent survey on contest theory and its various applications.

An important strand of research into contests focuses on the impact of risky rents in general. In practice, it is typically difficult, or even impossible, for contestants to fully insure themselves against the impact of random events on envisaged rent values. Since many real-world rent-seeking activities involve payoff uncertainties of various kinds, the modelling of risky rents in the analysis of contests thus enables a more realistic representation of the issues involved. There are then several different ways in which rent risk may be modelled. [6] considers the risk that cashflows may not be paid, or end up paid at points in time that are unknown in advance, due to an unreliable contest organiser such as a particular government. [2] similarly study the rent risk implied by the political uncertainty present in a regulated environment.

The present article contributes more specifically to studies of contests in which the *amount* of the risky rent is modelled as random. We cite [8] as an early example of the pertinent literature: in this paper, the rent is divisible and its amount known, but the proportion assigned to each contestant is subject to uncertainty. [5] models a uncertain rent whose distribution is known and discusses its impact on a rent-seeking game involving risk-neutral contestants. Again considering a risky rent disputed by risk-neutral players, the paper [9] develops a two-stage game where the first-stage winner receives a rent value realised in the second, stochastic stage. Finally, the article [14] examines the impact of selective uncertainty due to asymmetric information, which may cause one of two contestants to be unaware of the rent value v even though v is known to the second player.

In the presence of risky rent amounts and in opposition to some of the relevant literature exemplified in the above, it seems reasonable to explicitly consider risk averse contestants (see [1]), and our analysis will therefore do so. The analytical context closest to ours is found in [13], where a rent with a binomially distributed uncertain value is being contested in a winner-takes-all (WTA) game by symmetrically risk averse players. Unlike [13], however, we focus on a two-player contest and obtain equilibrium results for the general

*Corresponding author, jean-daniel.guigou@uni.lu, telephone: (+352) 46 66 44 6317, fax: (+352) 46 66 44 6811

†bruno.lovat@univ-lorraine.fr

‡marc.boissaux@gmail.com

case of heterogeneously risk averse players. We also concentrate on a proportional-prize contest (PPC), for which the rent is divisible and only the proportion in which it is shared is under contention. Whilst this represents another important type of game (see [8],[10]), it is comparably neglected in the literature: for examples of studies. We then study a comparable WTA contest as benchmark, and compare equilibria between the two.

The structure of the paper is then as follows. Section 2 introduces a proportional-prize contest with normally distributed risky rent which is of primary concern to our discussion. In Section 3, we show that a unique nontrivial Nash equilibrium exists for this contest and given a symmetrically heterogeneous setup. Section 4 quantifies individual and aggregate efforts seen in this equilibrium and for different levels of risk aversion. Section 5 establishes the existence of a Nash equilibrium for a winner-takes-all lottery contest specified using the same risky rent distribution as for the PPC we mainly focus on. Optimal efforts and expected utilities at equilibrium are compared between the two contest types. Section 6 concludes.

2 A proportional-prize contest

There are two risk-averse players, labelled player 1 and player 2, who compete in a contest for a share of a divisible but risky prize. Let x_1 and x_2 denote the (nonnegative) respective effort levels of players 1 and 2, and let $p_i(x_1, x_2)$ be the associated share of the prize received by player i , $i \in \{1, 2\}$. Player i 's uncertain payoff is:

$$y_i(x_1, x_2) = p_i(x_1, x_2)v - x_i, \quad (1)$$

where v is the uncertain value of the contested prize. Thus, in general, both players end up with a nonzero part of the allocated prize. Note that the cost of effort is measured in monetary units and obtained using the identity function.

Concerning p_i , we will use the following functional: $p_i = x_i/(x_1 + x_2)$ if $x_1 + x_2 > 0$, and $p_i = 0$ otherwise. This is the standard Tullock (1980) lottery contest success function (CSF), but here the CSF is interpreted as a share of the prize rather than as a probability of winning. For both interpretations, p_i equals player i 's effort relative to the total effort.

Let us now introduce both risk aversions and risky prizes in this basic framework. We make the following two main assumptions.

Assumption 1. Players have constant absolute risk-averse (CARA) preferences represented by the following negative exponential utility function:

$$u(y_i) = -\exp(-r_i y_i), \quad (2)$$

where $r_i > 0$ represents player i 's coefficient of absolute risk aversion ($r_i = -u''/u'$).

Assumption 2. The value of the contested prize v is normally distributed. More precisely, $v = V + \epsilon$, where V is a constant and ϵ is a normally distributed shock with mean 0 and variance σ^2 . Therefore, player i 's payoff y_i , which is a linear function of the contested prize, is also a normal variable with mean $\mu_i = p_i V - x_i$ and variance $\sigma_i^2 = p_i^2 \sigma^2$.

These assumptions are important in that they allow us to introduce a player's certainty equivalent payoff. Indeed, we can take expectations of $u(y_i)$ to see that

$$\begin{aligned} E[u(y_i)] &= \frac{1}{\sqrt{2\pi}\sigma_i} \int_{-\infty}^{+\infty} (-\exp(-r_i y_i)) \exp\left(-\frac{1}{2}\left(\frac{y_i - \mu_i}{\sigma_i}\right)^2\right) dy_i \\ &= -\exp(-r_i \mu_i + \frac{1}{2} r_i^2 \sigma_i^2) = u(\mu_i - \frac{1}{2} r_i \sigma_i^2). \end{aligned}$$

As utility is increasing in its argument, maximising expected utility is equivalent here to maximising $Y_i = \mu_i - \frac{1}{2} r_i \sigma_i^2$, which can be seen to equal the player's certainty equivalent payoff.

Replacing for μ_i and σ_i^2 , player i 's optimisation problem then takes the form

$$\arg \max_{x_i} Y_i \equiv \arg \max_{x_i} \left(p_i V - x_i - r_i^2 p_i^2 \frac{\sigma^2}{2} \right). \quad (3)$$

A player's certainty equivalent payoff Y_i is thus equal to that player's expected payoff net of both her effort cost and a risk premium which, for a given x_i , is increasing in the coefficient of risk aversion r_i and the variance of the random shock σ^2 . It is also decreasing in x_i : the larger x_i , the more the agent bears the risk associated to the prize sought v .

3 Existence and uniqueness of a Nash equilibrium

In this section, we establish the existence and uniqueness of a Nash equilibrium for the previously introduced proportional prize contest, in the case where the two players have respective risk aversion coefficients $r_1 = r - k$ and $r_2 = r + k$. Here we assume that $0 \leq k < r$, such that both players are indeed risk-averse. Also, the distribution of r_1 and r_2 is symmetrical with respect to r , with their sum equal to $r_1 + r_2 = 2r$, which makes it interesting to compare this setting with its natural benchmark equivalent where both players are homogeneous, that is $r_1 = r_2 = r$.

Proposition 1. *Consider the proportional-prize game previously discussed, with two heterogeneously risk averse players with respective risk aversion coefficients $r_1 = r - k$ and $r_2 = r + k$, where $r > k \geq 0$. If $V > 1/2(r - k)\sigma^2$, then this game has a unique interior Nash equilibrium defined by the optimal efforts*

$$x_{1,P}^* = \frac{1}{V} \left(k\sigma^2 + \sqrt{k^2\sigma^4 + V^2} \right) \frac{V + \frac{1}{V}(V + (k - r)\sigma^2) (k\sigma^2 + \sqrt{k^2\sigma^4 + V^2})}{\left(\frac{1}{V} (k\sigma^2 + \sqrt{k^2\sigma^4 + V^2}) + 1 \right)^3} \quad (4)$$

and

$$x_{2,P}^* = \frac{V + \frac{1}{V}(V + (k - r)\sigma^2) (k\sigma^2 + \sqrt{k^2\sigma^4 + V^2})}{\left(\frac{1}{V} (k\sigma^2 + \sqrt{k^2\sigma^4 + V^2}) + 1 \right)^3}. \quad (5)$$

Proof. Since we interest ourselves exclusively in possible interior equilibria, we may exclude the case where $x_1 = x_2 = 0$. Then the certainty equivalent payoffs for both players are

$$Y_1 = \frac{x_1}{x_1 + x_2} V - x_1 - \frac{(r - k)}{2} \left(\frac{x_1}{x_1 + x_2} \right)^2 \sigma^2 \quad (6)$$

and

$$Y_2 = \frac{x_2}{x_1 + x_2} V - x_2 - \frac{(r + k)}{2} \left(\frac{x_2}{x_1 + x_2} \right)^2 \sigma^2 \quad (7)$$

If a pure strategy interior Nash equilibrium exists, the corresponding optimal efforts x_1^* and x_2^* will verify the pair of necessary first-order conditions

$$\begin{cases} -Vx_1x_2 - Vx_2^2 + x_1^3 + 3x_1^2x_2 + 3x_1x_2^2 + x_2^3 + (r - k)\sigma^2x_1x_2 = 0 \\ -Vx_1^2 - Vx_1x_2 + 3x_1x_2^2 + x_2^3 + x_1^3 + 3x_1^2x_2 + (r + k)\sigma^2x_1x_2 = 0. \end{cases}$$

Subtracting the two above equations, we obtain

$$Vx_1^2 - 2\sigma^2kx_2x_1 - Vx_2^2 = 0, \quad (8)$$

which may be considered a second-order equation in x_1 . This has a positive discriminant and so allows for two real solutions. One of these is negative and so we ignore it, retaining the positive root

$$x_1 = \frac{1}{V} \left(k\sigma^2 + \sqrt{k^2\sigma^4 + V^2} \right) x_2.$$

Setting $k = \frac{V}{2\sigma^2}(e^t - e^{-t})$, we get the (necessarily positive) expression for e^t as

$$e^t = \frac{1}{V} \left(k\sigma^2 + \sqrt{k^2\sigma^4 + V^2} \right). \quad (9)$$

Clearly, this notation implies that

$$x_1 = e^t x_2. \quad (10)$$

We may now replace (10) into the first-order condition for player 2 to find expressions for the respective efforts. This gives after simplification by $x_2^2 \neq 0$

$$3e^t x_2 - V + 3(e^t)^2 x_2 + (e^t)^3 x_2 + x_2 - V e^t - k e^t \sigma^2 + r e^t \sigma^2 = 0,$$

that is

$$x_2 = \frac{V + V e^t + k \sigma^2 e^t - r \sigma^2 e^t}{(e^t + 1)^3}.$$

Replacing the expression (9) and then applying (10) gives the required expressions for the optimal effort candidates $x_{1,P}^*$ and $x_{2,P}^*$. Note from these expressions that the condition $V > 1/2(r - k)\sigma^2$ ensures that $x_{1,P}^* > x_{2,P}^* > 0$. As the only pair of efforts verifying the necessary first-order conditions for both players, $(x_{1,P}^*, x_{2,P}^*)$ is also the only pair of candidate efforts for an interior Nash equilibrium. We will now verify that they do indeed define such an equilibrium. Suppose, then, that player 2 invests $x_{2,P}^* = a = \frac{1}{2} \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$. Then the objective function for player 1 is

$$Y_1 = \frac{x_1}{x_1 + a} V - x_1 - \frac{(r - k)}{2} \left(\frac{x_1}{x_1 + a} \right)^2 \sigma^2 \quad (11)$$

and we can compute

$$\begin{aligned} \frac{\partial Y_1}{\partial x_1} &= - \frac{-V x_1 a - V a^2 + x_1^3 + 3x_1^2 a + 3x_1 a^2 + a^3 + x_1 \sigma^2 r a - x_1 \sigma^2 k a}{(x_1 + a)^3} \\ &= \frac{-(x_1 - e^t a) \left(x_1 e^t (a e^t + x_1 + 3a) + \frac{1}{2} a V + \sigma^2 r a e^t + 3a^2 e^t - \frac{1}{2} V a e^{2t} + 3a^2 e^{2t} + a^2 e^{3t} - V a e^t \right)}{e^t (x_1 + a)^3} \\ &= \frac{P(x_1)(x_1 - e^t a)}{e^t (x_1 + a)^3}, \end{aligned}$$

where

$$\begin{aligned} P(x_1) &= - \left(x_1 e^t (a e^t + x_1 + 3a) + \frac{1}{2} a V + \sigma^2 r a e^t + 3a^2 e^t - \frac{1}{2} V a e^{2t} + 3a^2 e^{2t} + a^2 e^{3t} - V a e^t \right) \\ &= - \left(x_1 e^t (a e^t + x_1 + 3a) + \frac{1}{2} a V + K \left(k \sigma^2 + \sqrt{(k^2 \sigma^4 + V^2)} \right) ((k - r) \sigma^2 + V) + V^2 \right) \end{aligned}$$

given

$$K = \frac{\sigma^2 r a t + 3a^2 t - \frac{1}{2} V a t^2 + 3a^2 t^2 + a^2 t^3 - V a t}{(k \sigma^2 + \sqrt{k^2 \sigma^4 + V^2}) (k \sigma^2 - \sigma^2 r + V) + V^2}.$$

The condition $V > 1/2(r - k)\sigma^2$ guarantees that $P(x_1)$ will always be negative. Since the sign of $\frac{\partial Y_1}{\partial x_1}$ is that of $(x_1 - e^t a)P(x_1)$, we may conclude that Y_1 is increasing on $\left[0, \frac{1}{2} e^t \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}\right]$ and decreasing on $\left[\frac{1}{2} e^t \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}, +\infty\right)$: it thus reaches a global maximum at $\frac{1}{2} e^t \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$. As a result, we may conclude that $x_{1,P}^* = \frac{1}{2} e^t \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$ is the best possible reply to $x_{2,P}^* = \frac{1}{2} \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$. Similarly, $x_{2,P}^* = \frac{1}{2} \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$ is the best possible response to $x_{1,P}^* = \frac{1}{2} e^t \frac{2V e^t + V - 2e^t \sigma^2 r + V e^{2t}}{(e^t + 1)^3}$. This then implies that $(x_{1,P}^*, x_{2,P}^*)$ defines an interior Nash equilibrium, which is the required result. $\square$

Remark. 1) The homogeneous case, where both players have the same risk aversion coefficient r , corresponds to the case $k = 0$ in what precedes. The above proof remains valid and the equilibrium efforts are obtained by simply replacing $k = 0$ into the generic formula. We obtain

$$x_{1,P}^* = x_{2,P}^* = \frac{1}{4} \left(V - \frac{1}{2} r \sigma^2 \right). \quad (12)$$

2) From expressions (12), (4) and (5), we may conclude that risk-averse rent-seekers always exert less effort than otherwise identical risk-neutral rent-seekers in contests where the rent is risky.

4 Equilibrium properties

Having established the existence of the Nash equilibrium for the type of problem under consideration, we now investigate its properties. In particular, a question of general interest in the literature is whether a less risk averse player exerts a greater effort than his more risk averse opponent or not. The paper [12] studies rent seeking games with riskless prizes when players have constant absolute risk (CARA) preferences, and shows that a less risk averse player will invest a larger effort than his more risk-averse opponent: they will therefore have a higher probability of winning the indivisible prize. [3] show, given two otherwise identical contestants (with CARA preferences), that the expenditure of the less risk averse contestant is greater (as with [12]) and that global reductions in risk aversion increase aggregate lobbying. [4] extends these earlier results by showing that, for asymmetric contests with prudent risk-averse contestants and equilibrium probabilities of winning smaller than 0.5, the aggregate lobbying effort is smaller in such a contest than in a second contest with the same contest success functions but risk-neutral contestants. Finally, and in the framework of contests between players with mean-variance preferences, [11] shows that, in asymmetric contests where players have different coefficients of variance aversion, the less variance-averse player will not necessarily exert more effort than the more variance-averse player.

We next establish an analogous result ordering individual efforts for the type of contest discussed.

Proposition 2. *Consider the proportional-prize contest with risky prize previously discussed. Write $x_{r_i,P}^*$ for the optimal effort of player i with risk aversion coefficient r_i . Then, if $r_i \leq r_j$, $x_{r_i,P}^* \geq x_{r_j,P}^*$, with equality only if $r_i = r_j$. In particular, $x_{r-k,P}^* > x_{r,P}^* > x_{r+k,P}^* \forall r > k > 0$.*

Proof. We initially show that, in the heterogeneous setting, $x_{r-k,P}^* \geq x_{r+k,P}^*$. This is a direct consequence of expression (9). Inspection of e^t shows that $e^t \geq 1$, with $e^t = 1$ only if $k = 0$ as long as the rent remains risky, that is $\sigma > 0$. Rewriting the relation (10) from the proof of proposition 1 as

$$x_{r-k,P}^* = e^t x_{r+k,P}^* \quad (13)$$

implies that $x_{r-k,P}^* \geq x_{r+k,P}^*$, with equality only if $k = 0$. Now define $k' = k/2$ and $r' = r + k/2 = r + k'$ for any $r > k \geq 0$. Then $r' > k' \geq 0$ and so we can apply the previous result to conclude that

$$\left(x_{r'-k',P}^* = x_{r,P}^* \right) \geq \left(x_{r'+k',P}^* = x_{r+k,P}^* \right),$$

with equality only if $k' = 0$, i.e. $k = 0$. Similarly, define $k' = k/2$ and $r' = r - k'$ to show that $x_{r-k,P}^* \geq x_{r,P}^*$, with equality only if $k = 0$. Thus, $x_{r-k,P}^* > x_{r,P}^* > x_{r+k,P}^* \forall r > k > 0$: we can order individual optimal efforts in the heterogeneous and homogeneous cases. Finally, consider any positive r_i and r_j . Say that $r_i \geq r_j$ with no loss in generality. Then there exists $k \geq 0$ such that $r_i = r_j + k$. The previous result allows us to conclude that $x_{r_i,P}^* \geq x_{r_j,P}^*$, with equality only if $r_i = r_j$. We have shown the most general statement given above, viz. that the less risk averse player will always exert a greater effort than his more risk-averse opponent. $\square$

Another related, but separate, question of interest is whether the total (aggregate) efforts invested in the contest between the present heterogeneous and the benchmark homogeneous cases can be compared. This amounts to choosing r and k such that $r > k > 0$ and analysing the sign of the effort differential expression

$$\begin{aligned} \Delta_{x^*} &= x_{r-k,P}^* + x_{r+k,P}^* - 2x_{r,P}^* = \left(\frac{V + Ve^t + k\sigma^2 e^t - r\sigma^2 e^t}{(e^t + 1)^3} \right) (e^t + 1) - 2 \left(\frac{1}{4}V - \frac{1}{8}r\sigma^2 \right) \\ &= \frac{1}{4(e^t + 1)^2} \left[4ke^t \sigma^2 + 2V(1 - e^{2t}) + \sigma^2 (e^t - 1)^2 r \right]. \end{aligned} \quad (14)$$

The sensitivities of this expression are interesting. Initially, it can be seen that Δ_{x^*} is strictly increasing in σ , such that total efforts in the heterogeneous context will outweigh those exerted in the homogeneous scenario by an ever larger amount as the risk of the offered rent increases. The monotonicity in V , on the other hand, is strictly decreasing given the coefficient of V is negative: $1 - (e^t)^2 < 1 - e^t < 0$ seeing as $e^t > 1$. Regarding r and k , Δ_{x^*} again strictly increases as either of those quantities increase. Summarising, we suggest that an

increase in either contest risk (viz. an increase in σ) or in contestant risk appetite (through r) increases the aggregate effort in the heterogeneous scenario with respect to the homogeneous benchmark. An increase in V can be interpreted to reduce risk in relative terms as the certain part of the prize is made proportionally more important with respect to the risky part. Finally, an increase in k increases the volatility in preferences (and, consequently, behaviour) seen across the contestant pool in the heterogeneous scenario: this corresponds to an increase in risk of yet another type, and it may thus be said that all sensitivities of Δ_{x^*} are coherent with respect to one another.

We can go further and obtain a proposition on the sign of Δ_{x^*} by studying its roots: when it is equal to zero, the summed optimal efforts in both scenarios are identical. We notice that Δ_{x^*} is linear in r , such that the critical value of r for which aggregate efforts in both scenarios are equal can directly be written down as

$$r^* = \frac{2(Ve^{2t} - 2ke^t\sigma^2 - V)}{\sigma^2(e^t - 1)^2}.$$

We consider the question whether such equality of efforts is achievable for realistic levels of the risk aversion parameters r and k , that is, such that $r^* - k > 0$ and the heterogeneous player with coefficient $r^* - k$ remains risk averse.

Proposition 3. *Consider again the proportional-prize game studied, given both a benchmark scenario where the two players have identical risk aversion coefficient r , and the heterogeneous scenario of proposition 1 with player risk aversion coefficients $r - k$ and $r + k$, where $r > k > 0$. The sum of player optimal efforts for the heterogeneous scenario is then always greater than the corresponding sum for the homogeneous benchmark scenario.*

Proof. For the aggregate efforts to meaningfully be equal, we interest ourselves in the inequality $r^* > k$ for some positive k . Writing $e^t \equiv X$, this translates to

$$(2V - k\sigma^2)X^2 - 2k\sigma^2X - (2V + k\sigma^2) > 0.$$

This can be viewed as a quadratic polynomial in X , where the coefficient of the square term is positive given¹ it has been assumed that $V > 1/2 \cdot k\sigma^2$. The required inequality is then met for any values of $X \equiv e^t$ outside the interval bounded by the polynomial's two real roots. These are easily found to equal, respectively, -1 and $(2V + k\sigma^2)/(2V - k\sigma^2)$, where, again because of the assumption $V > 1/2k\sigma^2$, the second root is always defined and positive. Concentrating on the positive root and replacing for e^t , the condition thus becomes

$$\frac{1}{V} \left(k\sigma^2 + \sqrt{k^2\sigma^4 + V^2} \right) > \frac{2V + k\sigma^2}{2V - k\sigma^2},$$

or

$$\sqrt{k^2\sigma^4 + V^2} > \frac{2V^2 + k\sigma^2V}{2V - k\sigma^2} - k\sigma^2 = \frac{2V(V - \frac{1}{2}k\sigma^2) + k^2\sigma^4}{2V - k\sigma^2}.$$

Since $V > 1/2 \cdot k\sigma^2$, the right-hand side of this condition is strictly positive, such that we may take squares to obtain a new equivalent condition

$$k^2\sigma^4 + V^2 > V^2 \left(\frac{2V + k\sigma^2}{2V - k\sigma^2} \right)^2 + k^2\sigma^4 - V \frac{4kV\sigma^2 + 2k^2\sigma^4}{2V - k\sigma^2}$$

or

$$1 > \frac{2V + k\sigma^2}{2V - k\sigma^2} \left[\frac{2V^2 - 3k\sigma^2V + 2k^2\sigma^4}{(2V - k\sigma^2)V} \right],$$

which in turn requires after simplification that

$$2k^3\sigma^6 < 0.$$

But this is not feasible if the two scenarios are to be distinct ($k > 0$) and the rent is to be risky ($\sigma > 0$). Thus the sign of Δ_{x^*} is constant if we vary $r > k$: since Δ_{x^*} is strictly increasing in r , this means that this sign is positive for any feasible $r > k$. $\square$

¹We need $r > k$ for $r - k > 0$. In that case, the hypothesis $V > 1/2r\sigma^2$ implies $V > 1/2k\sigma^2$.

5 Proportional-prize versus lottery contests

We now consider a more standard lottery variation of the contest discussed so far. With respect to the definition of section 2, this in particular implies that the prize, which remains risky, is now indivisible: the resulting game is generally called winner-takes-all (WTA). Accordingly and in line with the standard Tullock game, the contest success function (whose form we do not change) is now interpreted as giving a winning probability. All other notations and contest as well as player specifications, such as the negative exponential utility functions used, remain unchanged from the PPC introduced in section 2.

In the absence of risk aversion (for the limit values $r_i = 0$), the two games are strategically equivalent as their objective functions match and equal $E(y_i) = p_i V - x_i$ for all i in either case. In particular, both yield the same equilibrium effort ($x_L^* = x_P^* = V/4$) and the same expected payoff $E(y_i) = V/4$: given rent shocks in our model are symmetrically distributed around zero, the question whether the rent is risky or not is irrelevant to contest results in this particular case. In the present section, we show existence and uniqueness of an interior Nash equilibrium for the WTA contest with risk averse players and risky prizes. We then compare the PP and WTA contests for homogeneous player attitudes toward risk, and establish a very simple ordering for the optimal efforts and expected utilities generated by both contest types.

In a lottery contest a contestant wins the (risky) prize with a probability of success which is proportional to her effort. Player i 's expected utility function is then

$$\Pi_i = E[u(y_i)] = p_i E[u(V + \varepsilon - x_i)] + (1 - p_i)u(-x_i), \quad (15)$$

which can be written as

$$\Pi_i = -p_i e^{-r_i(V - x_i) + \frac{1}{2}r_i^2\sigma^2} - (1 - p_i)e^{r_i x_i} \quad (16)$$

by replacing the utility expectation by its certainty equivalent. We can then establish the following result.

Proposition 4. *Consider the WTA contest previously discussed, with two heterogeneously risk averse players with respective risk aversion coefficients $r_1 = r - k$ and $r_2 = r + k$, where $r > k \geq 0$. If $V > 1/2(r + k)\sigma^2$, then this game has a unique interior Nash equilibrium defined by the optimal efforts*

$$x_{1,L}^* = \frac{at}{(r - k)(-t + 1)((-t + 1)(r + k) + a)}$$

and

$$x_{2,L}^* = -tx_{1,L}^* = \frac{-at^2}{(r - k)(-t + 1)((-t + 1)(r + k) + a)}$$

with

$$\begin{aligned} a &= (r + k)(A - 1), \\ b &= (r - k)(-1 + B) + (r + k)B(-1 + A) \quad \text{and} \\ c &= (r - k)A(-1 + B), \end{aligned} \quad (17)$$

$$\begin{aligned} A &= \exp\left(\frac{1}{2}(r - k)((r - k)\sigma^2 - 2V)\right) \quad \text{and} \\ B &= \exp\left(\frac{1}{2}(r + k)(-(r + k)\sigma^2 + 2V)\right) \end{aligned} \quad (18)$$

as well as

$$t = \frac{1}{2a}(b + \sqrt{b^2 - 4ac}). \quad (19)$$

Proof. For both players, maximisation of expected utility, given the normally distributed shocks we consider, corresponds to maximisation expressions Π_1 and Π_2 . The first-order conditions that are necessarily verified by any candidate optimal efforts are then

$$\begin{cases} \frac{\partial}{\partial x_1}(\Pi_1) = \frac{\partial}{\partial x_1} \left(\frac{x_1}{x_1 + x_2} (-e^{-(r-k)(V-x_1) + \frac{(r-k)^2}{2}\sigma^2}) + \frac{x_2}{x_1 + x_2} (-e^{(r-k)x_1}) \right) & = 0 \\ \frac{\partial}{\partial x_2}(\Pi_2) = \frac{\partial}{\partial x_2} \left(\frac{x_2}{x_1 + x_2} (-e^{-(r+k)(V-x_2) + \frac{(r+k)^2}{2}\sigma^2}) + \frac{x_1}{x_1 + x_2} (-e^{(r+k)x_2}) \right) & = 0 \end{cases}$$

Now note that, given we only consider possible interior equilibria, $(x_1, x_2) \neq (0, 0)$. We can thus simplify by the nonzero quantities $e^{(r-k)x_1}$, $e^{(r+k)x_2}$ and $\frac{1}{(x_1+x_2)^2}$ as well as introduce the quantities A and B given in equations (18) to obtain the equivalent two conditions

$$\begin{cases} x_2 - x_2 A + (k-r)x_2^2 + (k-r)x_1^2 A + (k-r)x_1 x_2 + (k-r)x_1 x_2 A = 0 \\ x_1 \frac{1}{B} - x_1 + (k+r)x_2^2 \frac{1}{B} + (k+r)x_1^2 + (k+r)x_1 x_2 \frac{1}{B} + (k+r)x_1 x_2 = 0. \end{cases}$$

We now specify, without any loss of generality, t such that $x_2 = -tx_1$: thus necessarily $t < 0$ for both efforts to be strictly positive. Replacing in both above conditions and solving for x_1 , we obtain the two solutions

$$x_1 = \frac{t - At}{kt^2 - rt^2 + Ak - Ar - kt + rt - Akt + Art}$$

and

$$x_1 = \frac{B - 1}{kt^2 + rt^2 + Bk + Br - kt - rt - Bkt - Brt}.$$

But both first-order conditions have to lead to the same candidate values for optimality. Thus we require that

$$\frac{t - At}{kt^2 - rt^2 + Ak - Ar - kt + rt - Akt + Art} = \frac{B - 1}{kt^2 + rt^2 + Bk + Br - kt - rt - Bkt - Brt}.$$

Accordingly,

$$(t - At)(kt^2 + rt^2 + Bk + Br - kt - rt - Bkt - Brt) = (B - 1)(kt^2 - rt^2 + Ak - Ar - kt + rt - Akt + Art),$$

such that

$$\begin{aligned} & (t - 1)((k + r)t^2 - Ak + Ar + (k - r)t - Akt^2 - Art^2 + (k - r)AB - 2Bkt + (k + r)ABt) = 0 \\ \Leftrightarrow & (t - 1)((k + r - Ak - Ar)t^2 + (k - r - 2Bk + ABk + ABrt)t + ABk + Ar - ABrt - Ak) = 0 \\ \Leftrightarrow & (t - 1)(-(A - 1)(k + r)t^2 + ((r - k)(-1 + B) + (r + k)(-1 + A)B)t + A(k - r)(B - 1)) = 0. \end{aligned}$$

Next, we define the additional auxiliary quantities a, b and c from equations (17). Then the previous condition takes the form

$$(t - 1)(-at^2 + bt - c) = 0.$$

Since we disregard positive solutions for t , $t - 1 \neq 0$. Then the candidate solutions will solve the second-order equation $-at^2 + bt - c = 0$. Now consider that, based on the required condition $V > 1/2(r + k)\sigma^2$, $0 < A < 1$, $B > 1$, $a < 0$ and $c > 0$. Although we cannot say anything about the sign of b , this is nevertheless sufficient to conclude that $4ac < 0$ and so $b^2 - 4ac > b^2 \geq 0$: the given quadratic equation does have real roots. Also, for either $b \geq 0$ or $b < 0$, $b + \sqrt{b^2 - 4ac} > 0$ but $b - \sqrt{b^2 - 4ac} < 0$ so that

$$t = \frac{1}{2a}(b + \sqrt{b^2 - 4ac})$$

is the unique negative solution we want to retain. As a result, the candidate values for the equilibrium efforts are given by

$$\begin{aligned} x_1 &= \frac{t - At}{(k - r)t^2 + (r - k)t + A(k - r + (r - k)t)} \\ &= \frac{at}{(r - k)(-t + 1)((-t + 1)(r + k) + a)} \end{aligned} \tag{20}$$

and

$$x_2 = -tx_1 = \frac{-at^2}{(r - k)(-t + 1)((-t + 1)(r + k) + a)}. \tag{21}$$

We now move on to verify that the candidate efforts (x_1, x_2) given above do indeed specify an interior Nash equilibrium for the game considered. We consider the situation from the perspective of player 1 and assume

that player 2 puts in the candidate effort (21). Initially rewriting the first-order condition for player 1 as a quadratic equation in x_1 , we obtain

$$(k-r)Ax_1^2 + ((k-r)x_2 + (k-r)Ax_2)x_1 + x_2 - x_2A + kx_2^2 - rx_2^2 = 0.$$

Given the coefficient of the second-order term equals $(k-r)A < 0$, the above expression for $\frac{\partial \Pi_1}{\partial x_1}$ is always negative except between the two equation roots. These roots are

$$\frac{1}{2A(k-r)} \left((r-k)x_2 \pm \sqrt{x_2(k-r)(A-1)(4A-kx_2+rx_2+Akx_2-Arx_2)} + A(r-k)x_2 \right).$$

We are interested in the larger of these given that the derivative of the certainty equivalent for player 1, Π_1 , with respect to x_1 changes sign from positive to negative at that root: this implies that a maximum of Π_1 is attained at that root. Let s^+ be the solution with positive square root term and s^- that with negative square root term. To find the larger root of the two, we calculate $s^+ - s^-$ which simplifies to

$$s^+ - s^- = \frac{1}{A(k-r)} \sqrt{x_2(k-r)(A-1)(4A+(r-k)x_2+A(k-r)x_2)},$$

i.e. a strictly negative quantity. We thus retain the larger root s^- - if this equals $-x_2/t$, with x_2 given by (21), we have shown that effort (20) is the optimal response for player 1, which is the desired result. Now

$$s^- + x_2/t = t\sqrt{x_2(k-r)(A-1)(4A+x_2(k-r)(A-1))} + x_2(k-r)(-2A+t+At),$$

and this equals zero for the unique nonzero root

$$x_2 = \frac{At^2 - t^2}{(k-r)t^2 + (r-k)t + A(k-r + (r-k)t)}.$$

This is easily shown to correspond to the candidate expression (21) for player 2's effort. It may be proven in a similar fashion that (21) would be player 2's optimal response to (20), and we may conclude that the pair $(x_1, x_2) = (x_{1,L}^*, x_{2,L}^*)$ does indeed define a unique interior Nash equilibrium for the game under discussion. $\square$

Remark. 1) The case $k = 0$ leads to the optimal effort expressions in the case where the two players have homogeneous attitudes towards risk. Noticing that $k = 0$ implies $t = -1$ and $AB = 1$ and hence $b = 0$, we easily obtain

$$x_{1,L}^* = x_{2,L}^* = x_L^* = \frac{e^{Vr} - e^{\frac{1}{2}r^2\sigma^2}}{2re^{Vr} + 2re^{\frac{1}{2}r^2\sigma^2}} = \frac{1}{2r} \tanh\left(\frac{r}{2}(V - \frac{1}{2}r\sigma^2)\right). \quad (22)$$

The equilibrium effort thus depends on the expected value of the prize V , the (shared) coefficient of risk aversion r and the variance of the shock σ^2 . Using properties of the hyperbolic tangent function, the sensitivity of this expression with respect to the deterministic price level V is seen to be positive ($\partial x_L^*/\partial V > 0$): thus an increase in the expected value of the prize increases individual effort. Also, $\partial x_L^*/\partial r < 0$, such that an increase in the coefficient of risk aversion decreases individual effort. Finally, $\partial x_L^*/\partial \sigma^2 < 0$, which means that any increase in the level of noise decreases the corresponding equilibrium efforts - a result matching Treich(2010).

2) For the same case $k = 0$, the level of optimal expected utility for the WTA contest with risky prize can be obtained by replacing the above equilibrium effort x_L^* into expression (16) to obtain for either player

$$E[u(y_L^*)] = \frac{1}{2}(-e^{-r(V-x_L^*)+\frac{r^2}{2}\sigma^2}) + \frac{1}{2}(-e^{rx_L^*}),$$

that is

$$E[u(\pi_L^*)] = -\frac{1}{2} \frac{e^{\frac{1}{2}r^2\sigma^2 - Vr}}{e^{2(1+e^{\frac{1}{2}r^2\sigma^2 - Vr})}} (e^{\frac{r^2}{2}\sigma^2 - rV} + 1) = -\frac{e^{-\frac{1}{2}\tanh(-x_L^*)}}{2} (e^{-2x_L^*} + 1)$$

$E[u(\pi_L^*)]$ can be seen to be decreasing in σ^2 , which again corresponds to the results in Treich(2010).

We now compare the PP and WTA contests in the case where both players are homogeneously risk-averse with shared coefficient $r > 0$. In this case, we expect the two games to have distinct equilibria. In particular, uncertainty regarding the contest winner is introduced when moving from the PP to the WTA contest: this can be regarded as a second source of uncertainty in the WTA case. As a first intuitive guess, we may then suspect that the proportional-prize contest should always generate higher efforts than the lottery contest even when the rent is specified as risky. We now formally verify this intuition by directly comparing equilibrium efforts and expected utilities in both contest types using the expressions obtained in what precedes.

Proposition 5 (Effort and expected utility ordering between contest types). *Consider the proportional-prize contest, discussed in sections 3 and 4, and the winner-takes-all contest, discussed in the present section, with homogeneous risk aversion coefficients $r_1 = r_2 = r$.*

In this context, both optimal effort and expected utility levels are always larger for the proportional-prize contest than for the winner-takes-all contest.

Proof. From equations (12) and (22), the expressions for the optimal effort x_P^* for the proportional-prize contest and its analogue x_L^* for the winner-takes-all contest can be related through the expression

$$2rx_L^* = \tanh(2rx_P^*).$$

However, $x \geq \tanh x \ \forall x \geq 0$. We thus directly obtain that $2rx_L^* \geq \tanh(2rx_L^*)$, that is $\tanh(2rx_P^*) \geq \tanh(2rx_L^*)$, or

$$x_P^* \geq x_L^*$$

given that $\tanh$ is positive and strictly increasing on $\mathbb{R}_+$. Next, the difference between expected utility levels for the two contests is

$$E[u(y_P)] - E[u(y_L)] = \frac{1}{2} e^{(\frac{1}{2} \tanh(\frac{r}{2}(V - \frac{1}{2}r\sigma^2)))} (e^{-rV + \frac{r^2}{2}\sigma^2} + 1) - e^{-\frac{rV}{4}}.$$

Defining t as

$$t = \frac{r}{2} \left(V - \frac{1}{2} r \sigma^2 \right),$$

equality between the respective utility levels can be rephrased as

$$\frac{1}{2} e^{\frac{1}{2} \tanh t - 2t} + \frac{1}{2} e^{\frac{1}{2} \tanh t} = e^{-\frac{1}{2}t - \frac{1}{8}r^2\sigma^2},$$

that is

$$\frac{\frac{1}{2} e^{\frac{1}{2} \tanh t - 2t} + \frac{1}{2} e^{\frac{1}{2} \tanh t}}{e^{-\frac{1}{2}t}} = e^{-\frac{1}{8}r^2\sigma^2}.$$

However, the function in t given by the left-hand side of the above equality is increasing in t on the subdomain $[0, +\infty)$ and with image $[1, +\infty)$ such that we always have

$$\frac{\frac{1}{2} e^{\frac{1}{2} \tanh t - 2t} + \frac{1}{2} e^{\frac{1}{2} \tanh t}}{e^{-\frac{1}{2}t}} \geq 1 \geq e^{-\frac{1}{8}r^2\sigma^2},$$

implying that $E[u(y_P)] - E[u(y_L)] \geq 0$ in each case. This is the required result. $\square$

6 Conclusion

The present paper contains three main contributions to the literature. Having introduced a proportional-prize contest (PPC) with normally distributed risky rents in a heterogeneous, risk-averse two-player context, we first establish that it always allows for a unique non-null Nash equilibrium subject to a simple condition on the contest parameters and obtain a functional form for the optimal efforts. We then obtain ordering results concerning both individual and aggregate equilibrium efforts. Thus, less risk-averse individual players will always invest smaller efforts, and the sum of player efforts is studied to show that heterogeneous contests always give rise to larger aggregate efforts than their homogeneous analogues with the same overall risk aversion. Finally, we formulate a more standard benchmark winner-takes-all (WTA) lottery contest within the same framework involving risky rents. We show the existence of a unique interior Nash equilibrium in this case and establish that, in a setting of homogeneously risk-averse players, equilibrium efforts and expected utility levels, are always larger for the PPC than for the lottery.

References

- [1] Richard J. Allard. Risk aversion and the full costs of rent-seeking. *Bulletin of Economic Research*, 51(2):95, 1999.

- [2] Urs Steiner Brandt and Ole Jakob Bergfjord. *Regulations, risk, and rent seeking behaviour*. EAERE, 2008.
- [3] Richard Cornes and Roger Hartley. Risk aversion, heterogeneity and contests. *Public Choice*, 117(1-2):1–25, 2003.
- [4] Richard Cornes and Roger Hartley. Risk aversion in symmetric and asymmetric contests. *Economic Theory*, 51(2):247–275, 2012.
- [5] Ronald M. Harstad. Privately informed seekers of an uncertain rent. *Public Choice*, 83(1-2):81–93, 1995.
- [6] Nava Kahana and Shmuel Nitzan. Uncertain preassigned non-contestable and contestable rents. *European Economic Review*, 43(9):1705 – 1721, 1999.
- [7] K.A. Konrad. *Strategy and dynamics in contests*. LSE perspectives in economic analysis. Oxford University Press, 2009.
- [8] Ngo Van Long and Neil Vousden. Risk-averse rent seeking with shared rents. *The Economic Journal*, 97(388):pp. 971–985, 1987.
- [9] Ayse Öncüler and Rachel Croson. Rent-seeking for a risky rent: A model and experimental investigation. *Journal of Theoretical Politics*, 17(4):403–429, 2005.
- [10] Leonid Polishchuk and Alexander Tonis. Endogenous contest success functions: a mechanism design approach. *Economic Theory*, 52(1):271–297, 2013.
- [11] Alex Robson. Contests between players with mean-variance preferences. Discussion Papers in Economics economics:201207, Griffith University, Department of Accounting, Finance and Economics, July 2012.
- [12] Stergios Skaperdas and Li Gan. Risk aversion in contests. *The Economic Journal*, 105(431):pp. 951–962, 1995.
- [13] Nicolas Treich. Risk-aversion and prudence in rent-seeking games. *Public Choice*, 145(3-4):339–349, 2010.
- [14] Karl Wärneryd. Information in conflicts. *Journal of Economic Theory*, 110(1):121 – 136, 2003.