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Raouf Boucekkine, Aix-Marseille University, France
Fabien Prieur, Paris-X Nanterre University, France
Benteng Zou, CREA, University of Luxembourg

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For editorial correspondence, please contact: crea@uni.lu
University of Luxembourg
Faculty of Law, Economics and Finance
162A, avenue de la Faïencerie
L-1511 Luxembourg

UNIVERSITÉ DU LUXEMBOURG

Symmetric vs asymmetric equilibria and stochastic stability in a dynamic game of legislative lobbying *

Raouf Boucekkine † , Fabien Prieur‡ , Benteng Zou§

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Abstract

We study a 2-players stochastic dynamic symmetric lobbying differential game. Players have opposite interests; at any date, each player invests in lobbying activities to alter the legislation in her own benefit. The payoffs are quadratic and uncertainty is driven by a Wiener process. We prove that while a symmetric Markov Perfect Equilibrium (MPE) always exists, an asymmetric MPE only emerges when uncertainty is large enough. In the latter case, the legislative state converges to a stationary invariant distribution. Interestingly enough, the implications for the rent dissipation problem are much more involved than in the deterministic counterpart: the symmetric MPE still yields a limited social cost while the asymmetric may yield significant losses. We also characterize the most likely asymptotic state, in particular regarding the level of uncertainty.

Keywords: Political lobbying, symmetric versus asymmetric equilibrium, stochastic differential games, stochastic stability, social cost of lobbying

JEL classification: D72, C73

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†Aix-Marseille Univ., CNRS, EHESS, Centrale Marseille, AMSE and IMéRA. Senior member, Institut Universitaire de France. 2, Place Le Verrier, 13004 Marseille, France. E-mail: Raouf.Boucekkine@univ-amu.fr

‡University Paris X-Nanterre. E-mail: prieur@supagro.inra.fr

§CREA, University of Luxembourg. E-mail: benteng.zou@uni.lu

1 Introduction

As outlined by Gary Becker in his 1983 seminal contribution, "...political influence is not simply fixed by the political process, but can be expanded by expenditures of time and money on campaign contributions, political advertising, and in other ways that exert political pressure..." (page 372). In other words, the political outcomes are not only driven by the relationship between the politicians and the voters, they are also significantly affected by the deliberate actions of groups of individuals, say pressure groups or lobbies.

A traditional cause for political lobbying, in line with traditional rent-seeking literature (Tullock, 1967, or Kruger, 1974) has to do with the distortions in the use of resources. In particular, distorsionary taxes and subsidies, causing deadweight costs, may fuel political competition and foster lobbying. But as rightly pointed out by Becker (1983), lobbying activities may target many other fields like defense, environmental public policy, health policy etc...In short, all the domains in which public expenditures take place and which involve (as often) winners and losers. Perhaps the most interesting illustration of political lobbying nowadays is the so-called climate politics, worldwide and in the particular case of the US under the presidency of D. Trump, a notoriously "climato-skeptical" politician. Some theory already exists in this area (see for example Yu, 2005, and more recently Prieur and Zou, 2017). Related frameworks study environmental regulation as the equilibrium outcome of political competition between, say, industrialists vs the environmentalists. As explained in the two papers cited above, the pressure groups can either directly lobby the government to induce a change in the level of the environmental legislation or indirectly by engaging in information campaigns or public persuasion.

This paper is pretty much in line with the legislative environmental lobbying literature outlined above. In particular, it shares the dynamic game setting adopted in Prieur and Zou (2017), Yu (2005) (and, of course, Becker, 1983) being static. Our theory, however, differs from the latter in two essential ways. First, it tackles a generic dynamic legislative lobbying game in the sense that it is not specific to environmental and climate politics. Second, and more importantly, it introduces uncertainty into the game. In our view, uncertainty is a major ingredient of political games, and as such, it's critical to incorporate it into the theoretical analysis. Uncertainty is meant to cover internal political instability in all its forms, ranging from ordinary changes in the composition of the executive power during the electoral cycle or more drastic legislative moves of a constitutional nature. In both cases, the profitability of legislative lobbying is likely to be significantly altered (see Le Breton and Zaporozhets, 2007). External factors may also matter: for example, constraints involved by a country's commitment to enforce international treaties and agreements limit de facto the scope for lobbying. In such a case, the relevant level for lobbying turns to be international (see the case of the European Union as recently analyzed by Kluver, 2013). External economic shocks may also dramatically change the trade-off inherent in lobbying activities. A typical illustration is given by shocks on commodities' prices in resource-dependent economics: as reported in Boucekkine and Bouklia-Hassane (2011), the Algerian legislation with respect to foreign investment has been as volatile as the oil price in the three last decades (see also Boucekkine et al., 2014). As a result, the Algerian economy has been fiercely closed in the 70s in the times of high barrel price levels, and turned to be significantly liberalized from the mid-80s after the 1986 oil counter-shock and a subsequent acute external debt crisis...until 2008 with the resurgence of strongly nationalist policy coinciding with high price levels again for the oil barrel.

Our paper makes two sets of contributions. The first set concerns the literature of lobbying games and the broad concepts of symmetric (versus asymmetric) equilibrium and rent dissipation. Specifically, a symmetric equilibrium is an equilibrium where all players make the same lobbying effort; rent dissipation refers to the social cost of lobbying, first invoked by Tullock (1967). The second set of contributions is methodological and concerns the analytical treatment of continuous time stochastic dynamic games, and more precisely the computation of the corresponding Markov Perfect Equilibria (MPE), when uncertainty is driven by a brownian motion. In particular, in our linear-quadratic framework, we provide with the necessary tools to assess stochastic stability of the computed equilibria. To our knowledge, this is the first work solving a stochastic dynamic lobbying game.[1]

[1] Of course, there are a bunch of papers working out stochastic dynamic games. In Jorgsensen and Yeung (1996) for example, a stochastic dynamic game model of common property fishery is tackled. While the corresponding stationary distribution is computed, there is no related stochastic stability result. Clearly, our linear-quadratic setting allows to do so much more easily. Even more importantly, and differently from the latter fishing game, our players are not identical in the sense that their actions do not affect the state variable in the same way in contrast to the fishing game. The mechanisms behind the emergence of equilibria are therefore totally different. Finally, we do allow for the state variable to go to zero, the center of the political distribution in our case, while the counterpart-stock of fishes equal to zero- is dismissed in Jorgensen and Yeung, who consequently only focus on a unique asymmetric

As to existence of symmetric equilibria and the rent dissipation hypothesis, an abundant literature exists for the so-called (static) rent-seeking games (see Tullock, 1980, Hilman and Katz, 1984, or Leininger and Yang, 1994, for a repeated game version of the 1980 Tullock's model).[2] Two excellent systematic studies of the rent-seeking games literature can be found in Pérez-Castrillo and Verdier (1992) and Treich (2010). In both studies, one can see the great diversity of results obtained concerning the two key questions (that is, the structure of equilibria, and in particular existence of symmetric equilibria, and rent dissipation) depending on the rent-seeking technology, the type of competition and strategic interactions between players or behavioral characteristics towards risk.

We shall depart here from the traditional framework of rent-seeking games and move to a more natural and simpler setting, due to Wirl (1994), to study legislative lobbying games as those related to health or the environmental public policies. In contrast to rent-seeking games, players do not compete for a given prize but invest in lobbying activities to alter the legislation advantageously. As a result, the legislative state may change over time, which paves the way to a natural dynamic formulation of the lobbying decisions. Again this is in sharp contrast to the vast majority of rent-seeking games which are static. Wirl considers a symmetric dynamic game and computes the corresponding unique symmetric MPE. He shows that the social cost of rent-seeking (that's rent dissipation) is rather low because the threat of retaliation refrains the (Markov-like) players from investing a lot in lobbying.

As explained above, we shall incorporate uncertainty into this game in the form of a Brownian motion. As a result, an asymmetric MPE is shown to emerge in addition to a symmetric MPE. The emphasis here is not on the fact that an a priori symmetric (lobbying) game can give rise to asymmetric equilibria, this is a property which is referred to even in the early (static) rent-seeking literature (see Tullock, 1985). Our point is finer: first, we do prove that the asymmetric equilibrium only emerges when the level of uncertainty is large enough, in contrast to the symmetric equilibrium which exists whatever the latter level. In particular, asymmetric MPE cannot exist in a deterministic

equilibrium.

[2]The simplest rent-seeking games are typically modeled into the form of competition involving a given number of players which object is a given prize. Each agent makes a bet and his probability of wining the prize is an increasing function of the bet. More refined versions endogenize the number of players often through free entry.

world. Wirl (1994) does not discuss this point. We have it as a special case. Second, and more importantly, we prove that when the asymmetric MPE exists, the legislative state converges to a stationary invariant distribution, whose density can be analytically computed (and the corresponding most likely state characterized). Interestingly enough, the implications for the rent dissipation problem are now more involved than in the deterministic counterpart: while the symmetric MPE yields a limited social cost, the asymmetric may yield significant losses. We deeply investigate the distinctive properties of the asymmetric versus the symmetric equilibrium. A key difference comes from the fact to a marginal increase in the state variable, the legislative state, and while the two players react in opposite way to such a variation in both equilibria, the magnitude (that's the absolute value) of these marginal reactions in lobbying investments are equal along the symmetric equilibrium and significantly different under the asymmetric one. Given the linear-quadratic setting, the latter property determines the different asymptotic behaviours involved by the two equilibria. It also determines the main result of this paper: by supporting unbalanced lobbying investments, the asymmetric equilibrium may lead to a sizeable social cost in contrast to the symmetric. As outlined just above, such an outcome cannot arise in a deterministic framework as unbalanced investment will eventually lead to instability asymptotically. In our stochastic setting, uncertainty stabilizes somehow the dynamics induced by the asymmetric equilibrium leading to an invariant limit distribution. This is shown to happen only if uncertainty is strong enough.

Our analysis also allows to isolate various related interesting properties. Let's mention two of them here. One has to do with the impact of uncertainty on lobbying efforts. In contrast to the traditional lobbying literature under uncertainty as surveyed by Treich (2010), which relies on static games, we are able to distinguish between the short versus long-run effect of uncertainty. This is allowed by our dynamic setting and the distinction can be done either along the symmetric or the asymmetric equilibrium.[3] For example, along the former, uncertainty lowers lobbying efforts in the long-run but can perfectly increase them in the short-run depending mainly on initial conditions. Second, we can deliver a first assessment of the Algerian problem presented above: the volatility of oil prices impacting legislative uncertainty can lead to any level of social cost and any level of liberation if the players coordinate on the asymmetric equilibrium even though they

[3] In the latter case, the dynamics converge to a limit distribution but we can characterize the most likely asymptotic state and perform the comparative static exercise on this state.

have the same lobbying power.

The rest of the paper is organized as follows. Section 2 describes the stochastic legislative lobbying game and introduces some technical concepts. Section 3 deals with the symmetric MPE. Section 4 highlights the properties of the asymmetric MPE. Section 5 concludes.

2 Framework

2.1 The stochastic dynamic lobbying game

We consider two players, indexed by $i = 1, 2$, who have opposite interests on how legislation, as measured by the state variable z, should evolve. We postulate that a rising z is favorable to player 1 and harmful for player 2. Each player i invests x_i to push variable z in the most preferred direction. In addition, we extend Wirl (1994) to account for the uncertainty that surrounds the legislative process. Indeed, as explained in the introduction, for internal or external reasons and through direct or indirect mechanisms, the legislative state z is essentially a stochastic variable from the point of view of lobbyists, and should be treated as such. That is why we assume that the economy's state of legislation is governed by the following stochastic equation:

$$dz = [x_1 - x_2]dt + \sigma z \, dW, \tag{1}$$

where $W = (W_t)_{t\geq 0}$ is a standard Wiener process, and $z(0) = z_0$ is given. Parameter σ measures the volatility of z, that may originate in the wide set of internal and external factors discussed just above. To make an immediate sense of this model, consider the Algerian case cited in the introduction: z would then measure the level of liberalization of the economy, and player 1 (Resp. player 2) invests in lobbying to increase (Resp. decrease) z as much as possible, while uncertainty is driven in particular by oil revenues volatility. It's important to notice here that as the state equation (1) is specified, the two players have the same ex ante lobbying power: an increase in the lobbying effort by the same amount would lead the two players to have opposite but equal (in absolute value) impacts on the state of legislation. See related comments on this particular and sensitive point below in another context.

Players' payoffs have two components. They earn a direct benefit from the level of legislation, $\omega_i(z)$, but also have to incur a cost of lobbying, $\beta(x_i)$. Their objective is then to maximize the present value of benefit from their efforts of liberalization minus the associated cost:

$$\max_{x_i} \int_0^{\infty} e^{-rt} \left[\omega_i(z) - \beta(x_i)\right] dt, \tag{2}$$

with $r > 0$ the rate of time preference, taking as given the state constraint (1), and the lobbying strategy of the competitor.

To keep things as simple as possible, and in line with the literature, we take a linear-quadratic (LQ) specification example, i.e., benefit and cost have a quadratic form:

$$\begin{aligned} \omega_i(z) &= a_0 \pm a_1 z + \tfrac{a_2}{2} z^2, \\ \beta(x_i) &= \tfrac{b}{2} x_i^2, \end{aligned} \tag{3}$$

with $a_0, a_1 > 0$, $a_2 \leq 0$ and $b > 0$. What is important to note is the opposite sign of the term in z in the benefit. This reflects players' opposite interests with respect to the legislation. By convention, player 1 is the one pushing for a large z, i.e, we put a $+$ in front of a_1.

Several remarks are worth formulating at this stage.

First, corruption motives or office rents are left aside in our analysis. This allows us to focus on a game where the players are entirely devoted to push the legislation in the direction they wish, which is the essence of lobbying.

Second, players' lobbying efforts have the same marginal impact on z in absolute value (that's efforts x_1 and x_2 enter the state equation with opposite coefficients). In other words, we assume that they have identical lobbying powers. This needs not be always the case. For example, one may expect that the closer the lobbyists to the ideological line of the dominating party, the larger their lobbying power. Cultural aspects may matter too: clearly it is easier nowadays to lobby for a more environment-friendly legislation in the Netherlands than in Russia. But we prefer to keep our game as symmetric as possible. This will be a means to emphasize how uncertainty, in such a neutral framework, affects the number and nature – symmetric vs. asymmetric – of equilibria.

Third, it is worth summarizing the differences between our framework and the ones considered in the related literature. Our model is similar to Wirl (1994) except that he works with $\sigma = 0$: he does not take uncertainty into account. Boucekkine et al. (2014) do examine a stochastic lobbying problem but they choose a very different approach by assuming that a_1 is a discrete random variable that can take two values, with given probabilities (still sticking to Wirl's setting). This is their unique source of uncertainty in the model. Unfortunately, the latter modelling (usually referred to as piecewise deterministic game) turns out quite ineffective in the analysis of the stochastic stability of the induced equilibria. We show here below that moving to a continuous Wiener process modeling allows us to make a decisive move towards such an analysis.

Last but not least, one can readily see that our linear-quadratic modelling works for both positive and negative x_i. One can interpret $x_i > 0$ as the instantaneous investment by player i to push the legislation in a favorable direction while $x_i < 0$ can be understood as disinvestment by player i leading to a legislative move contrary to his preferred direction. For example, investment would consist in building a propaganda platform (website, radio, newspaper...), and disinvestment would correspond to shutting down the propaganda platform. Clearly, in the face of unexpected adverse shocks, which is inherent in stochastic models, disinvestment may result optimal. Hereafter, though keeping using the traditional expression "lobbying effort", we will not impose any positivity constraints in the mathematical treatment.[4]

2.2 Equilibrium and stochastic stability concepts

The LQ stochastic game, characterized by the objective (2) and the constraint (1), can be solved Markov perfect Nash equilibrium (MPE) as the solution concept. Two additional concepts will be central and therefore extensively used in the resolution.

The first concept refers to the **symmetric** vs **asymmetric** nature of the equilibrium. We define the symmetric MPE as follows:

Definition 1. *An MPE is said symmetric if the corresponding state z converges almost surely to zero. Otherwise, the MPE is said asymmetric.*

Definition 1 is a direct extension of Wirl's definition to a stochastic environment. In his deterministic game, he shows that there exists a unique symmetric MPE, i.e., the state

[4] If such a constraint is imposed, then the invariant distribution of the legislative state would be eventually truncated. This will not change at all the main result of the paper (that's, the rent dissipation depends on whether the symmetric versus the asymmetric equilibrium is selected) and eliminates trivial and unnecessary algebra.

variable converges to the neutral (or central) level $z = 0$ along the MPE. In other words, $z = 0$ is asymptotically stable along the MPE, which incidentally can only hold if the lobbying efforts are equal asymptotically (by the deterministic counterpart of equation (1)). In sum, the lobbying strategies lead to the center of the political spectrum asymptotically. Such an equilibrium is particularly natural when lobbying powers are equal. In the coming analysis, we will show, among others, that considering identical lobbying powers is not sufficient to rule out asymmetric MPEs in a stochastic framework.

The second important ingredient is the stochastic stability of the equilibrium. Here, we adopt Merton (1975)'s definition of the stability of stochastic dynamic processes:

Definition 2. *A stochastic process $z(t)$ is stable if there is stationary time invariant distribution of $z(t)$ for $t \to \infty$.*[5]

So according to (2), the z-process is said to be stable if and only if there is a unique distribution which is time and initial condition independent, and toward which the stochastic process tends. A major contribution of this paper will be to show that while we also get a unique symmetric MPE,[6] there exists a new asymmetric MPE which differs in many respects from the symmetric one. In particular, we will emphasize its different asymptotic behavior and examine the related economic implications.

The next sections are precisely devoted to the analysis of the possible outcomes of our stochastic game of lobbying. All the proofs are relegated to the Appendix.

3 Symmetric equilibrium

Our aim is first to concentrate on the symmetric outcome in order to emphasize the impact of uncertainty, both in the short and the long run. We start with a proposition that briefly states the symmetric equilibrium features:

Proposition 1. *The stochastic game of lobbying admits a unique symmetric MPE.*

[5] If this density distribution degenerates into a Dirac function, then the stochastic process converges to a unique point.

[6] Therefore we properly generalize Wirl's finding to a stochastic environment.

(i) Players' lobbying strategies are given by the following linear feedback rules:

$$x_1 = \frac{1}{b}\left(\frac{a_1}{br - C} + Cz\right), \quad x_2 = \frac{1}{b}\left(\frac{a_1}{br - C} - Cz\right), \tag{4}$$

with,

$$C = \frac{-b(\sigma^2 - r) - \sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}{6} < 0. \tag{5}$$

(ii) The stochastic process $z(t)$, whose dynamic behavior is given by

$$dz = \frac{2C}{b} z\, dt + \sigma z\, dW, \tag{6}$$

almost surely converges to the steady state $z_\infty = 0$

Let us first examine the shape and determinants of the strategies. An initial step to this end is to highlight the impact of strategic interaction by neutralizing the role of uncertainty. Taking $\sigma = 0$ boils down to assessing players' reactions to a change in z. Since C is always negative (for any value of $\sigma \geq 0$), player 1's feedback rule is decreasing in z whereas player's 2 feedback is increasing in the state ($\frac{\partial x_1}{\partial z} < 0$, $\frac{\partial x_2}{\partial z} > 0$). The reason why player 1 behaves this way while she is interested in large values of z is the fear that player 2 would exert an opposite lobbying effort in retaliation. So one gets the retaliation motive invoked by Wirl (1994) to argue that the social cost of lobbying is likely to be low.[7] A similar argument has been brought by Leininger and Yang (1994).

To understand why this type of interaction is compatible with a stable outcome, one can solve equation (6) to obtain:

$$z(t) = z_0 e^{\left(\frac{2C}{b} - \frac{\sigma^2}{2}\right)t + \sigma W(t)}.$$

Stochastic convergence to $z = 0$ holds if and only if $\frac{2C}{b} - \frac{\sigma^2}{2} < 0$ (see Boucekkine et al., 2015, for a simple mathematical exposition). At the symmetric equilibrium, this inequality is fulfilled irrespective of the value of σ because $C < 0$ and $b > 0$. So we get that the system stochastically converges to the central position of the legislation, $z = 0$, as in Wirl.

Uncertainty plays a role though as it affects the speed of convergence to the steady state. Stochastic convergence is exponential, and the (average) speed of convergence is captured

[7] The same logic is at work for player 2. Actually, with $\sigma = 0$, one exactly recovers the solution of the deterministic counterpart of the problem.

by the absolute value of the term $\frac{2C}{b} - \frac{\sigma^2}{2}$, not by the absolute value of $\frac{2C}{b}$ (as it would be in the absence of uncertainty). Clearly, the impact of uncertainty on convergence speed is ambiguous at first glance because C depends on σ. However, it follows from

$$\frac{\partial C}{\partial \sigma} = \frac{b\sigma}{3}\left[-1 - \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}\right] < 0, \tag{7}$$

that the speed of convergence is increasing in σ. In sum, the larger the uncertainty (that is the larger σ), the faster the convergence to the (stochastic) steady state.

So, uncertainty seems to be strongly stabilizing along the symmetric equilibrium. In fact, it turns out that the retaliation motive, which arises in the deterministic setting and is itself stabilizing, is further reinforced under uncertainty. A natural way to look at the stabilization role of uncertainty is to assess precisely its impact on the lobbying efforts at the MPE. In contrast to the traditional lobbying games literature surveyed in the Introduction, which uses static settings and can only evaluate the simultaneous impact of uncertainty, we are able to distinguish between short and long-term effects. We first restrict our attention to the steady state. From (4) and (7), we directly reach the conclusion that:

Corollary 1. *The larger the uncertainty, the lower the effort exerted by lobbyists at the stochastic steady state.*

This means that more uncertainty is always associated with lower lobbying effort in the long-run. That is to say, stationary rent dissipation due to lobbying is decreasing with uncertainty at the symmetric MPE. Admittedly, this is a well-known result in the traditional rent-seeking literature under uncertainty.[8] Even if we depart from the standard formulation of static rent-seeking games,[9] the intuition basically remains the same: a higher volatility surrounding the evolution of z makes the returns to lobbying more uncertain. This, in turn, is an incentive, for risk-averse players, to devote less resources to this activity.

Our dynamic game setting allows us to dive deeper into the examination of the impact of uncertainty in the short-run. Indeed, uncertainty also influences players' strategic choices

[8]Some crucial qualifications are needed though, see Konrad and Schlesinger (1997), and Treich (2010).

[9]In our setting, players' lobbying efforts are intended to change the balance of power between opposite interest groups and to push the state of the system toward their preferred direction. They do not affect the probability of "success", in contrast with Hillman and Katz (1984) for example.

indirectly, i.e., through the transitional dynamics. This is perfectly clear once we carry out the following decomposition, $i = 1, 2$:

$$\frac{dx_i(\sigma, z)}{d\sigma} = \underbrace{\frac{\partial x_i(\sigma, z)}{\partial \sigma}}_{\text{direct impact}} + \underbrace{\frac{\partial x_i(\sigma, z)}{\partial z}\frac{\partial z(\sigma)}{\partial \sigma}}_{\text{indirect impact}},$$

where the first term corresponds to the direct effect (the only one playing in the long run) whereas the second one precisely captures the indirect impacts channeling through the state variable z. The interplay between these two effects is difficult to assess in general, i.e., for any pair $(t, z(t))$. However, focusing on the starting point of the game, that is in the neighborhood of $t = 0$, we can establish that:

Corollary 2. *Suppose that $W(0) = 0$, and $z(0) = z_0$ is large enough in absolute value. Then the player who is adversely affected by the state of the system initially responds to a higher level of uncertainty by increasing her lobbying efforts.*

To understand why this adaptation to uncertainty can be optimal, it is sufficient to consider the case where z_0 is positive and remains positive for a while.[10] In this situation, one can observe that player 1's effort is decreasing in σ. In other words, the player who is in a good position initially reacts to a varying uncertainty in the same way as in the long-run. Things are more complicated for player 2 for whom the balance of power is unfavorable. Here, the indirect effect may go in the opposite direction as the pure direct effect. The initial position of the state of legislation then becomes critical. When z_0 is positive but relatively small, player 2 keeps finding it optimal to decrease her effort when the institutional environment turns riskier. So she displays the same qualitative behavior as player 1. However, when z_0 is large enough, the opposite logic is at work: a higher uncertainty is an incentive for player 2 to make larger efforts to turn the situation around. Actually, in this case, player 2 faces particularly adverse initial conditions. Then it is as if this player were (getting rid of her aversion to risk and) taking a gamble on a good realization of z ($dW < 0$) to rapidly shift the state of legislation in her preferred direction.

Overall, and in contrast with the usual conclusion drawn from static and long run analyses, uncertainty does not necessarily induce lobbyists to decrease their efforts in the short run. The discussion above clearly emphasizes the role of the initial condition z_0.

[10] A symmetric reasoning applies when $z_0 < 0$.

Nonetheless, as σ increases, the indirect effect becomes less persistent since convergence to the stochastic steady state becomes faster, and lobbying efforts decrease accordingly. Thus, while lobbying efforts may imply relatively large rent dissipation in the short-run, this can only occur in relatively short periods of time. The larger uncertainty, the shorter these periods.

In the next Section, we move to the analysis of the new class of – asymmetric – MPE (in the sense of Definition 1).

4 Asymmetric equilibrium

Besides the impact of uncertainty of lobbying efforts at the equilibrium, we are here particularly interested in its (de)stabilizing power. That is why, in the coming analysis, we need to decouple the issue of existence (and uniqueness) of the equilibrium from the one of stability, which deserves much more attention than in the previous study of the symmetric MPE.

4.1 Existence and distinctive properties

As for existence, and still focusing on linear feedback rules, Proposition 2 states that:

Proposition 2.

There exists a unique asymmetric MPE if and only if:

$$\sigma^2 \in (r + 2\sqrt{-\frac{a_2}{b}}, \;\; +\infty). \tag{8}$$

The MPE is characterized by the following lobbying efforts:

$$x_1 = \frac{B_1 + C_1 z}{b}, \;\; x_2 = -\frac{B_2 + C_2 z}{b}, \tag{9}$$

with,

$$C_1 = \frac{-b(\sigma^2 - r) - \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b}}{2} < 0, \; C_2 = \frac{-b(\sigma^2 - r) + \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b}}{2} < 0, \tag{10}$$

and,

$$B_1 = \frac{a_1 b \left(b\sigma^2 - C_1\right)}{b^2\sigma^4 - C_1 C_2} > 0, \quad B_2 = \frac{a_1 b \left(C_2 - b\sigma^2\right)}{b^2\sigma^4 - C_1 C_2} < 0. \tag{11}$$

The detailed proof is in the appendix. Two candidates for asymmetric MPE are identified. One has to be dismissed because it cannot be optimal. Because the players' problems are strictly concave in z and the feedbacks rules computed are linear in z, players' value functions should be concave in z. This concavity condition allows to rule out one of the two asymmetric equilibria. The same optimality argument can be invoked to restrict the set of admissible parameters as formulated in the Proposition above. In particular an asymmetric equilibrium only exists if and only if the level of uncertainty, as captured by the volatility parameter σ, is high enough. This means that the asymmetric equilibrium cannot exist in the deterministic case ($\sigma = 0$) or when σ is too small. This is not inconsistent with Wirl's analysis which identifies a single symmetric equilibrium in a deterministic world.

As far as the impact of z on efforts is concerned, we reach the same conclusion as in the symmetric MPE: players' efforts move in opposite direction following a change in z, player 1's effort being decreasing in z. The same outcome arises along the symmetric equilibrium. However, two crucial distinctive features of the asymmetric equilibrium have to be outlined here:

1. First of all, asymmetry reads in the fact that $C_1 \neq C_2$ in contrast to the symmetric equilibrium where $C_1 = C_2 = C$. Moreover, one can readily show that $\left|C_1\right| > \left|C_2\right|$.
2. Second, we can also readily demonstrate that inequality (7) holds for C_1, that's $\frac{\partial C_1}{\partial \sigma} < 0$, but not for C_2: $\frac{\partial C_2}{\partial \sigma} > 0$.

The first property means, as both coefficients are negative, that the first player's reaction to a marginal increase in z is larger in magnitude than the one of the second player. In other words, in response to a marginal increase in z, player 1 decreases more his investment than player 2 increases his. In a deterministic world, this systematic behavior would be sub-optimal yielding explosive behavior asymptotically. Here, this does not happen because the equilibrium only exists when the level of uncertainty is large enough, and, as the second property shows, uncertainty is in a way stabilizing. Indeed, one can see that:

$$\frac{\partial C_1}{\partial \sigma} = b\sigma \left[-1 - \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}} \right] < 0,$$

and

$$\frac{\partial C_2}{\partial \sigma} = b\sigma \left[-1 + \frac{b(\sigma^2 - r)}{\sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}} \right] > 0.$$

As uncertainty increases, player 1's reactivity is bigger while the reverse happens for player 2 (remember again that the two coefficients are strictly negative). Interestingly enough, the unique difference between the symmetric and asymmetric MPE is the unbalanced (marginal) reactions of the players in the latter. In both, player 1 decreases investment in response to a marginal increase in z in the fear of retaliation. When uncertainty becomes large enough, unbalanced reactions become optimal: the fear of retaliation combined with a large enough uncertainty about the future state leads player 1 to strengthen his reaction, player 2 acting in the opposite direction. This might not though lead to convergence towards a pointwise stochastic steady state as in the case of the symmetric equilibrium. We demonstrate here below that the asymmetric equilibrium leads to convergence to a limit invariant distribution.

4.2 Stochastic stability and characterization of the invariant limit distribution

Indeed, the asymmetric equilibrium cannot bring the legislative state almost surely to $z = 0$. We also get this point once we write down the dynamics of z at the MPE:

$$dz = [x_1 - x_2]dt + \sigma z\, dW = \left[\Gamma + (r - \sigma^2)z\right] dt + \sigma z\, dW, \tag{12}$$

with $\Gamma = \frac{a_1(C_2 - C_1)}{b(b\sigma^4 + a_2)}$ and $C_2 - C_1 = \sqrt{b^2(\sigma^2 - r)^2 + 4a_2 b} > 0$.

In the deterministic case, one can see that the stochastic differential equation degenerates into $\dot{z} = \Gamma + rz$. Therefore, no stable asymmetric equilibrium can arise. If uncertainty is low enough (say $\sigma^2 < r$), then the same explosive sub-optimal behaviour also emerges. Under the parametric assumptions of Proposition 2, it can be shown that while $z = 0$ is inaccessible for the $z-$process, it admits a stationary invariant distribution which density is given in the following proposition.[11]

Proposition 3. *At the asymmetric MPE, under the assumptions of Proposition 2, the density function of the stochastic process $z(t)$ almost surely converges to its long-run stationary (time and state invariant) density function $q(z), z \in \mathcal{R} \setminus \{0\}$, which is given by*

$$q(z) = \frac{M}{\sigma^2}\, z^{\,2\left(\frac{r}{\sigma^2}-2\right)}\, \exp\left\{-\frac{2\Gamma}{\sigma^2 z}\right\}, \tag{13}$$

where positive parameter M is chosen such that $\int_{\mathcal{R}\setminus\{0\}} q(z)dz = 1$.

The comparison between Propositions 1, second item, and 3 provides us with highly interesting and sharply contrasted results as to the asymptotic implications of symmetric vs asymmetric MPE. At the symmetric MPE, the outcome comes from the combination of two characteristics of equilibrium strategies. First, players' efforts move along opposite directions, see (9), i.e., one's efforts increase with z and the rival player's efforts decrease with z. Second, their efforts end up balancing each other and the legislative state converges almost surely to $z_\infty = 0$. At the asymmetric MPE, the former characteristic is still present but not the latter. This implies that the z-process will not in general reach a single value in the long run. Instead, it will exhibit an invariant distribution on the real line. As a consequence, the cost of lobbying might in the end be much higher than with the latter class of equilibrium.

To deepen our understanding of the features of this invariant distribution, we can next establish that

Corollary 3. *The invariant distribution satisfies:*

$$\frac{dq(z)}{dz} \lesseqqgtr 0, \quad when \;\; z \gtreqqless \widehat{z} = \frac{\Gamma}{2\sigma^2 - r}(>0).$$

The point $\widehat{z}$ is the most likely position of stochastic lobbying game, which is always positive under the current setting. In other word, applied to our working example, strategic interaction between lobbies, under uncertainty, leads the economy to a position which is mostly in favor of the liberal group.

A more politically-oriented interpretation of the result is that uncertainty in the dynamics of legislation, or institutions, itself may induce opposing groups to adopt strategies that

[11] The proof follows the same arguments as in the Appendix B of Merton (1975).

are not generally compatible with the convergence to the status quo $z_\infty = 0$, that arises at the symmetric MPE. Indeed, following the lobbying strategies of the asymmetric MPE, anything can happen asymptotically, that is, any level of legislation can be achieved with a positive probability, and the economy can end up with very bad or very good political and economic institutions. A second worthwhile result has to do with the sign of the most likely asymptotic state, $\hat{z}$. One can see that $\hat{z} > 0$. This is seemingly consistent with the shape of the asymmetric MPE as depicted here above: player 1 is the most reactive player and as such the most likely state ought to be more favorable to this player. Both players are (marginally) reacting in a stabilizing way: player 1, who loves high z, decreases investment, and player 2, who loves low z, increases investment. Along the unique asymmetric equilibrium, the most reactive player brings (consistently) the most likely state in its preferred region (that's positive z values). Nonetheless, as uncertainty increases, this player decreases more and more his investment while the other player does just the contrary. Clearly, the outcome should be less and less favorable to player 1 as uncertainty increases. This is proved hereafter.

Corollary 4. *Define* $\underline{\sigma^2} = r + 2\sqrt{\frac{-a_2}{b}}$. *For* $\sigma^2 > \underline{\sigma^2}$, *there exists* $\overline{\sigma^2}(> \underline{\sigma^2})$, *such that,*

$$\frac{d\widehat{z}(\sigma^2)}{d\sigma^2} \begin{cases} > 0, & \text{if } \underline{\sigma^2} < \sigma^2 < \overline{\sigma^2}, \\ < 0, & \text{if } \sigma^2 > \overline{\sigma^2}, \end{cases}$$

where $\overline{\sigma^2}$ *is the largest positive root of 3rd degree polynomial equation*

$$F'(y) = -12b^3y^3 + 33b^2ry^2 - 18b^3r^2y + br(2b^2r^2 + 12a_2b - 3a_2) = 0.$$

Therefore, the conclusion is mixed: while the most likely state is in the z-region favorable to player 1, this state becomes less favorable when uncertainty goes up (starting with large enough uncertainty levels), and ought to converge to zero when the level of uncertainty goes to infinity as one can check (see Proof of Corollary 4).

5 Conclusion

In this paper, we have designed a stochastic dynamic game in order to study legislative lobbying. The game is motivated by a series of examples taken from environmental and health policies lobbying in Western countries and from liberalization policies in resource-dependent economies. In both sets of examples, uncertainty and dynamics are essential ingredients which cannot be dismissed. In this context, we prove that many of the results identified for the deterministic counterpart can be reversed under uncertainty. This is due to the emergence of an asymmetric equilibrium for a large enough level of uncertainty. We show in particular that such an equilibrium delivers a much contrasted picture concerning the social cost of lobbying compared to the symmetric equilibria arising in the deterministic world. We also identify some interesting properties for the application cases: along the asymmetric equilibrium, the most likely asymptotic state lies in the region favorable to one player consistently with the asymmetric nature of the equilibrium. However, this position is weakened as uncertainty increases.

A Appendix

A.1 Proof of Propositions 1 and 2

The proof of Propositions 1and 2 can be done altogether, since they both come from the solution of the same stochastic dynamic game. Thus, in the following, we first present the general calculation and then show one by one of the proofs of the propositions.

To make it clear, we restate the the dynamic game as following: objective of player 1 is

$$\max_{x_1} \int_0^{+\infty} F_1(x_1, z)e^{-rt}dt = \int_0^{+\infty} e^{-rt}\left[a_0 + a_1 z + \frac{a_2}{2}z^2 - \frac{b}{2}x_1^2\right] dt$$

and the Objective of player 2 is

$$\max_{x_2} \int_0^{+\infty} F_2(x_2, z)e^{-rt}dt = \int_0^{+\infty} e^{-rt}\left[a_0 - a_1 z + \frac{a_2}{2}z^2 - \frac{b}{2}x_1^2\right] dt,$$

with constants a_0, a_1, b positive and a_2 non-positive. The common state constraint is

$$dz = (x_1 - x_2)dt + \sigma z dW.$$

It is easy to see this is a standard linear-quadratic stochastic differential game. To obtain the stationary MPE, we define the value function of player i as

$$V_i(z) = A_i + B_i z + \frac{C_i}{2}z^2, \quad i = 1, 2,$$

with A_i, B_i, C_i undetermined coefficients. C_i should be negative to ensure strict concavity of the value function as the objective functions are strictly concave in (x_i, z) and the state function is linear. Thus these value functions must check the following Hamilton-Jacob-Bellman equations:

$$rV_i(z) = \max_{x_i}\left[F_i(x_i, z) + \frac{dV_i}{dz}(x_1 - x_2) + \frac{\sigma^2 z^2}{2}\frac{d^2V_i}{dz^2}\right], \quad i = 1, 2. \tag{14}$$

The standard first order (necessary and sufficient) conditions on the right hand sides of (14) yield the optimal choice of player 1 and 2:

$$x_1^* = \frac{1}{b}\frac{dV_1}{dz} = \frac{B_1 + C_1 z}{b} \text{ and } x_2^* = -\frac{1}{b}\frac{dV_2}{dz} = -\frac{B_2 + C_2 z}{b}. \tag{15}$$

Substituting these optimal choices into the right hand sides of equation (14) and comparing the coefficients of term z on both left and right hand sides of (14), we obtain the following equation system for parameters:

$$\begin{cases} rA_1 = a_0 + \frac{B_1^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_1 = a_1 + \frac{B_1 C_1}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_1 = a_2 + \frac{C_1^2}{b} + \frac{2C_1 C_2}{b} + \sigma^2 C_1 \end{cases} \tag{16}$$

and

$$\begin{cases} rA_2 = a_0 + \frac{B_2^2}{2b} + \frac{B_1 B_2}{b}, \\ rB_2 = -a_1 + \frac{B_2 C_2}{b} + \frac{B_1 C_2 + B_2 C_1}{b}, \\ rC_2 = a_2 + \frac{C_2^2}{b} + \frac{2C_1 C_2}{b} + \sigma^2 C_2. \end{cases} \tag{17}$$

Combining the last equation of (16) and (17) together and rearranging terms, it yields

$$\left(r - \sigma^2\right)\ (C_1 - C_2) = \frac{(C_1 - C_2)(C_1 + C_2)}{b}.$$

Thus, two groups of solutions are possible:

$$C_1 = C_2$$

and

$$C_1 \neq C_2, \ \text{ then } \ C_2 = b(r - \sigma^2) - C_1.$$

A.1.1 Proof of Proposition 1

Substituting $C_1 = C_2 = C$ into the last equation of (16) (or (17)), it yields that

$$C_1 = C_2 = \frac{-b(\sigma^2 - r) \pm \sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}{6},$$

which is always real, given $a_2 < 0$. For shortening the notation, we denote the above $C_1 = C_2$ as $C^{(j)}$, $j = 1, 2$, with $C^{(1)}$ taking negative in front of the square root term, while $C^{(2)}$ taking the positive one.

Furthermore, substituting $C_1 = C_2$ into the second equations of (16) and (17), it yields

$$B_1 + B_2 = 0, \ \text{ or } \ B_1 = -B_2.$$

Thus, by the second equation of (16) again, we have

$$B_1 = \frac{a_1}{br - C} = -B_2.$$

Substituting B_i, C_i $(i = 1, 2)$ into the first equations of (16) and (17), we can obtain A_1 and A_2.

To complete the proof of Propositions 1, we notice that for $j = 2$, the value functions are, for $i = 1, 2$,

$$V_i^{(2)} = A_i^{(2)} + B_i^{(2)} z + \frac{C_i^{(2)}}{2} z^2$$

with $C_1^{(2)} = C_2^{(2)} = \frac{-b(\sigma^2 - r) + \sqrt{b^2(\sigma^2 - r)^2 - 12ba_2}}{6} > 0$, given $a_2 < 0$. Thus, the value functions, $V_i^{(2)}$, are strictly convex. Following the arguments in Section 9.2 and 9.4 of Stokey et al (1989), $j = 2$ cannot be an optimal choice to the linear-quadratic games. Therefore, the only optimal symmetric choice is $j = 1$ which is given by Propositions 1.

Substituting now the above two equilibrium strategies into the stochastic state equation, we have

$$dz = [x_1 - x_2]dt + \sigma z \, dW = \frac{2C^{(j)}}{b} z \, dt + \sigma z \, dW,$$

which is a linear homogenous stochastic differential equation with $z = 0$ as one long-run solution. From the $AK-$type model of Boucekkine et al. (2015), it is easy to check that $z = 0$ is almost surely stochastically stable if and only if

$$\frac{2C^j}{b} - \sigma^2 < 0.$$

That completes the proof of Propositions 1.

A.1.2 Proof of Corollary 1 & 2

Impact of uncertainty: long run

For $i = 1, 2$, we have

$$\frac{\partial x_i}{\partial \sigma} = \frac{a_1}{b} \frac{1}{(br - C)^2} \frac{\partial C}{\partial \sigma}.$$

As demonstrated above, we know that

$$\frac{\partial C}{\partial \sigma} < 0,$$

which yields the property highlighted in Corollary 1, $\forall i = 1, 2$,

$$\frac{\partial x_i}{\partial \sigma} < 0.$$

Impact of uncertainty: short run

For player 1, we have:

$$\frac{dx_1^{(j)}}{d\sigma} = \frac{1}{b}\left(\underbrace{\frac{a_1}{(br - C^{(j)})^2}}_{+} + z^{(j)}\right)\underbrace{\frac{\partial C^{(j)}}{\partial \sigma}}_{-} + \frac{C^{(j)}}{b}\, z^{(j)}\, \left[\left(\frac{2}{b}\frac{\partial C^{(j)}}{\partial \sigma} - \sigma\right) t + W_t\right].$$

And for player 2, it follows:

$$\frac{dx_2^{(j)}}{d\sigma} = \frac{1}{b}\left(\underbrace{\frac{a_1}{(br - C^{(j)})^2}}_{+} - z^{(j)}\right)\underbrace{\frac{\partial C^{(j)}}{\partial \sigma}}_{-} - \frac{C^{(j)}}{b}\, z^{(j)}\, \left[\left(\frac{2}{b}\frac{\partial C^{(j)}}{\partial \sigma} - \sigma\right) t + W_t\right].$$

A.1.3 Proof of Proposition 2

Substituting $C_2 = b(r - \sigma^2) - C_1$ into the last equation of (16) and rearranging terms, it follows:

$$C_1^{(j)} = \frac{-b(\sigma^2 - r) \pm \sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}}{2}, \quad j = 3, 4$$

with $C_1^{(3)}$ taking positive sign of the square root term and $C_1^{(4}$ taking negative one. Thus,

$$C_2^{(j)} = b(r - \sigma^2) - C_1^{(j)} = \frac{-b(\sigma^2 - r) \mp \sqrt{b^2(\sigma^2 - r)^2 + 4ba_2}}{2}, \quad j = 3, 4.$$

Remark. To guarantee that the square root term is real, some conditions on the parameters are needed, however, it is not essential for the current study, for example we can assume the absolute value of a_2 is not too large.

Combining the above explicit $C_i^{(j)}$ into the second equations of (16) and (17)), we obtain the $B_i^{(j)}$, $i = 1, 2$ and $j = 3, 4$.

Furthermore, it is easy to prove that $C_i^{(3)} < 0, C_i^{(4)} > 0$ for $i = 1, 2$. Thus, the value function for $j = 4$ is then given by

$$V_i^{(4)}(z) = A_i^{(4)} + B_i^{(4)} z + \frac{C_i^{(4)}}{2} z^2, \quad i = 1, 2,$$

which is strictly convex, similar to the above case of $j = 2$, following again Section 9.2 and 9.4 of Stokey et al (1989), $j = 4$ can not be an optimal choice. That finishes the proof of Proposition 2.

A.1.4 Proof of Proposition 3

Similar as the proof of Proposition 1, substituting the above optimal efforts of both players into the stochastic differential equation, and simplifying terms, we have

$$dz^{(3)} = [x_1^{(3)} - x_2^{(3)}]dt + \sigma z^{(3)}\, dW = \left[\frac{a_1(C_2^{(3)} - C_1^{(3)})}{b(b\sigma^4 + a_2)} + (r - \sigma^2)z^{(3)}\right] dt + \sigma z^{(3)}\, dW.$$

Following Merton (1975, Page 390) that "*a steady state distribution will always exist in the sense that*" $z(t)$ "*will either (1) be absorbed at one of the natural boundaries (i.e. a degenerate distribution with a dirac function for a density)" or (2) it will have a finite density function of the interval*" $(-\infty, +\infty)$ " *or (3) it will have a discrete probability mix of (1) and (2)*".

The same arguments as Merton (1975) by applying the results of Cox and Miller (1968, Page 223-225), that both natural boundaries $\pm\infty$, and middle point $z = 0$ are inaccessible, provided some simple parameter conditions are imposed on a_1, a_2, b, r and σ.

The rest of the proof can be done straightforward following the same arguments as Merton (1975, Page 389-390) by applying the Kolmogorov-Fokker-Planck "forward" equation.

Given $z(t)$ is a diffusion process, its transition density function will satisfy the Kolmogorov-Foller-Planck "forward" equation. Let $Q(z, t; z(0))$ be the conditional probability density for process $z(t)$ at time t, given initial condition $z(0)$. Then the corresponding Kolmogorov-Foller-Planck "forward" equation would be

$$\frac{\partial Q(z,t)}{\partial t} = \frac{1}{2}\frac{\partial^2}{\partial z^2}\left(\sigma^2 z^2 Q(z,t)\right) - \frac{\partial}{\partial z}\left[\left(\frac{a_1(C_2 - C_1)}{b(b\sigma^4 + a_2)} + (r - \sigma^2)z\right) Q(z,t)\right].$$

Suppose $z(t)$ has a steady state distribution, which is independent of initial condition $z(0)$. then

$$\lim_{t\to+\infty} Q(z,t) = q(z), \quad \lim_{t\to+\infty} \frac{\partial Q(z,t)}{\partial t} = 0,$$

and $q(z)$ satisfies

$$\frac{1}{2}\frac{d^2}{dz^2}(\sigma^2 z^2 q(z)) - \frac{d}{dz}\left[\left(\frac{a_1(C_2 - C_1)}{b(b\sigma^4 + a_2)} + (r - \sigma^2)z\right) q(z)\right] = 0.$$

The last equation can be easily solved by the standard variation of coefficient method in ordinary differential equation. Combining with the above arguments that boundaries $\pm\infty$ are inaccessible, then following Merton (1975, Page 390), the steady state density function must be given as in the Proposition 3.

Finally, differentiating the density function (13) with respect to z yields that

$$\frac{dq(z)}{dz} = \frac{M}{\sigma^2}\, \exp\left\{-\frac{2\Gamma}{\sigma^2 z}\right\}\, z^{\,2\left(\frac{r}{\sigma^2}-6\right)}\, \left[\frac{2\Gamma}{\sigma^2} + 2\left(\frac{r}{\sigma^2} - 2\right) z\right].$$

Thus,

$$\frac{dq(z)}{dz} \lesseqqgtr 0, \;\; when \;\; z \gtreqqless \widehat{z} = \frac{\Gamma}{2\sigma^2 - r}(> 0).$$

This leads to the statement in Corollary 3.

A.1.5 Proof of Corollary 4

Rewrite $\hat{z}(\sigma^2)$ as

$$\hat{z}(\sigma^2) = \frac{a_1}{b}\sqrt{\frac{b^2(\sigma^2 - r)^2 + 4a_2 b}{(2\sigma^2 - r)^2(b\sigma^4 + a_2)^2}} = \frac{a_1}{b}\sqrt{X(\sigma^2)}$$

with

$$X(\sigma^2) = \frac{b^2(\sigma^2 - r)^2 + 4a_2 b}{(2\sigma^2 - r)^2(b\sigma^4 + a_2)^2}.$$

So,

$$\frac{d\hat{z}(\sigma^2)}{d\sigma^2} = \frac{a_1}{b}\,\frac{1}{2X(\sigma^2)}\,\frac{dX(\sigma^2)}{d\sigma^2}$$

and

$$sign\left(\frac{d\hat{z}(\sigma^2)}{d\sigma^2}\right) = sign\left(\frac{dX(\sigma^2)}{d\sigma^2}\right).$$

Direct calculation yields that

$$\begin{aligned}\frac{dX(\sigma^2)}{d\sigma^2} &= \frac{2}{(2\sigma^2-r)^3(b\sigma^4+a_2)^3}\Bigg[\underbrace{b^2(\sigma^2-r)(2\sigma^2-r)(b\sigma^4+a_2)}_{+} \\ &\qquad - \underbrace{2(b^2(\sigma^2-r)^2+4a_2b)(b\sigma^2(3\sigma^2-r)+a_2)}_{+}\Bigg] \\ &\equiv \frac{2}{(2\sigma^2-r)^3(b\sigma^4+a_2)^3}\; F(\sigma^2).\end{aligned}$$

It is easy to check that the 4th degree polynomial $F(y)$ checks

$$F((\underline{\sigma^2}) = b^2(\underline{\sigma^2}-r)(2\underline{\sigma^2}-r)(b\underline{\sigma^4}+a_2) > 0 \text{ and } \lim_{y\to\pm\infty} F(y) = -\infty.$$

Furthermore,

$$\frac{dX(\underline{\sigma^2})}{d\sigma^2} = \frac{2b^2(\underline{\sigma^2}-r)}{(2\underline{\sigma^2}-r)^2(b(\underline{\sigma^2})^2+a_2)^2} > 0 \tag{18}$$

shows that $\underline{\sigma^2}$ is located on some increasing part of curve $\hat{z}(\sigma^2)$.

We can conclude from the above analysis that:

Remark 1. $F(\sigma^2)$ has at least 2 real roots: one is larger than $\underline{\sigma^2}$ and one is smaller than $\underline{\sigma^2}$.

Moreover, it is straightforward that

$$\hat{z}(\underline{\sigma^2}) = 0 \text{ and } \lim_{\sigma^2\to+\infty} \hat{z}(\sigma^2) = 0. \tag{19}$$

Therefore, the above Remark 1, condition (18) and (19) yield that, for $\sigma^2 > \underline{\sigma^2}$, it only can happen that

$$\frac{dX(\sigma^2)}{d\sigma^2}\begin{cases} > 0, & \text{if } \underline{\sigma^2} < \sigma^2 < \overline{\sigma^2}, \\ < 0, & \text{if } \sigma^2 > \overline{\sigma^2}, \end{cases}$$

where $\overline{\sigma^2}$ is the largest positive root of 3rd degree polynomial equation

$$F'(y) = -12b^3y^3 + 33b^2ry^2 - 18b^3r^2y + br(2b^2r^2 + 12a_2b - 3a_2) = 0.$$

That finishes the proof.

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