FACULTY OF LAW, ECONOMICS AND FINANCE

CREA
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Paper

2010-15

Center for Research in Economic Analysis
University of Luxembourg

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available online : http://wwwfr.uni.lu/recherche/fdef/crea/publications2/discussion_papers/2010

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December 2010

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Prices and Deadweight Loss in Multi-Product Monopoly[1]

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JEL Classification Number: L12, D42

Key words: multi-product monopoly prices, interdependent products, pricing complements, pricing substitutes, linear demand

Abstract: This paper investigates prices and deadweight loss in multi-product monopoly with linear interrelated demands and constant marginal costs. We show that with commonly used models for linear demand, such as the Bowley demand, vertically or horizontally differentiated demand, the price for each good is independent of demand cross-effects and of the number and characteristics of other goods. This elementary finding contrasts with the existing perception in a segment of the literature that prices critically depend on demand cross-effects. We also show that for these linear models, the deadweight loss due to monopoly pricing always amounts to half the total monopoly profit.

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[1] Financial support from EGQ project F2R-CRE-PUL-10EGQH at the University of Luxembourg is gratefully acknowledged.

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1. Introduction

Most firms sell not one, but many interrelated products. For example supermarkets sell a multitude of substitute, complement or independent goods. Airlines and railway companies sell tickets with different conditions for the same route and oil corporations sell gasoline in petrol stations that differ by their locations.

This paper examines multi-product monopoly (MPM) facing linear demand for differentiated goods and constant unit costs and shows that optimal prices and welfare can be expressed in a very simple way: As in the textbook case of a single product monopoly, *the monopoly price of each good is the average of its own inverse demand intercept and its own marginal cost, and is independent of the characteristics of other products, the interactions between products and the number of products sold.* In contrast to this simple conclusion, a common view in economics and in marketing is that monopoly prices critically depend on cross demand effects.[2] In particular, there seems to be a somewhat widespread misconception that MPM prices should be lower for complements and higher for substitutes, relative to independent goods (see Section 2). Though intuitively plausible at first sight, this conclusion is actually invalid!

We obtain these elementary results for MPM facing three commonly used linear demand structures, corresponding to the three examples cited above: The standard Bowley or Shubik-type models for demand with heterogeneous products, vertically (quality) differentiated products and horizontally (spatially) differentiated products. As seen below, this multitude of demand models is motivated by the diversity of economic settings where the issue of MPM pricing has been historically analyzed, often by founding fathers of modern industrial economics. We demonstrate that our basic insights also carry over qualitatively to MPM pricing under non-linear demand.

Our result on MPM prices is interesting in its own right, but it can also be used to shed light on a number of further issues. In particular, we show that this result can be of help in the complex but important problem of estimating deadweight loss in multiproduct monopoly. In a single product monopoly with linear demand, deadweight loss is half the monopoly profit. Exploiting the property that prices of existing goods do not change when a new product is added to a product line, we can show that that monopoly profit and the deadweight loss always rise proportionally. Consequently deadweight loss in MPM will also be half the monopoly profit regardless of how many, or what types of, products are added. This surprising result holds in all the models of linear demand we consider, despite the significant differences in their welfare functions.

Based on an extensive literature search going back to the beginnings of neo-classical economics, we believe that these simple properties of linear MPM have not been uncovered. An early reference to the problem of MPM is from Wicksell (1901, 1934) who argues that "*every retailer possesses, within his immediate circle, what we call an actual sales monopoly*" (Wicksell, 1934, p. 87). Without solving for the monopoly prices, he recognizes that they are "*complex and … difficult to unravel*" (p. 86). Similarly, Edgeworth (1925) analyzes railway fares of different classes but does not give an explicit solution. Hotelling (1932) provides a numerical example where a monopoly chooses the profit-maximizing prices for first and second-class railway tickets facing different consumer groups, but he does not solve for a general price rule

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[2] For example Reibstein and Gatignon (1984) argue in a seminal marketing paper that "The optimal price

since this was not the main focus of his paper. Robinson (1933) formally solves the problem of a monopolist selling in different markets, but explicitly excludes price interdependence such as in "*the case of first- and third-class railway fares, analyzed by Edgeworth*" (Robinson, 1933, p. 181).

Coase (1946) goes beyond Robison's analysis and examines monopoly prices for two interrelated products using verbal and graphical arguments but does not provide a mathematical solution. Using a similar approach, Holton (1957) considers the MPM problem in the context of a supermarket selling interrelated products. He does not solve for optimal prices either, but argues that *"supermarket operators do indeed establish prices with not only price elasticities but cross-elasticities in mind"*, in contrast to our main conclusion. Finally Selten (1970) addresses the problem of MPM pricing with linear demand without recognizing the simple properties of monopoly prices.[3]

While the complete solution of MPM pricing seems to have eluded attention, partial results have emerged in the marketing literature. Shugan and Desiraju (2001) show that monopoly prices of two vertically differentiated products do not depend on each other's costs. Moorthy (2005) and Besanko et. al. (2005) find that with linear demand MPM prices do not respond to cost changes of other products. Neither of these studies derives the general solution for prices or explores the full scope of MPM pricing.

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is extremely sensitive to the inclusion or the exclusion of the cross-elasticities" (p.266).

[3] Several authors have pointed out the similarity of the MPM problem to the Ramsey tax (Ramsey 1927), which maximizes social welfare for a certain level of tax revenue. Yet the Ramsey problem is not identical to unconstrained monopoly profit maximization. For example Ten Raa (2009) shows that the structure of monopoly prices often differs from that induced by the Ramsey tax.

We think that our results can be useful in various contexts. A straightforward practical implication is that even in the presence of strong product interactions, neglecting such relations is part of good pricing practice for a monopolist. Cross-subsidization will normally not be optimal for a non-regulated monopolist, at least under linear demand (Baumol, Panzar and Willig, 1982).[4] Besides its managerial relevance this insight also provides a theoretical justification for research in economics and marketing analyzing retail prices in a single product context.

Our welfare results could serve as a simple and practical benchmark helping antitrust authorities estimate the social loss of MPM. If demand functions can be considered as approximately linear, one need not analyze in detail each product's price elasticity and its cross-elasticities with all other products. The social cost of MPM can simply be estimated by looking at the company's profit. For example this approach could help to evaluate the social cost of a local retail monopoly. Similarly the deadweight loss caused by a railway monopoly can be estimated from the company's profit without having to analyze the qualities and prices of the different tickets offered.

Finally our results have implications for joint profit maximization by oligopoly firms. Jointly maximizing the total profit is mathematically equivalent to MPM pricing. Our findings indicate that with linear demand, even if products exhibit strong interdependences, oligopoly firms do not need any information about their competitors' products and costs in order to set the prices that jointly maximize the total profit. This gives static collusion a decentralized flavor that might be relevant in practice.

[4] Note that it might be optimal to price goods below marginal costs if their demand intercept is negative i.e. if they cannot be sold independently without complement goods.

The paper is organized as follows. Section 2 provides examples in the literature of misconceptions regarding MPM pricing. Section 3 derives the profit-maximizing prices for general linear MPM. In Section 4, this result is applied to three different demand models to show that in each case MPM prices are independent of product interactions. Section 5 analyzes the relation between the deadweight loss and monopoly profits, and Section 6 deals with the case of non-linear demand. Section 7 briefly concludes.

2. Illustrative Examples

An elementary fallacy in basic monopoly theory holds that a firm selling two complementary products will charge less for each than if each product were sold alone. Alternatively, it claims that the monopoly price of a given good is lower when it is sold alone than when it is sold together with a substitute. In its most succinct form, this conventional wisdom can be presented within the standard two-good paradigm.

Consider a representative consumer with utility function $U(x_1,x_2) = a(x_1 + x_2) - 0.5b(x_1^2 + x_2^2) + gx_1x_2 + y$, where y is income, and $|g| < b$. This gives rise to the standard symmetric inverse demand function $p_i = a - bx_i + gx_j$ (Bowley, 1924). The corresponding direct demand is then $x_i = a/(b - g) - (bp_i + gp_j)/(b^2 - g^2)$, $i, j = 1, 2$. While this can also be written as (Singh and Vives, 1984) $x_i = \alpha - \beta p_i + \gamma p_j$, with $\alpha = a/(b - g)$, $\beta = b/(b^2 - g^2)$ and $\gamma = -g/(b^2 - g^2)$, it is important to observe that the constants α, β and γ are not autonomous here. In contrast, a, b and g are, except for the restriction that $|g| < b$.

Using the demand functions in the form $x_i = \alpha - \beta p_i + \gamma p_j$, $i, j = 1, 2$, and unit cost c for both products, one obtains both monopoly prices as $p^* = 0.5[c + \alpha/(\beta - \gamma)]$.

Then, so goes the fallacy, this price is higher with substitute goods ($\gamma > 0$) and lower with complements ($\gamma < 0$), relative to the case of independent goods ($\gamma = 0$).[5] This would be correct if α and β remained constant when γ changes, which however is not the case. Indeed, using the relations between Greek and Roman letters (Singh and Vives, 1984), we obtain $\alpha/(\beta - \gamma) = a$, which is the intercept of inverse demand, or the consumer's willingness to pay at zero consumption i.e., $\partial U(0,0)/\partial x_i$, indeed a primitive constant.[6] Expressing the prices with the parameters of the inverse demand function we obtain $p^* = 0.5(c + a)$, which is also the optimal price for a monopolist selling only good i (facing inverse demand $p_i = a - bx_i$ and unit cost c). With linear demand, a monopolist selling two goods sets each price as if it were the only good sold. In other words, *pricing is fully independent of the (substitute/complement) relations between the two goods*.

On the other hand, optimal outputs do depend on these relationships. Indeed, maximizing total profits with respect to outputs, one gets $x^* = (a - c)/2(b - g)$, which is higher for complements ($g > 0$) and lower for substitutes ($g < 0$), relative to independent products ($g = 0$). With two perfect substitutes (i.e. $g = -b$), the optimal output is half the usual monopoly optimal output. This is fully in line with standard economic intuition. [7]

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[5] For example Motta (2004, p. 537) relies on this analysis to argue that, "*relative to the benchmark case where the two products are independent (...) the monopolist reduces the price of its products when they are complements (...) and it increases them when they are substitutes (...). The intuition for this result is straightforward. When the products are complements they exercise a positive externality on each other and the monopolist internalizes it by decreasing its prices*".

[6] In contrast, α is the quantity demanded under zero prices. In the presence of substitutes (complements) one would expect it to be lower (higher), relative to the case of independent products.

[7] If output is expressed in terms of the parameters of the direct demand function we obtain $x^* = 0.5[\alpha - (\beta - \gamma)c]$, which seems to imply that output is higher for "substitutes" ($\gamma > 0$). In particular, if two

A key broader implication of this elementary example is that one should a priori view with suspicion any comparative statics with respect to changes in a parameter of the direct demand function. Such changes cannot be viewed in isolation, without due consideration of the interdependences with other demand parameters.

With the two standard ways of solving the monopoly problem using direct or indirect demand functions, the simple expression for monopoly prices can be easily overseen. Allen (1938) uses the direct demand function to solve for the monopoly prices in the linear two-good example but fails to see that the solution can be simplified if it is expressed in terms of the parameters of the inverse demand function. Selten (1970) discovered that in general linear MPM the optimal quantities can be "expressed in a surprisingly simple way" (Selten, 1970, p. 52) as half of the socially optimal level[8] but he too does not derive the full solution for monopoly prices. Note, that contrary to our result on prices, Selten's result does not allow to derive simple comparative statics, as optimal quantities still depend in a complex way on cross demand effects. Similarly, when the indirect demand functions are used to solve for the optimal monopoly quantities, our simple solutions will only appear if one then substitutes these quantities back into the demand function. Varian (2006, p. 455) uses this approach for solving a numerical example without identifying the simple formula for prices.

Another line of reasoning that is sometimes used to justify the conventional wisdom is as follows. Considering MPM with general non-linear demand, one easily

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goods are close to perfect substitutes, the monopoly will sell at least twice as much as it sells one good alone, clearly in violation of standard intuition.

[8] A similar result has been obtained by Ramsey (1927) for revenue maximizing taxes in a competitive market with linear demand, but the connection to Selten's result has not been recognized. In fact the similarity breaks down when marginal costs are not constant.

derives the optimal Lerner index, $(p_i - c_i)/p_i$, as $1/\varepsilon_{ii} - \sum_{j\neq i}(p_j - c_j)x_j\varepsilon_{ij}/(\varepsilon_{ii}p_ix_i)$, where ε_{ii} and ε_{ij} are the own and cross price elasticities.[9] When every cross elasticity ε_{ij} is zero, we obtain the single-good monopoly condition $(p_i - c_i)/p_i = 1/\varepsilon_{ii}$. When the goods are substitutes, we have $\varepsilon_{ij} < 0$, and the price p_i would appear to be higher compared to the corresponding price in a separate (single-good) market.[10] Again this argument would be correct only if elasticity ε_{ii} were to remain the same as we add new products. In the presence of substitute goods, the quantity demanded for a given product will fall and the value of ε_{ii} will rise. A higher ε_{ii} offsets the impact of ε_{ij}'s, and pushes p_i in the opposite direction. With linear demand, these two opposite effects exactly offset each other. *This provides the key economic intuition behind the simple result on MPM pricing*. We will show that with non-linear demand, a MPM may indeed *price complementary goods even higher than independent goods* (see example in Appendix E)!

A further complicating issue regarding MPM pricing is that the term "monopoly" has historically also been applied to firms selling differentiated goods. Cournot (1838, p. 80) and Allen (1938, p. 361) compare a multiproduct monopoly to what we would now call a duopoly selling the same (complement or substitute) products, and refer to the duopoly as "two independent monopolists". Obviously, relative to MPM prices, prices set by a duopoly are lower for substitutes and usually higher for complements, but this is not the question addressed in this paper. Rather, we

9 See e.g., Tirole (1988, p. 70).

10 For example Betancourt (2004, p. 94) carries out this analysis and concludes: *"... if two items are gross substitutes (ε_{21} >0), the price of item 1 will be higher in the multiproduct setting than it would have been in the single product one (...). On the other hand if they are gross complements (ε_{21} >0), the price of item 1 in the multiproduct setting will be lower than it would have been in the single product one (...)."*

compare the monopoly price of a given product when it is sold alone to the price when it is sold together with a complement or substitute, in all cases by the same monopoly firm.

We now begin a general investigation of MPM with linear demand, the ultimate aim being to show that the conclusions of Section 2 and our general welfare result hold for three commonly used linear demand structures.

3. The General Linear Model

We start by analyzing MPM pricing for a general linear demand model, which is seen in Section 4 to encompass three commonly used but different linear demand structures. We refrain at this stage from specifying a precise microeconomic foundation for such a demand system, as we know of none that would cover all of our applications.

We consider a monopoly selling n products with constant marginal costs. Prices, quantities and marginal cost are denoted by p_i, x_i, and c_i respectively, $i = 1,\ldots, n$. The corresponding vectors for all n products are written as bold $\boldsymbol{p}$, $\boldsymbol{x}$ and $\boldsymbol{c}$. The linear demand function is specified by a constant Jacobian matrix $\partial \mathbf{x}/\partial \boldsymbol{p} = \boldsymbol{A}$, and a constant $n\times1$ vector $\boldsymbol{\alpha}$, representing the vector of quantity demanded when all prices are zero, as

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p} \qquad (1)$$

When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{x}(\boldsymbol{c})$. We assume $\boldsymbol{A}$ is negative definite and symmetric, i.e., its elements $a_{ij} = a_{ji}$ for all i, j. The diagonal elements of $\boldsymbol{A}$ are all negative, i.e., $a_{ii} < 0$ for all $i,$ as the demand for every good is downward sloping. The off-diagonal elements, however, can be positive, negative or zero, according to products being substitutes, complements or independent. Given the demand function (1) and marginal costs vector $\boldsymbol{c} = (c_1,\ldots, c_n)$, we can write the monopoly profit as

$$\pi(\boldsymbol{p}) = (\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}) \qquad (2)$$

We will demonstrate that the monopoly prices can be expressed in a simple way using the vector of demand intercepts $\boldsymbol{p}^0$, which is the (minimal) price vector that exactly reduces demand for all products to zero. As matrix $\boldsymbol{A}$ is invertible, this vector is uniquely defined by $\boldsymbol{x}(\boldsymbol{p}^0) = \boldsymbol{\alpha} + \boldsymbol{A}\boldsymbol{p}^0 = \boldsymbol{0}$, i.e., $\boldsymbol{p}^0 = -\boldsymbol{A}^{-1}\boldsymbol{\alpha}$.

PROPOSITION 1: The profit-maximizing prices are $\mathbf{p}^* = 0.5(\mathbf{p}^0 + \mathbf{c})$. *Under these prices only half of the socially optimal quantity of every good is sold.*

Proof: see Appendix A.

The impact of all the parameters of the demand function (1) on the monopoly price vector $\boldsymbol{p}^*$ is summarized in the demand intercept $\boldsymbol{p}^0$. This is similar to the single product case with linear demand, where the monopoly price does not depend on the slope. Hence, Proposition 1 can be interpreted as a generalized version of the solution to a single-good or a two-good monopoly (see Section 2). It implies that, in general linear monopoly, only 50% of a product's cost change is passed on to its price, and that a cost change for one product does not affect the prices of other products. This conclusion corroborates the findings by Moorthy (2005) and Besanko et al (2005).

The optimal price vector satisfies our key property - independence of inter-product relations (substitutes or complements) - whenever $\boldsymbol{p}^0$ has the same property. The latter in turn depends on the microeconomic model invoked to derive demand. As seen in Section 2, for the standard quadratic utility with two products, $\boldsymbol{p}^0$ indeed has the desired property. In the next section, we argue that this property holds generally for three widely used distinct models of product differentiation in industrial organization.

The intuition behind Proposition 1 can be seen more clearly when the problem is set up in quantities. A small change Δx_1 for good 1 (say), keeping the other quantities constant, has two effects on profits: a direct quantity effect $(p_1 - c_1)\Delta x_1$, and an indirect effect through price changes $\Delta x_1 \sum_{i=1}^{n} (\partial p_i / \partial x_1) x_i$. If profit is maximized, the two effects must exactly offset each other, i.e., $p_1 - c_1 + \sum_{i=1}^{n} (\partial p_i / \partial x_1) x_i = 0$. Given our definition of $\boldsymbol{p}^0$, good 1's inverse demand function is $p_1 = p_1^0 + \sum_{i=1}^{n} (\partial p_1 / \partial x_i) x_i$. By demand symmetry, $\partial p_1 / \partial x_i = \partial p_i / \partial x_1$, so that we have $p_1 - p_1^0 = \sum_{i=1}^{n} (\partial p_i / \partial x_1) x_i$. Hence, $p_1 - c_1 + p_1 - p_1^0 = 0$, which directly yields the optimal price $p_1^* = 0.5(p_1^0 + c_1)$.

4. Price Independence

In this section we show that the result derived in Section 3 can be applied to MPM facing three commonly used models of linear demand for differentiated products, each having its own microeconomic foundation. In all three cases we obtain simple profit-maximizing prices, which are independent of other products and product relations.

4.1 Heterogeneous products:

We first look at one of the standard models for heterogeneous products, a generalized Bowley-type demand with a mixture of substitute, complement and independent goods. There is a continuum of consumers indexed by a parameter λ, which can be interpreted as the marginal utility of income, with density function $f(\lambda)$. Each consumer λ has a utility function $h + (\boldsymbol{a}'\boldsymbol{y} - 0.5\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})/\lambda$, where h is the numeraire good whose price is 1, $\boldsymbol{y}$ is the consumption bundle of the monopoly products, $\boldsymbol{a}$ is an $n\times 1$ positive vector and $\boldsymbol{B}$ is an $n\times n$ matrix. Without loss of generality, let $\boldsymbol{B}$ be symmetric. We assume it to be positive definite so that the utility function is concave.

Each consumer chooses $\boldsymbol{y}$ to maximize utility subject to a budget constraint $h + \boldsymbol{p}'\boldsymbol{y} = m$. The first-order condition of utility maximization, $\boldsymbol{a} - \boldsymbol{B}\boldsymbol{y} - \lambda\boldsymbol{p} = \boldsymbol{0}$ yields an individual demand function $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{p})$. We denote the average λ, $\int \lambda f(\lambda)d\lambda$ by $\bar{\lambda}$. Integrating all $\boldsymbol{y}$ we get the aggregate demand function:

$$\boldsymbol{x}(\boldsymbol{p}) = \boldsymbol{B}^{-1}(\boldsymbol{a} - \bar{\lambda}\boldsymbol{p}) \qquad (3)$$

This demand function (3) follows our general version (1) with $\boldsymbol{B}^{-1}\boldsymbol{a} = \boldsymbol{\alpha}$, and $\bar{\lambda}\boldsymbol{B}^{-1} = -\boldsymbol{A}$. To ensure an interior solution we require the following condition.

ASSUMPTION 1: *For any λ, $\boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{c}) > \boldsymbol{0}$.*

Assumption 1 implies that when all prices are equal to marginal costs, every consumer has a positive demand for every product. This ensures that the demand function (3) is valid under the monopoly price. As $\boldsymbol{B}$ is symmetric and positive definite the demand function (3) satisfies the requirement of Proposition 1. The vector $\boldsymbol{p}^0$ of maximum prices is also easy to determine. As $\boldsymbol{B}^{-1}$ has full rank, $\boldsymbol{x}(\boldsymbol{p})$ is zero when $\boldsymbol{a} - \bar{\lambda}\boldsymbol{p} = \boldsymbol{0}$, so the maximum price vector $\boldsymbol{p}^0 = \boldsymbol{a}/\bar{\lambda}$. We obtain:

PROPOSITION 2: The MPM price for good i is $p_i^* = 0.5(c_i + a_i/\bar{\lambda})$.

As $a_i/\bar{\lambda}$ and c_i only depend on good i, product relations do not affect the optimal price. Note that $a_i/\bar{\lambda}$ can be interpreted as the marginal utility of product i to an average consumer when her consumption of all products is zero. This should not depend on product relations. If the monopolist can estimate this value, he can choose the optimal price easily, independently of how many goods he sells and how large their cross-elasticities are, in full contrast to the conclusions reached by Holton (1957).

If the monopoly sells each good in an independent market, good *i*'s demand function will reduce to $(a_i - \bar{\lambda} p_i)/b_i$ and the optimal price will be $0.5(c_i + a_i/\bar{\lambda})$, which is identical to the MPM price. So the MPM achieves optimal price coordination when it acts as if it were selling *n* products in *n* separate markets. Product interdependence does not have any influence on the prices.[11]

Unfortunately we cannot apply this result to situations such as "*first- and third-class railway fares*" analyzed by Edgeworth (1925) and Hotelling (1932). Demand with vertically differentiated products relies on different micro foundations and we cannot be sure that the demand intercept $\boldsymbol{p}^0$ is independent of inter-product relations and the characteristics of other products. However, in the next sub-section we show that this is indeed the case and thus our result also applies to vertically differentiated products.

4.2 Vertically differentiated products

We consider a model of *n* (≥ 2) vertically differentiated products where product *i* has quality q_i.[12] Without loss of generality, let $q_{i+1} > q_i$ for all *i*, so that q_n indicates the highest quality and q_1 the lowest. There is a continuum of consumers indexed by θ, which is uniformly distributed on [0,1]. Each consumer θ purchases at most one good.

[11] Linearity is not always necessary for this result. For instance, if $p_i = a - bx_i^{\sigma} - rx_i^{0.5(\sigma-1)}x_j^{0.5(\sigma+1)}$, $\sigma > 0$, the price is $(\sigma a + c)/(1 + \sigma)$, same as in a separate market ($p_i = a - bx_i^{\sigma}$).

If he buys good i at price p_i, he obtains a surplus of $\theta q_i - p_i$. The interpretation of $1/\theta$ is similar to the previous marginal utility of income λ. Each consumer chooses the product with the highest surplus, provided it is non-negative. Mussa and Rosen (1978) study the monopoly price problem in a more general version of our model where the monopolist also determines quality. They do not obtain explicit price solutions, in view of their non-linear demand functions.

We need some assumptions to avoid technical difficulties. To ensure that every product has positive demand, we assume that the marginal cost of any product increases with its quality, while the consumer benefit increases more. Also the returns of quality are diminishing, i.e. marginal costs increase with quality at an increasing rate. If we write the marginal cost c_i of a product with quality q_i as $c(q_i)$, this translates into:

ASSUMPTION 2: *For any q,* $0 < c'(q) < 1$ *and* $c''(q) > 0$.

We can determine the demand for a given good with quality q_i by identifying the highest and lowest type of consumers buying this good. The marginal consumer who is indifferent between buying product 1 and buying nothing gets a surplus $\theta_1 q_1 - p_1 = 0$, so all consumers with an index lower than $\theta_1 \equiv p_1/q_1$ will not buy any product. For consumer θ_i indifferent between buying products i and $i-1$ we have $\theta_i q_{i-1} - p_{i-1} = \theta_i q_i - p_i$, so $\theta_i \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$. If $\theta_i < \theta_{i+1}$ for all $i < n$, and $\theta_n < 1$, we obtain positive demand for all goods as $x_i = \theta_{i+1} - \theta_i$ for $i < n$ and $x_n = 1 - \theta_n$. We will show that these conditions hold at the MPM prices. Substituting θ_is into these demand functions we get:

[12] See Mussa and Rosen (1978), Gabszewicz and Thisse (1979), and Shaked and Sutton (1982).

$$x_1 = \frac{p_2 - p_1}{q_2 - q_1} - \frac{p_1}{q_1}, \qquad x_n = 1 - \frac{p_n - p_{n-1}}{q_n - q_{n-1}},$$

$$x_i = \frac{p_{i+1} - p_i}{q_{i+1} - q_i} - \frac{p_i - p_{i-1}}{q_i - q_{i-1}} \qquad \text{for } 1 < i < n \tag{4}$$

It is easy to see that (4) is linear with a symmetric Jacobian matrix, so Proposition 1 applies. To find the MPM price the only information we need is the vector of demand intercepts $\boldsymbol{p}^0$. One can see that the demand for each product is zero when $p_i = q_i$ for all i. So $\boldsymbol{p}^0$ is equal to the vector of product qualities $\boldsymbol{q}$. In Appendix B we verify that the Jacobian matrix is negative definite and each good has a positive demand.

PROPOSITION 3: The price for good i in MPM with vertically differentiated products is $p_i^* = 0.5(c_i + q_i)$.

Proof: see Appendix B.

The monopoly price is simply the average of a product's quality and cost. It is again independent of other products' characteristics. Hence, the prices for "*first- and third-class railway fares*" only depend on the quality and cost of the service offered, not on those of other classes. In particular the prices are the same as the single good monopoly prices, i.e. the price if the monopoly only offers one class of tickets. In this case demand is $x_i = 1 - p_i/q_i$, and its optimal price is $0.5(c_i + q_i)$, which is identical to the MPM price.

According to Proposition 1 the monopoly only sells half the quantities sold in a competitive market. In a model of vertically differentiated products every consumer acquires at most one product. This means that compared to a competitive market, in monopoly some consumers switch to lower quality goods and in total fewer consumers will be served. While in the previous model each consumer buys half of the quantity of the social optimum, here the number of customers being served falls by half.

4.3 Horizontally differentiated products

We analyze a model of spatially (horizontally) differentiated products. The Hotelling model and its various extensions have been widely used to analyze oligopoly competition and location choices. Tirole (1988, p. 140) discusses spatial discrimination by a monopolist selling one product. However, little seems to be known about how a monopolist sets prices for a fixed number of products with predetermined locations. We show that each of these prices are again independent of the features of other products.

We construct an extended version of the Hotelling model, which can be nested in our linear framework. Our model can be visualized as a star-shaped city with n (≥ 2) selling locations owned by a monopolist. The city has $n - 1$ roads radiating from the center and stretching indefinitely into suburbs. There is one shop at the city center and one branch shop along each road with one unit distance from the center. We do not address the question of how to choose locations but simply examine how a MPM sets profit-maximizing prices at these different shops. We assume that the central shop offers consumers a value v_1 at a price p_1, while branches offer v_i at p_i, for $i > 1$. Consumers reside along each road with uniform density. Each consumer incurs a unit travel cost τ, and maximizes his surplus $v_i - p_i - \tau s$, where s is distance.[13]

To ensure an interior solution where every shop has a positive demand under MPM prices, we need certain conditions. On one hand the shops' net values need to be

[13] Chen and Riordan (2007) analyze an oligopoly version of this model with full symmetry across firms, and hence no firm at the center.

sufficiently high relative to the travel cost so that all consumers between the centre and branch shops are covered. On the other hand, the differences between the net values of the centre and branch shops should be sufficiently small so that every shop can sell something. These requirements lead to the following conditions.

ASSUMPTION 3. $|v_1 - c_1 - v_i + c_i| < \tau < 0.2(v_1 - c_1 + v_i - c_i)$ *for all* $i > 1$.

In equilibrium no shop can charge a price higher than the value it offers, so we have $p_i < v_i$ for all i. If a branch shop can sell anything, we must have $v_i - p_i + \tau > v_1 - p_1$. Under these conditions we can derive the demand functions by identifying marginal consumers indifferent between buying at the center or a branch shop and those indifferent between a branch and buying nothing. For the former marginal consumers, we have $v_1 - p_1 - \tau y_i = v_i - p_i - \tau(1 - y_i)$, where y_i is the distance to the centre. Thus demand for the central shop $y_i = 0.5(v_1 - p_1 + p_i - v_i + \tau)/\tau$. Shop i serves the remaining $1 - y_i$ customers, but also attracts clients from the suburb up to a distance z_i, which is determined by $v_i - p_i - \tau z_i = 0$, so $z_i = (v_i - p_i)/\tau$. If $0 < y_i < 1$ for all $i > 1$, the demand function for the center $x_1 = \sum_{i=2}^{n} y_i$, and for branch shop i, $x_i = 1 - y_i + z_i$, i.e.,

$$x_1 = \frac{n-1}{2\tau}(\tau + v_1 - p_1) - \sum_{i=2}^{n} \frac{v_i - p_i}{2\tau},$$

$$x_i = \frac{\tau + 3v_i - 3p_i - v_1 + p_1}{2\tau} \qquad \text{for } i > 1. \qquad (5)$$

Again (5) is linear in prices and the Jacobian matrix is symmetric. We also prove that this matrix is negative definite (Appendix C). One can verify that the demand for every good is zero when $p_1^0 = v_1 + 2\tau$ and $p_i^0 = v_i + \tau$ for any $i > 1$.[14] With these demand intercepts we can apply Proposition 1 and obtain the monopoly prices.

PROPOSITION 4: The MPM prices with horizontally differentiated products are $p_1^* = 0.5(v_1 + c_1) + \tau$, and $p_i^* = 0.5(v_i + c_i + \tau)$ for $i > 1$.

Proof: see Appendix C.

The monopoly prices cannot be characterized by a single formula here, as the center shop differs from the others. Nonetheless, all prices again only depend on shop-specific parameters, not on other shops' values or costs. In fact, this property can be generalized to a model with different distances between the centre and branch shops.[15]

Similarly to the previous two cases, every shop only sells half of the socially optimal quantity. This is somehow surprising, because the market always covers all consumers between the center and branch shops. Only suburban residents stop buying any products due to monopoly pricing.

In the previous two models, every price is equal to the "naïve" monopoly prices, charged in independent markets or for a single product monopoly. In this case, if a branch shop is the only seller along its road, its price would be $0.5(v_i + c_i + \tau)$, which is again exactly the MPM price. However, if the central shop is the only seller, its price would be $0.5(v_1 + c_1)$, lower than the MPM price by τ. This result indicates that the MPM price is not always equal to separate monopoly prices. Nonetheless, for $n \geq 2$, the

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[14] These hypothetical prices lie outside the permissible price range as demand should vanish when $p_i \geq v_i$.

[15] If we let s_i be the distance between the center and shop i, and normalize the average distance to 1, p_i^* will change slightly, with τ multiplied by $(s_i + 2)/3$, while p_1^* remains the same.

introduction of any new product/road will not affect the existing prices. In this sense we can still say that the MPM prices are independent of each other.

5. Welfare Loss

Estimating the deadweight loss in MPM with complex product relations might at first sight appear quite challenging. In this section we will show that the optimal prices determined in the previous section can be used to establish a simple relation between deadweight loss and monopoly profits. Since profits are usually observable, this relation provides an easy way to estimate the social loss caused by MPM.

Again our result can be understood as a generalization of a well-known property of the textbook example of a single product monopoly with linear demand: Deadweight loss equals half the monopoly profit. This relation remains valid in our three MPM models. This is unexpected because the welfare functions are fundamentally different across the three models and cannot be presented in a unified framework.

In the standard model with heterogeneous products every consumer's demand vector is $\boldsymbol{y} = \boldsymbol{B}^{-1}(\boldsymbol{a} - \lambda\boldsymbol{p})$. Substituting this into his utility function $(\boldsymbol{a}'\boldsymbol{y} - 0.5\boldsymbol{y}'\boldsymbol{B}\boldsymbol{y})/\lambda$ we obtain $0.5(\boldsymbol{a} + \lambda\boldsymbol{p})'\boldsymbol{B}^{-1}(\boldsymbol{a}/\lambda - \boldsymbol{p}) = 0.5\boldsymbol{a}'\boldsymbol{B}^{-1}\boldsymbol{a}/\lambda - 0.5\lambda\boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$. Since the first term is independent of prices, we only need to consider the second term. Integrating it for all λ the consumer total utility is determined as $-0.5\overline{\lambda}\boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$. Subtracting the total cost $\boldsymbol{c}'\boldsymbol{x}$ from the utility, we get the social welfare. The deadweight loss can be obtained by comparing the welfare under marginal cost pricing and MPM pricing.

The deadweight loss is more complicated with vertically differentiated products. Recall that in this case we have $\theta_1 \equiv p_1/q_1$, and $\theta_1 \equiv (p_i - p_{i-1})/(q_i - q_{i-1})$ for $i > 1$.

These θ_i's define the consumer demand for each product. Consumers purchasing product $i < n$ obtain utility $q_i \int_{\theta_i}^{\theta_{i+1}} \theta d\theta = 0.5 q_i(\theta_{i+1}^2 - \theta_i^2)$. Those purchasing product n obtain utility $q_n \int_{\theta_n}^{1} \theta d\theta = 0.5 q_n(1 - \theta_n^2)$. The total utility of all consumers is $0.5 \sum_1^{n-1} q_i (\theta_{i+1}^2 - \theta_i^2) + 0.5 q_n(1 - \theta_n^2)$. Subtracting from this function the total cost $\boldsymbol{c}'\boldsymbol{x}$, we obtain social welfare. With $\boldsymbol{p} = \boldsymbol{c}$ and the MPM price vector $\boldsymbol{p}^*$, we obtain the maximum welfare and its value under MPM. Their difference is the deadweight loss.

Finally in the case of horizontally differentiated products, the calculation of the deadweight loss involves transportation costs. We first consider the utility obtained by consumers residing along one road. The utility from the center is $\int_0^{y_i} (v_1 - \tau s) ds = v_1 y_i - 0.5 \tau y_i^2$, where $y_i = 0.5(v_1 - p_1 - v_i + p_i + \tau)/\tau$, which is the position of marginal consumers who are indifferent between purchasing at the centre or branch shop i. Consumers who purchase from shop i obtain utility $\int_0^{1-y_i} (v_i - \tau s) ds + \int_0^{z_i} (v_i - \tau s) ds = v_i(1 - y_i) - 0.5\tau(1 - y_i)^2 + v_i z_i - 0.5 \tau z_i^2$, where $z_i = (v_i - p_i)/\tau$. After adding the two utilities and subtracting the cost $c_1 y_i + c_i(1 - y_i + z_i)$, we obtain the social welfare along road i. Adding this welfare for all $i > 1$, we have the total social welfare.

Despite the fundamental structural differences in the welfare functions explained above, the following results holds for all the three different models:

PROPOSITION 5. *In all three MPM models, the deadweight loss is equal to half the monopoly profit.*

Proof: see Appendix D.

The simple relationship known from the linear single product monopoly survives in multiproduct monopoly. As long as demand and cost functions are linear, the relation between the deadweight loss and profits remains unchanged regardless of how many products or what kinds of goods are introduced.[16]

The intuition of this result can be best seen in the horizontal model. According to Proposition 4, adding a new road does not affect the prices along existing roads, and hence will not affect the relationship between welfare loss and monopoly profits there. But the additional deadweight loss along the new road is also half the additional profits, so the overall loss remains as half of the monopoly profits. For the other two models, these relations are more complex, as a new product affects the monopoly profits and deadweight loss from existing products. Nevertheless, the simple relation is always valid. If the linear model is a good approximation, this relation provides a good indication for the deadweight loss due to MPM pricing.

6. Non-linear Demand

Linear demand and cost functions are widely used in industrial economics and often a good approximation of real market conditions. However, even if demand is approximately linear, the monopoly prices obtained for the linear model may not be approximately correct if they are very sensitive to non-linearity. In this section we show that this is not the case. Our results are still approximately true if the Hessian matrix of the consumer utility function does not vary significantly. We focus on the standard model for heterogeneous goods. We assume that the representative consumer's utility

[16] Again linearity is not always necessary. For example, given the inverse demand function in footnote 10, the deadweight loss is equal to the profit multiplied by $\sigma/(1+\sigma)$, if either one or two goods are sold.

function is $h + u(\boldsymbol{x})$, where h is the numeraire good, $u(\boldsymbol{x})$ is continuously twice differentiable and strictly concave in $\boldsymbol{x}$, so that the Hessian matrix $u''(\boldsymbol{x})$ is negative definite and its determinant $|u''(\boldsymbol{x})| \neq 0$. The first-order condition for the utility maximization implies an inverse demand function $p(\boldsymbol{x}) = u'(\boldsymbol{x})$. As in our earlier result, the choke-off price vector $\boldsymbol{p}^0$ corresponds to zero demand, i.e., $\boldsymbol{p}^0 = u'(\boldsymbol{0})$. It equals the marginal utility at zero consumption, which is vector $\boldsymbol{a}/\bar{\lambda}$ in the linear case. If we used our simple rule we would obtain an estimated monopoly price as $\boldsymbol{p}^{\mathrm{m}} = 0.5[u'(\boldsymbol{0}) + \boldsymbol{c}]$. However, the true optimal price p^* should maximize the monopoly profit $[p(\boldsymbol{x}) - \boldsymbol{c}]'\boldsymbol{x}$. As $p'(\boldsymbol{x}) = u''(\boldsymbol{x})$, the first-order condition for the optimal $\boldsymbol{x}^*$ is as $p^* - \boldsymbol{c} + u''(\boldsymbol{x}^*)\boldsymbol{x}^* = \boldsymbol{0}$.

How far will the estimated price $\boldsymbol{p}^{\mathrm{m}}$ be from the true optimum p^*? Given $\boldsymbol{x}^*$, we can always find a non-negative vector $\boldsymbol{w}_1 \leq \boldsymbol{x}^*$, such that $u'(\boldsymbol{x}^*) = u'(\boldsymbol{0}) + u''(\boldsymbol{w}_1)\boldsymbol{x}^*$. We can then write $u''(\boldsymbol{x}^*)\boldsymbol{x}^*$ as $\boldsymbol{p}^* - u'(\boldsymbol{0}) + [u''(\boldsymbol{x}^*) - u''(\boldsymbol{w}_1)]\boldsymbol{x}^*$. Substituting this into the first-order condition for $\boldsymbol{x}^*$, we obtain:

PROPOSITION 6: The optimal price in non-linear MPM with heterogeneous products can be expressed as $\boldsymbol{p}^* = \boldsymbol{p}^m + 0.5[u''(\boldsymbol{w}_1) - u''(\boldsymbol{x}^*)]\boldsymbol{x}^*$.

Proposition 6 shows that our simple rule yields a price that differs from the true optimum only by the last error term. If the Hessian matrix of the utility function, $u''(\boldsymbol{x})$, does not vary significantly, this error term will be close to zero and our simple pricing rule will yield prices that are close to the true optimal prices. A small change in $u''(\boldsymbol{x})$ cannot lead to a significant gap between these two prices. Interestingly, the error term is not clearly linked to products being substitutes or complements, which implies that even in the non-linear case there is no clear-cut relationship between product relations and monopoly prices. In Appendix E we provide an elementary closed-form example showing that with non-linear demand the prices of complementary goods may indeed be higher than those of independent goods.

A similar approach can be used to show that the monopoly output $\boldsymbol{x}^{\mathrm{m}}$ under the estimated price $\boldsymbol{p}^{\mathrm{m}}$ is roughly half of the socially optimal level $\boldsymbol{x}^{\mathrm{c}}$ when $\boldsymbol{p} = \boldsymbol{c}$. We can always find a vector $\boldsymbol{w}_2$ ($\boldsymbol{x}^{\mathrm{m}} \leq \boldsymbol{w}_2 \leq \boldsymbol{x}^{\mathrm{c}}$) such that $u'(\boldsymbol{x}^{\mathrm{c}}) = u'(\boldsymbol{x}^{\mathrm{m}}) + u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. On the other hand, we can find a non-negative vector $\boldsymbol{w}_3 \leq \boldsymbol{x}^{\mathrm{m}}$, such that $u'(\boldsymbol{x}^{\mathrm{m}}) = u'(\boldsymbol{0}) + u''(\boldsymbol{w}_3)\boldsymbol{x}^{\mathrm{m}}$. Since $u'(\boldsymbol{x}^{\mathrm{c}}) = \boldsymbol{c}$ and $u'(\boldsymbol{x}^{\mathrm{m}}) = 0.5[u'(\boldsymbol{0}) + \boldsymbol{c}]$, these two equations imply $u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) = u''(\boldsymbol{w}_3)\boldsymbol{x}^{\mathrm{m}}$. Then we have

PROPOSITION 7: *The relation between the socially optimal output and the monopoly output under* $\boldsymbol{p}^m$ *is:* $\boldsymbol{x}^c = 2\boldsymbol{x}^m + [u''(\boldsymbol{w}_2)]^{-1}[u''(\boldsymbol{w}_3) - u''(\boldsymbol{w}_2)]\boldsymbol{x}^m$.

If $u''(\boldsymbol{w}_3) - u''(\boldsymbol{w}_2)$ is very small and the elements of $[u''(\boldsymbol{w}_2)]^{-1}$ are finite, $\boldsymbol{x}^{\mathrm{c}}$ must be close to $2\boldsymbol{x}^{\mathrm{m}}$. These conditions hold if the Hessian matrix does not change significantly and $|u''(\boldsymbol{x})|$ is not close to zero. The latter is generally true when the demand for each product is significantly downward sloping, rather than horizontal. This does not seem to be a very restrictive requirement for a monopoly[17]. Again a smooth variation in $u''(\boldsymbol{x})$ should not radically change the simple relation $\boldsymbol{x}^{\mathrm{c}} = 2\boldsymbol{x}^{\mathrm{m}}$, which we find in the linear case.

Finally, the deadweight loss (*DL*) caused by the estimated monopoly price $\boldsymbol{p}^{\mathrm{m}}$ remains close to half of the monopoly profit. Given the definitions of $\boldsymbol{x}^{\mathrm{c}}$ and $\boldsymbol{x}^{\mathrm{m}}$, $DL = u(\boldsymbol{x}^{\mathrm{c}}) - u(\boldsymbol{x}^{\mathrm{m}}) - \boldsymbol{c}'\boldsymbol{x}^{\mathrm{c}} + \boldsymbol{c}'\boldsymbol{x}^{\mathrm{m}}$. We can find a vector $\boldsymbol{w}_4$, $\boldsymbol{x}^{\mathrm{m}} \leq \boldsymbol{w}_4 \leq \boldsymbol{x}^{\mathrm{c}}$, and write $u(\boldsymbol{x}^{\mathrm{c}})$ as $u(\boldsymbol{x}^{\mathrm{m}}) + u'(\boldsymbol{x}^{\mathrm{m}})(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) + 0.5(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})'u''(\boldsymbol{w}_4)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. Substitute this into *DL* we get

$(\boldsymbol{p}^{\mathrm{m}} - \boldsymbol{c})'(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}}) + 0.5(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})'u''(\boldsymbol{w}_4)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$. Substitute $\boldsymbol{p}^{\mathrm{m}} - \boldsymbol{c} = -u''(\boldsymbol{w}_2)(\boldsymbol{x}^{\mathrm{c}} - \boldsymbol{x}^{\mathrm{m}})$ into this function and rearrange its terms, we find the following result.

PROPOSITION 8: *The deadweight loss due to the simple monopoly price* $\boldsymbol{p}^m$ *is* $0.5\pi^m + 0.5(\boldsymbol{x}^c - \boldsymbol{x}^m)'\{[u''(\boldsymbol{w}_4) - u''(\boldsymbol{w}_2)](\boldsymbol{x}^c - \boldsymbol{x}^m) - u''(\boldsymbol{w}_2)(\boldsymbol{x}^c - 2\boldsymbol{x}^m)\}$.

In the last term, we know that both $u''(\boldsymbol{w}_4) - u''(\boldsymbol{w}_2)$ and $\boldsymbol{x}^{\mathrm{c}} - 2\boldsymbol{x}^{\mathrm{m}}$ are small when the Hessian matrix $u''(\boldsymbol{x})$ does not vary significantly. Hence, the deadweight loss is close to half the monopoly profit, even if there are minor variations of $u''(\boldsymbol{x})$.

Our analysis shows that in each case the error term introduced by our simple estimation based on the linear model is limited by the variation of the Hessian matrix. If we can estimate the consumer utility function, we should be able to roughly estimate whether our simple results offer reasonable solutions, without having to precisely solve the non-linear problem. Similar approaches can be used to demonstrate that our findings for vertically and horizontally differentiated products hold approximately with nonlinear demand, if the Jacobian matrix of demand does not vary significantly.

7. Concluding Remarks

The paper analyzes pricing and welfare effects of MPM with linear demand and cost functions. Our main result is that the MPM price for each good depends only on the marginal cost and the inverse demand intercept of that good, the nature of any number

[17] This requirement may not be necessary. In the case of footnote 7, the determinant $|u''(\boldsymbol{x})|$ can approach zero when r is close to 1. But $\boldsymbol{x}^{\mathrm{c}}$ is still nearly equal to $2\boldsymbol{x}^{\mathrm{m}}$ so long as σ is not far from 1.

of other goods being immaterial. This conclusion is at odds with much literature, old and new in industrial organization and marketing, stressing the role of substitute/complement products and cross-elasticities in MPM pricing. The underlying oversight in the literature seems to originate from a general tendency to interpret the parameters of direct demand functions as being autonomous, which unfortunately has led to questionable comparative statics conclusions in other settings not covered here.

Our pricing result can be used to show that deadweight loss in monopoly is half of the MPM profit. In other words, relations known from the simple one-product textbook linear model generalize verbatim to three workhorse linear models of interdependent products: heterogeneous, vertically and horizontally differentiated.

Due to their basic nature, the present results can be relevant to a wide range of contexts, covering theoretical and policy issues. Some examples are emerging areas in industrial organization such as bundling and tying. More broadly, the simple insights from this paper could contribute to fields as different as antitrust theory, regulation, urban and spatial economics and marketing. Our welfare results are potentially useful in regulatory design, as a way to estimate deadweight loss in complex situations.

While we limited our analysis mainly to linear demand, our results in Section 6 indicate that our main insight is robust, the linear demand being special only insofar as it leads to the two effects of adding a substitute or complement product to an existing product line being clearly identified and exactly canceling out. We hope that this paper might lead to renewed interest in the topic of monopoly pricing, which was addressed by economists in the early years, but seems to have been largely ignored recently.

Appendix A: We differentiate the profit function (2), $(\boldsymbol{p} - \boldsymbol{c})'(\boldsymbol{\alpha} + \boldsymbol{Ap})$ and get the first-order condition $d\pi/d\boldsymbol{p} = \boldsymbol{\alpha} + \boldsymbol{A}'(\boldsymbol{p} - \boldsymbol{c}) + \boldsymbol{Ap} = \boldsymbol{0}$. Since $\boldsymbol{A}$ is symmetric, we have $\boldsymbol{\alpha} + \boldsymbol{A}(2\boldsymbol{p} - \boldsymbol{c}) = \boldsymbol{0}$. The Hessian matrix of the profit function is equal to $2\boldsymbol{A}$. So the second-order condition holds since $\boldsymbol{A}$ is negative definite. Then the optimal price can be solved from the first-order condition, as $\boldsymbol{p}^* = 0.5(\boldsymbol{c} - \boldsymbol{A}^{-1}\boldsymbol{\alpha})$.

If we plug $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ into the demand function (1), we get $\boldsymbol{x} = \boldsymbol{0}$. So $-\boldsymbol{A}^{-1}\boldsymbol{\alpha}$ is the demand intercept vector $\boldsymbol{p}^0$. The optimal price $\boldsymbol{p}^*$ can be written as $0.5(\boldsymbol{c} + \boldsymbol{p}^0)$.

Putting $\boldsymbol{p}^*$ into the demand function (1), we get $\boldsymbol{x}^* = 0.5(\boldsymbol{\alpha} + \boldsymbol{Ac})$. When $\boldsymbol{p} = \boldsymbol{c}$, we get the socially optimal output $\boldsymbol{\alpha} + \boldsymbol{Ac}$, which is twice of the monopoly output $\boldsymbol{x}^*$. ||

Appendix B: (i) The demand function (3) is clearly linear in prices. To apply Proposition 1, we need to show that Jacobian matrix $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_{i+1} = 1/(q_{i+1} - q_i) = \partial x_{i+1}/\partial p_i$ for all i, and $\partial x_i/\partial p_j = 0$ for any $j \neq i$ and $|j - i| > 1$, the matrix is indeed symmetric.

To show it is negative definite, we see the sum of the first row of $\partial\boldsymbol{x}/\partial\boldsymbol{p}$ is equal to $-1/q_1$, and the sum of every other row is zero. Hence the matrix has a quasi-dominant diagonal and must be negative definite (McKenzie 1960, Theorem 2).

(ii) We then need to find $\boldsymbol{p}^0$. Substituting $\boldsymbol{p} = \boldsymbol{q}$ into the demand function (3), we get $\boldsymbol{x} = \boldsymbol{0}$. So $\boldsymbol{p}^0 = \boldsymbol{q}$, and the MPM price $p_i^* = 0.5(c_i + q_i)$. We also have $\boldsymbol{x}(\boldsymbol{p}^*) = 0.5\boldsymbol{x}(\boldsymbol{c})$ from Proposition 1.

(iii) To complete the proof, we need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, we need to show $\frac{c_2 - c_1}{q_2 - q_1} \geq \frac{c_1}{q_1}$, or $\frac{c_2}{q_2} \geq \frac{c_1}{q_1}$. This holds since $c''(q) > 0$. For $x_n \geq 0$, we must have $c_n - c_{n-1} \leq q_n - q_{n-1}$. This is true given $c'(q) < 1$.

For $1 < i < n$, $x_i \geq 0$ holds if $\frac{c_{i+1} - c_i}{q_{i+1} - q_i} \geq \frac{c_i - c_{i-1}}{q_i - q_{i-1}}$. To prove this, we write $c_{i+1} - c_i$ as $(q_{i+1} - q_i)c'(\omega_i)$ and $c_i - c_{i-1} = (q_i - q_{i-1})c'(\omega_{i-1})$, where $q_{i-1} \leq \omega_{i-1} \leq q_i \leq \omega_i \leq q_{i+1}$. As $c''(q) > 0$, $\omega_{i-1} \leq \omega_i$, we get $c'(\omega_i) \geq c'(\omega_{i-1})$, so $x_i \geq 0$.

Finally, we show that no consumer receives a negative surplus under $\boldsymbol{p}^*$. The marginal consumer buying from good 1 receives a zero surplus. For $i > 1$, the marginal consumer $\theta_i = (p_i - p_{i-1})/(q_i - q_{i-1})$, receives a positive surplus if $\theta_i q_i \geq p_i$ or $p_i q_{i-1} \geq p_{i-1} q_i$. Using p_i^* and p_{i-1}^*, it becomes $c_i/q_i \geq c_{i-1}/q_{i-1}$, which holds given $c''(q) > 0$. ||

Appendix C: (i) To apply Proposition 1, again we need to show that the $n \times n$ Jacobian matrix $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is symmetric and negative definite. As $\partial x_i/\partial p_1 = \partial x_1/\partial p_i = 0.5/\tau$ for all $i > 1$, and $\partial x_i/\partial p_j = 0$ for i and $j \neq 1$, it is indeed symmetric.

Moreover since $\partial x_1/\partial p_1 = -0.5(n-1)\tau$, $\partial x_i/\partial p_i = -1.5/\tau$, the sum of the first row of $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ is 0, and the sum of any other row is $-1/\tau < 0$. By McKenzie (1960) $\partial \boldsymbol{x}/\partial \boldsymbol{p}$ must be negative definite.

(ii) One can verify that demand function (5) is zero when $p_1 = 2\tau + v_1$, and $p_i = \tau + v_i$. So the MPM prices $p_1^* = 0.5(v_1 + c_1 + 2\tau)$, and $p_i^* = 0.5(v_i + c_i + \tau)$ for $i > 1$.

We need to show $\boldsymbol{x}(\boldsymbol{c}) > \boldsymbol{0}$. For $x_1 \geq 0$, it suffices to show $\tau + v_1 - c_1 \geq v_i - c_i$. For $x_i \geq 0$, we need $v_i - c_i + 3\tau \geq v_1 - c_1$. Assumption 3 guarantees both of them.

Finally, every marginal consumer must receive a non-negative surplus. For a consumer indifferent between the center and shop i, her surplus from the center is $v_1 - p_1 - \tau y_i = v_1 - p_1 - 0.5(v_1 - p_1 - v_i + p_i + \tau) = 0.25(v_1 - c_1 + v_i - c_i - 5\tau)$. It is positive given Assumption 3. A marginal consumer outside of shop i receives a zero surplus. ||

Appendix D: (i) Substituting the monopoly price $\boldsymbol{p}^*$ and marginal cost pricing $\boldsymbol{c}$ into function $-0.5\bar{\lambda}\boldsymbol{p}'\boldsymbol{B}^{-1}\boldsymbol{p}$, and subtracting one from the other, we get the total utility loss as $0.5\bar{\lambda}(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{B}^{-1}(\boldsymbol{p}^* + \boldsymbol{c})$. But we know $\bar{\lambda}(\boldsymbol{p}^* - \boldsymbol{c}) = 0.5(\boldsymbol{a} - \bar{\lambda}\boldsymbol{c})$, and $\boldsymbol{B}^{-1}(\boldsymbol{a} - \bar{\lambda}\boldsymbol{c}) = \boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, so the utility loss is equal to $0.5(\boldsymbol{p}^* + \boldsymbol{c})'\boldsymbol{x}^*$. On the other hand, since the monopoly pricing reduces the outputs by half, the total cost falls by $\boldsymbol{c}'\boldsymbol{x}^*$. Thus, the deadweight loss, the sum of the utility and cost changes, is equal to $0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^*$, which is half of the total monopoly profit.

(ii) Vertically differentiated products: We first write double the total utility of all consumers as $2u = \sum_1^{n-1} q_i\,(\theta_{i+1}^2 - \theta_i^2) + q_n - q_n\theta_n^2$. Regrouping the summation items, it becomes $q_n - q_1\theta_1^2 - \sum_2^n \theta_i^2 (q_i - q_{i-1})$. Substituting $\theta_1 \equiv p_1/q_1$ and $\theta_i \equiv$

$(p_i - p_{i-1})/(q_i - q_{i-1})$, it becomes $q_n - \theta_1 p_1 - \sum_2^n \theta_i (p_i - p_{i-1})$. Regrouping the summation items again, it changes to $q_n + \sum_1^{n-1} p_i (\theta_{i+1} - \theta_i) - \theta_n p_n$. As $\theta_{i+1} - \theta_i = x_i$ and $1 - \theta_n = x_n$, we get $2u = \sum_1^n p_i x_i + q_n - p_n$.

Moreover, we write $q_n - p_n$ as $q_n - \sum_2^n (p_i - p_{i-1}) - p_1 = q_n - \sum_2^n \theta_i (q_i - q_{i-1}) - \theta_1 q_1$. Regrouping the summation items, we get $q_n - p_n = q_n + \sum_1^{n-1} q_i (\theta_{i+1} - \theta_i) - \theta_n q_n = \sum_1^n q_i x_i$. Substitute this into $2u$ expression, we get $u = 0.5 \sum_1^n (p_i + q_i) x_i = 0.5(\boldsymbol{p} + \boldsymbol{q})'\boldsymbol{x}$.

Social welfare (SW) $u - \boldsymbol{c}'\boldsymbol{x} = 0.5(\boldsymbol{p} + \boldsymbol{q} - 2\boldsymbol{c})'\boldsymbol{x}$. When $\boldsymbol{p} = \boldsymbol{c}$, we get $SW(\boldsymbol{c}) = 0.5(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}(\boldsymbol{c})$; when $\boldsymbol{p} = \boldsymbol{p}^* = 0.5(\boldsymbol{q} + \boldsymbol{c})$, $SW^* = 0.75(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}^*$. Given $\boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, the deadweight loss $SW(\boldsymbol{c}) - SW^* = 0.25(\boldsymbol{q} - \boldsymbol{c})'\boldsymbol{x}^*$.

As $\boldsymbol{q} - \boldsymbol{c} = 2(\boldsymbol{p}^* - \boldsymbol{c})$, the deadweight loss is $0.5(\boldsymbol{p}^* - \boldsymbol{c})'\boldsymbol{x}^* = 0.5\pi^*$.

(ii) Horizontally differentiated products: The utility obtained by consumers along one road is $u_i = [v_1 - 0.5\tau y_i]y_i + [v_i - 0.5\tau(1 - y_i)](1 - y_i) + [v_i - 0.5(v_i - p_i)z_i]z_i$, where $z_i = (v_i - p_i)/\tau$ and $y_i = 0.5(v_1 - p_1 - v_i + p_i + \tau)/\tau$. Substitute y_i and z_i in brackets [], $u_i = 0.25(3v_1 + p_1 + v_i - p_i - \tau)y_i + 0.25(3v_i + p_i + v_1 - p_1 - \tau)(1 - y_i) + 0.5(v_i + p_i)z_i$, which simplifies to $0.5(v_1 + p_1)y_i + 0.5(v_i + p_i)(1 - y_i + z_i) + 0.25(v_1 - p_1 + v_i - p_i - \tau)$.

Note that $v_1 - p_1 + v_i - p_i = \tau(2y_i + 2z_i - 1)$, and $1 - y_i + z_i = x_i$, we write this utility as $u_i = 0.5(v_1 + p_1)y_i + 0.5(v_i + p_i)x_i + 0.5\tau(y_i + z_i - 1)$. Replacing z_i by $y_i + x_i - 1$, we get $u_i = 0.5(v_1 + p_1 + 2\tau)y_i + 0.5(v_i + p_i + \tau)x_i - \tau$. One-road welfare is $u_i - c_1 y_i - c_i x_i$.

Recall that $\sum_2^n y_i = x_1$. The welfare from all roads is $0.5x_1(v_1 + p_1 + 2\tau - 2c_1) + 0.5\sum_2^n x_i(v_i + p_i + \tau - 2c_i) - n\tau$. When $\boldsymbol{p} = \boldsymbol{c}$, we get the maximum total welfare as $0.5x_1(\boldsymbol{c})(v_1 + 2\tau - c_1) + 0.5\sum_2^n x_i(c)(v_i + \tau - c_i) - n\tau$. Since $\boldsymbol{x}(\boldsymbol{c}) = 2\boldsymbol{x}^*$, $v_1 + 2\tau - c_1 = 2(p_1^* - c_1)$, and $v_i + \tau - c_i = 2(p_i^* - c_i)$, we can write this welfare as $2x_1^*(p_1^* - c_1) + 2\sum_2^n x_i^*(p_i^* - c_i) - n\tau = 2\pi^* - n\tau$. When $\boldsymbol{p} = \boldsymbol{p}^*$, the welfare is $0.5x_1^*(1.5v_1 + 3\tau - 1.5c_1) + 0.75\sum_2^n x_i^*(v_i + \tau - c_i) - n\tau$, which is equal to $1.5x_1^*(p_1^* - c_1) + 1.5\sum_2^n x_i^*(p_i^* - c_i) - n\tau = 1.5\pi^* - n\tau$. Subtracting two welfare values, we get the deadweight loss $0.5\pi^*$. ||

Appendix E: Consider a two-good monopolist with zero costs facing a demand function $p_i = a - bx_i^{\sigma} - rx_j$, $i, j = 1, 2$, $\sigma > 0$. Profit is $\pi = (a - bx_1^{\sigma} - rx_2)x_1 + (a - bx_2^{\sigma} - rx_1)x_2$.

The first-order condition for x_i implies: $a - b(\sigma + 1)(x^*)^{\sigma} - 2rx^* = 0$. (i)

The monopoly price is $p^* = a - b(x^*)^{\sigma} - rx^* = \sigma b(x^*)^{\sigma} + rx^*$. (ii)

In two independent markets, demand and profit are: $p_i = a - bx_i^{\sigma}$ and $\pi = (a - bx_i^{\sigma})x_i$.

The first order condition for x_i is $a - b(\sigma + 1)(x^0)^{\sigma} = 0$, (iii)

The separate price satisfies $p^0 = a - b(x^0)^{\sigma} = \sigma b(x^0)^{\sigma}$. (iv)

Subtracting (iv) from (ii), we get: $p^* - p^0 = \sigma b[(x^*)^{\sigma} - (x^0)^{\sigma}] + rx^*$. (v)

Subtracting (iii) from (i), we get: $b(\sigma + 1)[(x^*)^{\sigma} - (x^0)^{\sigma}] = -2rx^*$. (vi)

Substituting (vi) into (v), we obtain:

$$p^* - p^0 = \frac{1-\sigma}{1+\sigma} rx^*. \qquad \text{(vii)}$$

If $\sigma < 1$, (vii) implies that $p^* > p^0$ if and only if goods are substitutes ($r > 0$), in line with conventional wisdom. However, if $\sigma > 1$, (vii) implies that $p^* > p^0$ if and only if goods are complements ($r < 0$), in total violation of conventional wisdom. ||

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