FACULTY OF LAW, ECONOMICS AND FINANCE

CREA Discussion Paper 2010-21

Center for Research in Economic Analysis
University of Luxembourg

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available online : http://wwwfr.uni.lu/recherche/fdef/crea/publications2/discussion_papers/2010

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December 14, 2010

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STEADY-STATE GROWTH AND THE ELASTICITY OF SUBSTITUTION

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Abstract: In a neoclassical economy with endogenous capital- and labor-augmenting technical change the steady-state growth rate of output per worker is shown to increase in the elasticity of substitution between capital and labor. This confirms the assessment of Klump and de La Grandville (2000) that a greater elasticity of substitution allows for faster economic growth. However, unlike their findings my result applies to the steady-state growth rate. Moreover, it does not hinge on particular assumptions on how aggregate savings come about. It holds for any household sector allowing savings to grow at the same rate as aggregate output.

Keywords: Capital Accumulation, Elasticity of Substitution, Direction of Technical Change, Neoclassical Growth Model.

JEL-Classification: E22, O11, O33, O41.

This Version: December 14, 2010

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This is a revised and extended version of my discussion paper Irmen (2010). Most of this research was written while I was visiting MIT during the Fall 2008. I would like to thank MIT for its kind hospitality. Moreover, I would like to thank two anonymous referees for their comments.

1 Introduction

Is the measured degree of factor substitution an indicator for an economy's growth potential? The debate surrounding this question began with the contributions by de La Grandville (1989) and Klump and de La Grandville (2000). These authors study the link between the elasticity of substitution, being treated as a parameter of an aggregate CES production function, and economic growth in the neoclassical economy of Solow (1956). They conclude that the degree of factor substitution is a powerful engine of economic growth in the sense that a higher elasticity of substitution between capital and labor leads to a higher growth rate along the transition and a higher steady-state level of output per worker. This assessment has been challenged by Miyagiwa and Papageorgiou (2003). These authors emphasize the role of the underlying savings hypothesis of Solow (1956) and find cases in a model of overlapping generations where a higher elasticity of substitution is an impediment to growth.[1]

Such conflicting results arise in a neoclassical setting since the elasticity of substitution affects the two main pillars on which aggregate one-sector models of economic growth are based. First, there is a direct impact on aggregate production since differing degrees of factor substitution affect the shape of the aggregate production function. Second, there is an indirect effect on aggregate savings and capital accumulation since the degree of factor substitution affects

[1]Irmen and Klump (2009) reconcile these findings by pointing out that the positive growth effects of a high elasticity of substitution arise as long as the propensity to save out of capital income exceeds the propensity to save out of wage income. Xue and Yip (2009) provide a comprehensive discussion of the growth effects of the elasticity of substitution in one-sector models of economic growth. Such effects are also at the heart of Papageorgiou and Saam (2008) who study the growth process under different savings hypotheses and two-level CES production functions.

the functional income distribution.

This paper extends the neoclassical framework of the aforementioned works and studies the relationship between the elasticity of substitution and economic growth in a multi-sector environment where both technical change and its direction are endogenous. More precisely, I allow for endogenous capital- and labor-augmenting technical change in an otherwise neoclassical production sector. For this setting, my main result is that a greater elasticity of substitution means faster steady-state growth of per-worker variables. This result is shown to rely on the *efficiency effect* of the elasticity of substitution established by Klump and de La Grandville (2000): ceteris paribus, an increase in this elasticity increases output.

Although this finding confirms the spirit of the claim of Klump and de La Grandville, it is not subject to the above mentioned criticism. In fact, my steady-state result holds for any household sector that allows for a constant aggregate consumption growth rate equal to the growth rate of the economy. Hence, the effect of the elasticity of substitution through savings and capital accumulation on the steady state, i. e., the second pillar of aggregate one-sector growth models, is mute. To the best of my knowledge, the present paper is the first to establish such a result.

I derive my findings in a *neoclassical economy with endogenous capital- and labor-augmenting technical change*. It is *neoclassical* since it maintains the assumptions of perfect competition, of an aggregate production function with constant returns to scale and positive and diminishing marginal products, and of capital accumulation. It has *endogenous growth* since economic growth results from innovation investments undertaken by profit-maximizing firms. To allow for innovation investments in *capital- and labor-augmenting technical change*, I introduce two intermediate-good sectors, one producing a capital-intensive intermediate, the other a labor-intensive intermediate.[2] Innovation investments increase the productivity of capital and labor at the level of these intermediate-good firms. Moreover, they feed into aggregate productivity indicators that evolve cumulatively, i. e., in a way often referred to as 'standing on the shoulders of giants'.

Competitive final-good firms use both intermediates and produce according to the normalized CES production function of de La Grandville (1989) and Klump and de La Grandville (2000). In equilibrium, the quantity of either intermediate-good input is equal to the amount of capital and labor in efficiency units, respectively. This has several important implications. First, the economy's equilibrium production function of the final good coincides with the normalized CES production function employed by Klump, McAdam, and Willman (2007). Hence, my analysis may be seen as providing a micro-foundation for the production function on which these authors base their empirical analysis.[3]

Second, there is a need to distinguish the exogenous *partial* from the endogenous *total* elasticity of substitution between capital and labor. Throughout, I shall use the acronyms PES and TES to denote the partial and the total elasticity of substitution, respectively. While the former concept leaves the efficiency of capital and labor constant, the latter accounts for the effect of changes in the capital-labor ratio on the incentives to engage in capital- and labor-augmenting

[2]The production sector extends and complements the one devised in Irmen (2005) by allowing for capital-augmenting technical change. In turn, the latter builds on Hellwig and Irmen (2001) and Bester and Petrakis (2003). See Acemoglu (2003) for an alternative model of endogenous capital- and labor-augmenting technical change where innovation investments are financed through rents that accrue in an environment with monopolistic competition.

[3]Indeed, with the caveat that in my framework capital- and labor-augmenting technical change is endogenous, the aggregate production function stated in their equation (4) and in equation (3.1) below coincide.

technical change. Hence, my analysis suggests that the determinants of technical change will also affect the TES of an economy.

The endogeneity of the TES begs two questions. The first is about the relationship between the PES and the TES. I find that these two elasticities differ unless the PES is equal to unity. Moreover, I establish that capital and labor are either gross complements or gross substitutes independent of whether one considers the PES or the TES, i. e., if the PES is greater (smaller) than unity then the TES is greater (smaller) than unity, too. Moreover, gross complementarity implies that the TES exceeds the PES, whereas the opposite holds for gross substitutes.

The second question concerns the evolution of the TES during the process of economic development. To address this question, I add a household sector as in Solow (1956) and Swan (1956) to the neoclassical economy with endogenous capital- and labor-augmenting technical change. For this economy, numerical computations indicate that the PES determines, inter alia, whether there is spiral or monotonic convergence to the steady state. The evolution of the TES inherits these properties. Hence, it varies with the state of the economy.

This last result contributes to a growing literature studying the reasons and the implications of an endogenous TES. On the one hand, my analysis emphasizes that endogenous technical change may be behind the endogeneity of the TES.[4] On the other hand, it confirms a claim made by Arrow, Chenery, Minhas, and Solow (1961), p. 247, according to which the process of economic development

[4]In macroeconomic one-sector growth models, the elasticity of substitution may simply be endogenous because the aggregate production function is not of the CES class. Examples include Revankar (1971) and Jones and Manuelli (1990). The role of the inter-sectoral allocation of inputs for an endogenous TES is emphasized in Miyagiwa and Papageorgiou (2007) for closed and in Saam (2008) for open economies.

may shift the total aggregate elasticity of substitution.[5] However, it contrasts with findings derived in Miyagiwa and Papageorgiou (2007). These authors study a Solow economy with a production sector comprising a final-good and two intermediate-good sectors. They find that the aggregate elasticity of substitution (AES) is endogenous and determined by the evolution of the economy's capital endowment and its inter-sectoral equilibrium allocation.[6] Contrary to my numerical computations, their results suggest that the AES converges monotonically towards the steady state.

The remainder of this paper is organized as follows. In Section 2, I lay out the details of the competitive production sector, define its equilibrium, and introduce the concepts of the PES and the TES. In Section 3, I deal with the analysis of the steady state. The positive effect of a greater elasticity of substitution on the steady-state growth rate appears here as Theorem 1. I present the numerical example in Section 4. Section 5 concludes. All proofs are relegated to Appendix 6.1. Appendix 6.2 details the analysis of the local stability of the dynamical system underlying the numerical example of Section 4.

2 The Competitive Production Sector

The production sector has a final-good sector and an intermediate-good sector in an infinite sequence of periods $t = 1, 2, ..., \infty$. The *manufactured final good* can be

[5]This claim is also in line with recent empirical evidence. See, e. g., Duffy and Papageorgiou (2000) or Pereira (2002).

[6]Conceptually, the AES of Miyagiwa and Papageorgiou (2007) corresponds to my TES, whereas their primary elasticities of substitution (ES) resemble my partial elasticity of substitution.

consumed or invested. If invested it may either become future capital or serve as an input in current innovation activity undertaken by intermediate-good firms. Intermediate-good firms produce one of two types of intermediates and sell it to the final-good sector. The production of the *labor-intensive intermediate good* uses labor as the sole input, the only input in the production of the *capital-intensive intermediate good* is capital. Labor- and capital-augmenting technical change is the result of innovation investments undertaken by intermediate-good firms. *Labor* and *capital* are supplied to the intermediate-good sector. Capital needs to be installed one period before its use. The final good serves as numéraire.

2.1 Production

The final-good sector produces with the following CES production function $F : \mathbb{R}^2_+ \to \mathbb{R}_+$,

$$Y_t = F\left(Y_{K,t}, Y_{L,t}\right) = \Gamma \left[\gamma \, Y_{K,t}^{\psi} + (1-\gamma) \, Y_{L,t}^{\psi}\right]^{1/\psi}, \tag{2.1}$$

where Y_t is aggregate output in t, $Y_{K,t}$ is the aggregate amount of the capital-intensive intermediate input, and $Y_{L,t}$ denotes the aggregate amount of the labor-intensive intermediate input.[7] The parameters satisfy $\Gamma > 0$, $1 > \gamma > 0$, and $1 > \psi > -\infty$. Moreover, $\sigma = 1/\left(1-\psi\right)$ is the elasticity of substitution between $Y_{K,t}$ and $Y_{L,t}$. I show below that σ is also the PES between capital and labor. The key question is then how changes in σ affect the steady-state growth rate. In

[7] I shall stick as close as possible to the notation of Klump and de La Grandville (2000). For reasons that become obvious below, I have to replace their constants A and a by Γ and γ, respectively. See Chapter 3 in de La Grandville (2009) for a careful derivation of the normalized CES production function.

terms of the labor-intensive intermediate-good input, let $y_t = F(\kappa_t, 1) \equiv f(\kappa_t)$, where $\kappa_t \equiv Y_{K,t}/Y_{L,t}$. Then

$$y_t = f(\kappa_t) = \Gamma \left[\gamma \kappa_t^{\psi} + (1-\gamma) \right]^{1/\psi}. \tag{2.2}$$

To identify the effect of the elasticity of substitution on otherwise identical economies, I follow de La Grandville (1989) and Klump and de La Grandville (2000) and normalize (2.2) by choosing some baseline values for the following variables: $\bar{\kappa}$, $\bar{y} = f(\bar{\kappa})$, and $\bar{m} = [f(\bar{\kappa}) - \bar{\kappa} f'(\bar{\kappa})] / f'(\bar{\kappa}) > 0$, which is the marginal rate of substitution. The normalized CES production function that satisfies these criteria is then equal to (see, Klump and de La Grandville (2000), eq. 5)

$$f_\sigma(\kappa) = \Gamma(\sigma) \left[\gamma(\sigma) \kappa^{\psi} + (1 - \gamma(\sigma)) \right]^{1/\psi} \tag{2.3}$$

with

$$\Gamma(\sigma) \equiv \bar{y} \left(\frac{\bar{\kappa}^{1-\psi} + \bar{m}}{\bar{\kappa} + \bar{m}} \right)^{1/\psi} \quad \text{and} \quad \gamma(\sigma) = \frac{\bar{\kappa}^{1-\psi}}{\bar{\kappa}^{1-\psi} + \bar{m}}. \tag{2.4}$$

Also, I follow Klump and de La Grandville (2000) and denote partial derivatives of f with respect to κ by a prime so that $f'_\sigma \equiv \partial f_\sigma / \partial \kappa$ and $\partial f'_\sigma / \partial \sigma \equiv \partial^2 f_\sigma / \partial \kappa \partial \sigma$.

In units of the final good of period t the profit in t of the final-good sector is

$$Y_t - p_{K,t} Y_{K,t} - p_{L,t} Y_{L,t}, \tag{2.5}$$

where $p_{j,t}$, $j = K, L$, is the price of the respective intermediate factor. The final-good sector takes the sequence $\{p_{K,t}, p_{L,t}\}_{t=1}^{\infty}$ of factor prices as given and maximizes the sum of the present discounted values of profits in all periods. Since it simply buys both intermediates in each period, its maximization problem is equivalent to a series of one-period maximization problems. Focussing on configurations where both intermediates are used, the profit-maximizing first-order conditions for $t = 1, 2, ...$ are

$$Y_{K,t} \quad : \quad p_{K,t} = f'_{\sigma}\left(\kappa_t\right), \tag{2.6}$$

$$Y_{L,t} \quad : \quad p_{L,t} = f_{\sigma}\left(\kappa_t\right) - \kappa_t f'_{\sigma}\left(\kappa_t\right). \tag{2.7}$$

There are two different sets of intermediate-good firms, each represented by the set $\mathbb{R}_+$ of nonnegative real numbers with Lebesgue measure. Intermediate-good firms may either belong to the sector that produces the labor- or the capital-intensive intermediate. In other words, they are either part of the labor- or of the capital-intensive intermediate-good sector.

At any date t, all firms of a sector have access to the same sector-specific technology with production function

$$y_{l,t} = \min\left\{1, a_t l_t\right\} \quad \text{or} \quad y_{k,t} = \min\left\{1, b_t k_t\right\}, \tag{2.8}$$

where $y_{l,t}$ and $y_{k,t}$ is output, 1 a capacity limit,[8] a_t and b_t denote the firms' labor and capital productivity in period t, l_t and k_t is the labor and the capital input. The firms' respective labor and capital productivity is equal to

$$a_t = A_{t-1}(1 - \delta + q_t^A) \quad \text{or} \quad b_t = B_{t-1}(1 - \delta + q_t^B); \tag{2.9}$$

[8]The assumption of a capacity constraint is by no means restrictive for my results. The capacity choice may be endogenized along the lines of Hellwig and Irmen (2001).

here $A_{t-1} > 0$ and $B_{t-1} > 0$ denote aggregate indicators of the level of technological knowledge to which innovating firms in period t have access for free. Naturally, $\delta \in (0,1)$ is the rate of depreciation of technological knowledge in both sectors, and q_t^A and q_t^B are indicators of productivity growth gross of depreciation.

To achieve a productivity growth rate $q_t^j > 0$, $j = A, B$, a firm must invest $i(q_t^j)$ units of the final good in period t. The function $i : \mathbb{R}_+ \to \mathbb{R}_+$ is the same for both sectors, time invariant, $\mathcal{C}^2$, and strictly convex. Moreover, with the notation $i'\left(q^j\right) \equiv di\left(q^j\right)/dq^j$ for $j = A, B$, it satisfies

$$i(0) = \lim_{q^j \to 0} i'\left(q^j\right) = 0, \quad i'\left(q^j\right) > 0 \text{ for all } q^j > 0, \quad \lim_{q^j \to \infty} i\left(q^j\right) = \infty. \qquad (2.10)$$

Hence, higher rates of productivity growth require ever-growing investments.

If a firm innovates, the assumption is that an innovation in period t is proprietary knowledge of the firm only in t, i. e., in the period when the innovation materializes. Subsequently, the innovation becomes embodied in the sector specific productivity indicators (A_t, B_t), (A_{t+1}, B_{t+1}), ..., with no further scope for proprietary exploitation. The evolution of these indicators will be specified below.[9]

Per-period profits in units of the current final good are

$$\pi_{L,t} \;=\; p_{L,t}\, y_{l,t} - w_t l_t - i(q_t^A), \quad \pi_{K,t} = p_{K,t}\, y_{k,t} - R_t k_t - i(q_t^B), \qquad (2.11)$$

[9] As will become clear below, all firms innovate in equilibrium since $i(0) = \lim_{q^j \to 0} i'\left(q^j\right) = 0$. To save space, I shall disregard throughout the discussion of what would happen if firms did not innovate. Details on this are available from the author upon request.

where $p_{L,t}\, y_{l,t}$, $p_{K,t}\, y_{k,t}$ is the respective firm's revenue from output sales, $w_t l_t$, $R_t k_t$ its wage bill at the real wage rate w_t and its capital cost at the real rental rate of capital R_t, and $i(q_t^j)$, $j = A, B$, its investment outlays.

Firms choose a production plan $(y_{l,t}, l_t, q_t^A)$ or $(y_{k,t}, k_t, q_t^B)$ taking the sequence $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$ of real prices and the sequence $\{A_{t-1}, B_{t-1}\}_{t=1}^{\infty}$ of aggregate productivity indicators as given. They choose a production plan that maximizes the sum of the present discounted values of profits in all periods. Because production choices for different periods are independent of each other, for each period t, they choose the plan $(y_{l,t}, l_t, q_t^A)$ and $(y_{k,t}, k_t, q_t^B)$ that maximizes the profit $\pi_{L,t}$ and $\pi_{K,t}$, respectively.

If a firm innovates, it incurs an investment cost $i(q_t^j) > 0$ that is associated with a given innovation rate $q_t^j > 0$ and is independent of the output $y_{l,t}$ or $y_{k,t}$. An innovation investment is only profit-maximizing if the firm's margin is strictly positive, i. e., if $p_{L,t} > w_t/a_t$ or $p_{K,t} > R_t/b_t$. Then, there is a positive scale effect, namely if the firm innovates, it wants to apply the innovation to as large an output as possible and produces at the capacity limit, i. e., $y_{l,t} = 1$ or $y_{k,t} = 1$. The choice of (l_t, q_t^A) and (k_t, q_t^B) must then minimize the costs of producing the capacity output. Assuming $w_t > 0$ and $R_t > 0$ these input combinations satisfy

$$l_t = \frac{1}{A_{t-1}(1-\delta+q_t^A)}, \quad k_t = \frac{1}{B_{t-1}(1-\delta+q_t^B)}, \tag{2.12}$$

and

$$\begin{aligned} \hat{q}_t^A &\in \arg\min_{q^A \geq 0} \left[\frac{w_t}{A_{t-1}(1-\delta+q^A)} + i\left(q^A\right) \right], \\ \hat{q}_t^B &\in \arg\min_{q^B \geq 0} \left[\frac{R_t}{B_{t-1}(1-\delta+q^B)} + i\left(q^B\right) \right]. \end{aligned} \tag{2.13}$$

Given the convexity of the innovation cost function and the fact that $\lim_{q^j \to 0} i'(q^j) = 0$, the conditions (2.13) determine a unique level $\hat{q}_t^A > 0$ and $\hat{q}_t^B > 0$ as the solution to the first-order conditions

$$\frac{w_t}{A_{t-1}(1-\delta+\hat{q}_t^A)^2} = i'\left(\hat{q}_t^A\right) \quad \text{and} \quad \frac{R_t}{B_{t-1}(1-\delta+\hat{q}_t^B)^2} = i'\left(\hat{q}_t^B\right). \tag{2.14}$$

The latter relate the marginal reduction of a firm's wage bill/capital cost to the marginal increase in its investment costs.

Recall that the set of each intermediate-good sector is $\mathbb{R}_+$ with Lebesgue measure. Then, the maximum profit of any intermediate-good firm producing the labor- or the capital-intensive intermediate must be zero at any t. Indeed, since the supply of labor and capital is bounded in each period, the set of intermediate-good firms employing more than some $\varepsilon > 0$ units of labor or capital must have bounded measure and hence must be smaller than the set of all intermediate-good firms. Given that inactive intermediate-good firms must be maximizing profits just like the active ones, I need that maximum profits of all active intermediate-good firms at equilibrium prices are equal to zero.

Using (2.11), (2.12), and (2.14), it holds for profit-maximizing intermediate-good firms earning zero profits in equilibrium that

$$p_{L,t} = (1-\delta+\hat{q}_t^A)i'(\hat{q}_t^A) + i(\hat{q}_t^A), \quad p_{K,t} = (1-\delta+\hat{q}_t^B)i'(\hat{q}_t^B) + i(\hat{q}_t^B), \tag{2.15}$$

i. e., the price is equal to variable costs plus fixed costs when w_t/a_t and R_t/b_t are consistent with profit-maximization as required by (2.14). Upon combining the equilibrium conditions of the final-good sector and both intermediate-good sectors I find the following proposition.

Proposition 1 *If (2.6), (2.7), and (2.15) hold, then there are maps,* $g^A : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ *and* $g^B : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$*, such that* $\hat{q}^A_t = g^A(\kappa_t, \sigma) > 0$ *with* $g^A_\kappa(\kappa_t, \sigma) > 0$ *and* $\hat{q}^B_t = g^B(\kappa_t, \sigma) > 0$ *with* $g^B_\kappa(\kappa_t, \sigma) < 0$ *for any* $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$*.*

Proposition 1 emphasizes three important properties of the production sector. First, the equilibrium incentives to engage in labor- and capital-augmenting technical change depend on the factor intensity of the final-good sector and on the elasticity of substitution. Second, for all $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$, I have $\hat{q}^A_t > 0$ and $\hat{q}^B_t > 0$. Finally, $\hat{q}^A_t$ increases whereas $\hat{q}^B_t$ decreases in κ_t.

The first and the third property are due to the fact that $p_{K,t}$ and $p_{L,t}$ depend on κ_t and σ according to (2.6) and (2.7). This dependency feeds back onto $\hat{q}^A_t$ and $\hat{q}^B_t$ through the zero-profit condition (2.15). The second property hinges on the characteristics of the input requirement function given in (2.10). If $i(0) > 0$, then maximum profits of innovating firms could be strictly negative even at low levels of $\hat{q}^j_t > 0$. Then, these firms would not enter and the intermediate-good production of the respective sector would collapse. If $i'(0) > 0$, then the first marginal unit of q^j would no longer be costless. Consequently, the cost-minimization problem (2.13) would not necessarily admit an interior solution. Indeed, firms would choose $\hat{q}^j_t = 0$ if the marginal reduction of their wage bill/capital cost at $q^j = 0$ was smaller than $i'(0)$.

2.2 Evolution of Technological Knowledge

The evolution of the economy's level of technological knowledge is given by the evolution of the aggregate indicators $(A_t, B_t) \in \mathbb{R}^2_{++}$. Labor- and capital-augmenting productivity growth occurs at the level of those intermediate-good firms that produce at t. Denoting the measure of these firms by n_t and m_t, respectively, their contribution to A_t and B_t is equal to the highest level of labor and capital productivity attained by one of them, i. e.,

$$
\begin{aligned}
A_t &= \max\{a_t(n) = A_{t-1}\left(1-\delta+q_t^A(n)\right) \mid n \in [0, n_t]\} \\
B_t &= \max\{b_t(m) = B_{t-1}\left(1-\delta+q_t^B(m)\right) \mid m \in [0, m_t]\}.
\end{aligned}
\tag{2.16}
$$

Since in equilibrium $q_t^A(n) = q_t^A$ and $q_t^B(m) = q_t^B$, I have $a_t = A_{t-1}\left(1-\delta+q_t^A\right)$ and $b_t = B_{t-1}\left(1-\delta+q_t^B\right)$. Hence, for all $t = 1, 2, ...$

$$
A_t = A_{t-1}\left(1-\delta+q_t^A\right) \quad \text{and} \quad B_t = B_{t-1}\left(1-\delta+q_t^B\right) \tag{2.17}
$$

with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

2.3 Dynamic Competitive Equilibrium of the Production Sector

For given sequences of capital $\{K_t\}_{t=1}^{\infty}$, $K_t \in \mathbb{R}_{++}$, and labor $L_t = L_1 (1+g_L)^{t-1}$, $g_L > (-1)$, $L_1 > 0$, the dynamic competitive equilibrium of the production sector determines a sequence of prices $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$, a sequence of allocations $\{Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, y_{l,t}, y_{k,t}, q_t^A, q_t^B, a_t, b_t, l_t, k_t\}_{t=1}^{\infty}$, and a sequence of indicators of the level of technological knowledge $\{A_t, B_t\}_{t=1}^{\infty}$.

Definition 1 *In a dynamic competitive equilibrium of the production sector, the above mentioned sequences satisfy the following conditions for $t = 1, 2, ..., \infty$:*

(E1) At all t, all firms maximize profits and earn zero-profits.

(E2) At all t, the market for both intermediates clears, i. e.,

$$
Y_{L,t} = n_t \quad \textit{and} \quad Y_{K,t} = m_t. \tag{2.18}
$$

(E3) At all t, there is full employment of labor and capital, i.e.,

$$n_t l_t = L_t \quad \textit{and} \quad m_t k_t = K_t. \tag{2.19}$$

(E4) The productivity indicators A_t and B_t evolve according to (2.17) with $A_0 > 0$ and $B_0 > 0$ as initial conditions.

Condition (E1) is satisfied if Proposition 1 holds. (E2) and (E3) require market clearing of the market for both intermediates and both factors. To avoid more complicated notation, the market-clearing conditions (2.18) and (2.19) use the fact that all entering intermediate-good firms at t produce the capacity output and hire the same amount of workers and capital, respectively.

By (2.12) the equilibrium amount of labor and capital employed by some intermediate-good firm is $l_t = 1/A_{t-1}\left(1-\delta+q_t^A\right)$ and $k_t = 1/B_{t-1}\left(1-\delta+q_t^B\right)$. Then, from (E3), $n_t = A_{t-1}\left(1-\delta+q_t^A\right)L_t$ and $m_t = B_{t-1}\left(1-\delta+q_t^B\right)K_t$. With (E2) and (E4) I have

$$\begin{aligned} Y_{L,t} &= A_{t-1}\left(1-\delta+q_t^A\right)L_t = A_t L_t, \\ Y_{K,t} &= B_{t-1}\left(1-\delta+q_t^B\right)K_t = B_t K_t. \end{aligned} \tag{2.20}$$

Hence, at a semantic level, technical change is capital- and labor-saving at the level of the individual firm and capital- and labor-augmenting at the level of economic aggregates. If, ceteris paribus, A_{t-1} and B_{t-1} increase, the capacity output is produced with less labor and less capital. At the aggregate level, these gains in productivity translate into more entry through the requirement of full employment of labor and capital. Accordingly, aggregate output of each intermediate good is equal to the respective input in efficiency units, and a higher A_t or B_t is equivalent to having more labor or capital, respectively.

Moreover, from (2.20) I have in equilibrium

$$Y_t = F\left(B_t K_t, A_t L_t\right) \quad \text{and} \quad \kappa_t = \frac{B_t K_t}{A_t L_t}. \tag{2.21}$$

Hence, aggregate output of the final good is produced using the efficient capital stock and efficient labor. The variable κ_t has an interpretation as the efficient capital intensity of the economy. Moreover, it may be used as the state variable to study the evolution of the production sector:

Proposition 2 *Let κ_t be the state variable that determines the behavior of the production sector in t. Given a sequence $\{\kappa_t\}_{t=1}^{\infty}$, $\kappa_t \in \mathbb{R}_{++}$, there is a unique dynamic equilibrium that satisfies Definition 1.*

2.4 The Partial and the Total Elasticity of Substitution

One may use (2.3) and (2.21) to derive the ratio MPK_t/MPL_t, i. e., the relative marginal product of capital, as

$$\frac{MPK_t}{MPL_t} = \frac{\gamma(\sigma)}{1-\gamma(\sigma)} \left(\frac{B_t}{A_t}\right)^{\frac{\sigma-1}{\sigma}} \left(\frac{K_t}{L_t}\right)^{\frac{-1}{\sigma}}. \tag{2.22}$$

It decreases in the capital intensity, K_t/L_t. This is the substitution effect, and σ is the PES between capital and labor. 'Partial' refers to the fact that B_t/A_t remains constant. However, changes in K_t/L_t induce technical change. To see how, write κ_t of (2.21) as

$$\kappa_t = \frac{B_{t-1}\left(1-\delta+g^B\left(\kappa_t,\sigma\right)\right)}{A_{t-1}\left(1-\delta+g^A\left(\kappa_t,\sigma\right)\right)} \frac{K_t}{L_t}. \tag{2.23}$$

The latter implicitly defines a functional relationship between κ_t and K_t/L_t characterized by the elasticity

$$\frac{d \ln \kappa_t}{d \ln (K_t/L_t)} = \frac{1}{1+\varepsilon_\kappa^A(\kappa_t,\sigma) - \varepsilon_\kappa^B(\kappa_t,\sigma)} \in (0,1). \tag{2.24}$$

Here,

$$\varepsilon_\kappa^A(\kappa_t,\sigma) \equiv \frac{g_\kappa^A(\kappa_t,\sigma)\kappa_t}{1-\delta+g^A(\kappa_t,\sigma)} > 0 \quad \text{and} \quad \varepsilon_\kappa^B(\kappa_t,\sigma) \equiv \frac{g_\kappa^B(\kappa_t,\sigma)\kappa_t}{1-\delta+g^B(\kappa_t,\sigma)} < 0, \tag{2.25}$$

are the elasticities of the equilibrium growth factors of A_t and B_t with respect to κ_t. According to (2.24), κ_t increases in K_t/L_t. Moreover, as $\varepsilon_\kappa^A > 0$ and $\varepsilon_\kappa^B < 0$ such an increase induces more labor- and less capital-augmenting technical change. It is in this sense that induced technical change is linked to the relative abundance of factors of production as envisaged by Hicks (1932).

With these relationships at hand, it is straightforward to derive the TES, $\hat{\sigma}$, as a measure of the relative change in the relative marginal product of capital to the relative change in the relative abundance of capital taking induced innovation into account, i. e.,

$$\hat{\sigma}(\kappa_t,\sigma) \equiv -\left[\frac{d \ln (MPK_t/MPL_t)}{d \ln (K_t/L_t)}\right]^{-1} = \sigma \left[\frac{1+\varepsilon_\kappa^A - \varepsilon_\kappa^B}{1+\sigma\left(\varepsilon_\kappa^A - \varepsilon_\kappa^B\right)}\right], \tag{2.26}$$

where the argument of ε_κ^j, $j = A, B$, is (κ_t,σ). In view of Proposition 1, the relationship between $\hat{\sigma}$ and σ is as follows.

Corollary 1 *For any $(\kappa_t,\sigma) \in \mathbb{R}^2_{++}$ it holds that*

$$\sigma \lesseqqgtr 1 \quad \Leftrightarrow \quad \hat{\sigma} \lesseqqgtr 1 \quad \Leftrightarrow \quad \sigma \lesseqqgtr \hat{\sigma}. \tag{2.27}$$

The first part of (2.27) states that capital and labor are either gross complements or substitutes independent of whether reference is made to the PES or to the TES. The second part of (2.27) says that the TES is strictly greater (smaller) than the PES if capital and labor are gross complements (substitutes). For the Cobb-Douglas case, where the term B_t/A_t vanishes in (2.22), I find $\sigma = \hat{\sigma} = 1$.

Corollary 1 obtains since $\hat{\sigma}$ depends on κ through ε_κ^A and ε_κ^B. As κ_t is a state variable of the production sector, the conclusion is that the TES depends on the state of the economy. The numerical exercise presented in Section 4 suggests that the relationship between both variables may be increasing or decreasing depending on the considered value of the PES (see Figures 3 and 4 below).

3 Steady-State Analysis

I define a steady state, or equivalently a balanced growth path, as a path along which all variables mentioned in Definition 1 grow at constant exponential rates (possibly zero) for all $t \geq \tau \geq 1$.

It is immediate from Proposition 1 and 2 that in a steady state $\kappa_t = \kappa^*$ and both rates, $\hat{q}_t^A$ and $\hat{q}_t^B$, are constant. Yet, the dynamic competitive equilibrium of the production sector on its own does not pin down κ^*. Therefore, I embed the production sector into a richer macroeconomic environment that accounts for capital investment and a resource constraint. I refer to this environment as the *neoclassical economy with endogenous capital- and labor-augmenting technical change*. This environment delivers a single steady-state condition for κ^*. With this condition at hand, I study the role of the PES for steady-state growth.

Definition 2 *The neoclassical economy with endogenous capital- and labor-augmenting technical change is defined by the following environment:*

1. *The (normalized) CES production function (2.1)*

$$Y_t = F_\sigma\left(B_tK_t, A_tL_t\right) = \Gamma\left(\sigma\right)\,\left[\gamma\left(\sigma\right)\,\left(B_tK_t\right)^\psi + \left(1-\gamma\left(\sigma\right)\right)\left(A_tL_t\right)^\psi\right]^{1/\psi}. \quad (3.1)$$

2. *Capital accumulation according to*

$$K_{t+1} = I_t^K + (1-\delta^K)K_t, \quad K_1 > 0, \quad (3.2)$$

where $I_t^K > 0$ is gross investment of current output in the capital stock, $\delta^K \in [0,1]$ is the depreciation rate of capital, and $K_1 > 0$ the initial condition.

3. *Two indicators of technological knowledge, A_t and B_t, that evolve according to (2.17).*

4. *Innovation investments of current output, $I_t^A > 0$ and $I_t^B > 0$, are necessary and sufficient for $q_t^A > 0$ and $q_t^B > 0$. Moreover,*

$$I_t^A = A_tL_t\, i\left(q_t^A\right) \quad \text{and} \quad I_t^B = B_tK_t\, i\left(q_t^B\right). \quad (3.3)$$

5. *A resource constraint according to which consumption, $C_t > 0$, gross investment in the capital stock, $I_t^K > 0$, and innovation investments, $I_t^A > 0$ and $I_t^B > 0$, add up to aggregate output, i. e.,*

$$C_t + I_t^K + I_t^A + I_t^B = Y_t. \quad (3.4)$$

6. *The labor force grows at a constant rate $g_L > (-1)$, i. e., $L_t = L_1(1+g_L)^{t-1}$ with $L_1 > 0$ as initial condition.*

Definition 2 adds capital accumulation according to (3.2) and the resource constraint (3.4) to the production sector of Section 2. Moreover, it uses equilibrium conditions. In accordance with (2.20), I replace $Y_{L,t}$ and $Y_{K,t}$ by $A_t L_t$ and $B_t K_t$ in (3.1) and use (E2) to conclude that $I_t^A = n_t i\left(q_t^A\right) = A_t L_t i\left(q_t^A\right)$ and $I_t^B = m_t i\left(q_t^B\right) = B_t K_t i\left(q_t^B\right)$ in (3.3). In addition to consumption, the three ways to invest current output show up on the left-hand side of the resource constraint (3.4).

The following proposition establishes the key properties of a steady state in a neoclassical economy with endogenous capital- and labor-augmenting technical change.

Proposition 3 *Suppose the neoclassical economy with endogenous capital- and labor-augmenting technical change exhibits a steady state starting at period τ with $I_t^K > 0$, $I_t^A > 0$, $I_t^B > 0$ for $t \geq \tau$. Then, for all $t \geq \tau$*

$$Y_t = F_\sigma\left(B_\tau K_t, A_t L_t\right) = \Gamma(\sigma)\left[\gamma(\sigma)\left(B_\tau K_t\right)^\psi + (1-\gamma(\sigma))\left(A_t L_t\right)^\psi\right]^{1/\psi}, \quad (3.5)$$

and output per worker grows at rate

$$g^* \equiv q^A - \delta. \quad (3.6)$$

Proposition 3 states that in a steady state the growth rate of output per worker coincides with the net growth rate of labor-saving technical change g^*; there is no capital-saving technical progress, i. e., $B_t = B_\tau$ for all $t \geq \tau$. These findings mimic the predictions of the so-called Steady-State Growth Theorem of Uzawa (1961). In fact, the proof of Proposition 3 shows that in a steady state, the neoclassical economy of Definition 2 is isomorphic to the environment to which Uzawa's theorem applies. If such an economy is equipped with the production sector of Section 2, Proposition 3 means that

$$\delta = q^B = g^B(\kappa^*, \sigma) \quad \text{and} \quad g^* = q^A - \delta = g^A(\kappa^*, \sigma) - \delta. \tag{3.7}$$

The first of these conditions pins down the steady-state capital intensity κ^*, the second gives the steady-state growth rate of output per worker. This is quite remarkable: the steady-state efficient capital intensity and the steady-state growth rate of the economy depend only on parameters that characterize the economy's production sector. The following theorem exploits this fact.

Theorem 1 *Consider two neoclassical economies with endogenous capital- and labor-augmenting technical change equipped with a production sector set out in Section 2. Let these economies differ only with respect to σ. Then, the economy with the greater σ experiences faster steady-state growth of output per worker.*

Hence, a greater PES is associated with faster steady-state growth. To grasp the intuition of Theorem 1, denote $\kappa^*(\sigma)$ the implicit function defined by $\delta = g^B(\kappa^*, \sigma)$ and consider the prices of the capital- and the labor-intensive intermediate of (2.6) and (2.7) at the steady state, i. e.,

$$p_K^* = f_\sigma'(\kappa^*(\sigma)) \quad \text{and} \quad p_L^* = f_\sigma(\kappa^*(\sigma)) - \kappa^*(\sigma) f_\sigma'(\kappa^*(\sigma)). \tag{3.8}$$

From Proposition 3 and (3.7), any change in σ must be accompanied by a change in $\kappa^*(\sigma)$ such that $\delta = g^B(\kappa^*(\sigma), \sigma)$, i. e., the incentive to engage in capital-augmenting technical progress must remain unchanged. From Proposition 1 and (3.8), this means that $p_K^* = f_\sigma'(\kappa^*(\sigma))$ must remain constant. The point of Theorem 1 is that under these circumstances p_L^* increases in σ. Then, in accordance with Proposition 1 and (3.7), $g^A(\kappa^*(\sigma), \sigma)$ and g^* increase.

To see that this is indeed the case consider p_L^* as stated in (3.8). Since $f'_\sigma(\kappa^*(\sigma))$ remains constant, changing σ has two effects on $f_\sigma(\kappa^*(\sigma)) - \kappa^*(\sigma) f'_\sigma(\kappa^*(\sigma))$. First, there are two (indirect) effects through an adjustment of $\kappa^*(\sigma)$ which cancel out. Second, there is the (direct) efficiency effect identified by Klump and de La Grandville (2000), i. e., $\partial f_\sigma(\kappa^*(\sigma)) / \partial\sigma > 0$ for $\bar{\kappa} \neq \kappa^*$. Hence, it is due to the efficiency effect that the economy with the greater elasticity of substitution has faster steady-state growth of output per worker.[10]

Observe that Theorem 1 does not depend on the assumption that the requirement functions, i, are the same in both sectors. If, for some reason, the innovation process in one sector is more difficult than in the other such that $i^A(q^A) = \alpha\, i(q^A) \neq i^B(q^B) = \beta\, i(q^B)$ with $\alpha \neq \beta$, then α becomes a parameter of g^A and β one of g^B. Accordingly, the steady-state efficient capital intensity and the steady-state growth rate of output per worker depend on these parameters in accordance with (3.7). However, the qualitative effect of the PES on steady-state growth remains unaffected. A similar reasoning reveals that the depreciation rates of technological knowledge may differ without affecting my qualitative results.

Theorem 1 is also robust with respect to modifications in the way the indicators A_t and B_t evolve. For instance, I assume in (2.16) that their evolution depends only on the innovation activity of the respective sector. To relax this assumption, one may allow for spillovers. As long as these are not too strong, e. g., such

[10] It is worth noting that a greater elasticity of substitution may also be associated with faster sustained growth in models where technical change is absent. For instance, de La Grandville (1989), p. 479, computes for the Solow model with a normalized CES production function a critical value of the elasticity of substitution above which there is sustained growth of per-worker variables. Klump and Preisser (2000), p. 48, show that the economy's asymptotic growth rate increases in the elasticity of substitution. However, unlike here, the capital intensity tends to infinity as the economy approaches its balanced growth path.

that the evolution of A_t depends only on the innovation investments in capital-augmenting technical change and vice versa, the steady-state growth effects of the PES remain valid.[11]

Finally, the growth effect of Theorem 1 may also be related to the total elasticity of substitution between capital and labor.

Corollary 2 *If the production technology of two economies is characterized by $\sigma_2 > \sigma_1$ and both partial elasticities of substitution are close to unity, then $\hat{\sigma}_2 > \hat{\sigma}_1$ and the economy with the greater total elasticity of substitution between capital and labor has the higher steady-state growth rate.*

Hence, if one believes to measure the TES rather than the PES, the prediction of faster steady-state growth under a greater elasticity of substitution remains valid. Though, since little is known about the derivatives $\partial \varepsilon_\kappa^j / \partial \sigma$, $j = A, B$, this result may only be locally valid.

4 A Numerical Example

This section adds a household sector as in Solow (1956) and Swan (1956) to the neoclassical economy with endogenous capital- and labor-augmenting technical change set out in Definition 2 and provides a numerical simulation. The purpose is twofold. First, it allows me to probe the results derived in Corollary 1,

[11]For instance, one may replace (2.16) by $A_t = A_{t-1}\left[\left(1-\delta+q_t^A\right)\right]^{\phi}\left[\left(1-\delta+q_t^B\right)\right]^{\lambda}$ and $B_t = B_{t-1}\left[\left(1-\delta+q_t^B\right)\right]^{\phi}\left[\left(1-\delta+q_t^A\right)\right]^{\lambda}$, with $\phi > 0$ and $\lambda \in [0, \phi]$. Here, λ measures the strength of the spillover from current research in one sector on the productivity indicator of the other sector. Details are available upon request.

Theorem 1, and Corollary 2 quantitatively. Second, I use this framework to address the question of whether and how the TES may vary during the process of development.

Let aggregate savings, S_t, be a constant fraction, $s \in (0,1)$, of total income accruing to capital and labor. Then, in equilibrium I have for all $t = 1, 2, ..., \infty$

$$\begin{aligned} S_t &= s\left(w_t L_t + R_t K_t\right) \\ &= s\left[F_\sigma\left(B_t K_t, A_t L_t\right) - A_t L_t i\left(g^A\left(\kappa_t, \sigma\right)\right) - B_t K_t i\left(g^B\left(\kappa_t, \sigma\right)\right)\right]. \end{aligned} \tag{4.1}$$

Using $I_t^K = S_t$ in (3.2), I express this difference equation in terms of the state variables κ_t and B_t as

$$\begin{aligned} \kappa_{t+1}\left(\frac{1-\delta+g^A\left(\kappa_{t+1},\sigma\right)}{1-\delta+g^B\left(\kappa_{t+1},\sigma\right)}\right) &= \frac{B_t s\left[f_\sigma\left(\kappa_t\right) - i\left(g^A\left(\kappa_t,\sigma\right)\right) - \kappa_t\, i\left(g^B\left(\kappa_t,\sigma\right)\right)\right]}{1+g_L} \\ &+ \frac{1-\delta^K}{1+g_L}\kappa_t. \end{aligned} \tag{4.2}$$

From Proposition 1 and (2.17), I obtain

$$B_{t+1} = B_t\left(1-\delta+g^B\left(\kappa_{t+1},\sigma\right)\right). \tag{4.3}$$

The dynamical system of the model consists of (4.2) and (4.3) and the initial conditions $\{K_1, L_1, A_0, B_0\}$. Given, $L_t = L_1(1+g_L)$, $g_L > (-1)$, $L_1 > 0$, it determines a sequence $\{\kappa_t, B_t\}_{t=1}^{\infty}$, and hence sequences of prices $\{p_{L,t}, p_{K,t}, w_t, R_t\}_{t=1}^{\infty}$, allocations $\{Y_t, Y_{K,t}, Y_{L,t}, n_t, m_t, y_{l,t}, y_{k,t}, q_t^A, q_t^B, a_t, b_t, l_t, k_t, I_t^K, I_t^A, I_t^B, C_t, S_t, K_{t+1}, L_{t+1}\}_{t=1}^{\infty}$, and of the indicators of the level of technological knowledge $\{A_t, B_t\}_{t=1}^{\infty}$. Notice that κ_1 and B_1 are pinned down by

$$\kappa_1 = \frac{B_0\left(1-\delta+g^B\left(\kappa_1,\sigma\right)\right)K_1}{A_0\left(1-\delta+g^A\left(\kappa_1,\sigma\right)\right)L_1} > 0, \quad B_1 = B_0\left(1-\delta+g^B\left(\kappa_1,\sigma\right)\right). \tag{4.4}$$

I apply the de La Grandville normalization to the CES production function of (2.1) for the special case where $\bar{\kappa} = 1$. It follows that $\bar{y} = \Gamma$, $\bar{m} = (1-\gamma)/\gamma$, $\Gamma(\sigma) = \Gamma$, and $\gamma(\sigma) = \gamma$. As to the remaining parameters I choose the following values: $\gamma = 1/3$, $s = 0.2$, $g_L = 0$, $\Gamma = 1$, $v_0 = 1$, $v = 2$, $\delta = 0.225$, and $\delta^K = 0.2$.[12] Key results for $\sigma \in \{0.1, 0.25, 0.5, 0.75, 1, 2, 5, 8, 10\}$ are shown in Table 1.[13]

The second column of Table 1 states the TES at the steady state. As σ increases so does $\hat{\sigma}(\kappa^*(\sigma),\sigma)$. However, in accordance with Corollary 1, $\sigma \lesseqgtr \hat{\sigma}(\kappa^*(\sigma),\sigma)$ as long as $\sigma \lesseqgtr 1$ with equality only in the Cobb-Douglas case. Column 3 confirms Theorem 1: the greater σ, the greater is the steady-state growth rate of the economy. In addition, it suggests that the function $g^*(\sigma) = g^A\left(\kappa\left(\sigma\right),\sigma\right) - \delta$ is concave. Taken together, the first three columns also suggest that the validity of Corollary 2 may not be confined to a neighborhood of $\sigma = 1$: an economy with a greater σ has greater values for $\hat{\sigma}(\kappa^*(\sigma),\sigma)$ and g^*.

Columns 4 and 5 show that the steady state is locally stable for all considered values of σ. However, σ may affect the local transitional dynamics. For $\sigma \in \{0.1, 0.25, 0.5, 0.75, 1, 2, 5\}$ the distinct complex roots with modulus strictly

[12] These choices are not made to get close to a particular economy. For this purpose, more information would be needed, e. g., to justify the values chosen for v_0, v, or δ. See Klump and Saam (2008) for a proposal how to calibrate the normalized CES in dynamic one-sector macroeconomic models.

[13] All computations presented in this section use *Mathematica*. The notebooks are available upon request. I detail the method used to analyze the local stability of the steady state as well as the computations underlying Figures 1 - 4 in Appendix 6.2.

smaller than unity indicate spiral convergence towards the steady state. For $\sigma \in \{8, 10\}$ both roots are distinct, real, strictly positive, and smaller than unity such that convergence is monotonic.

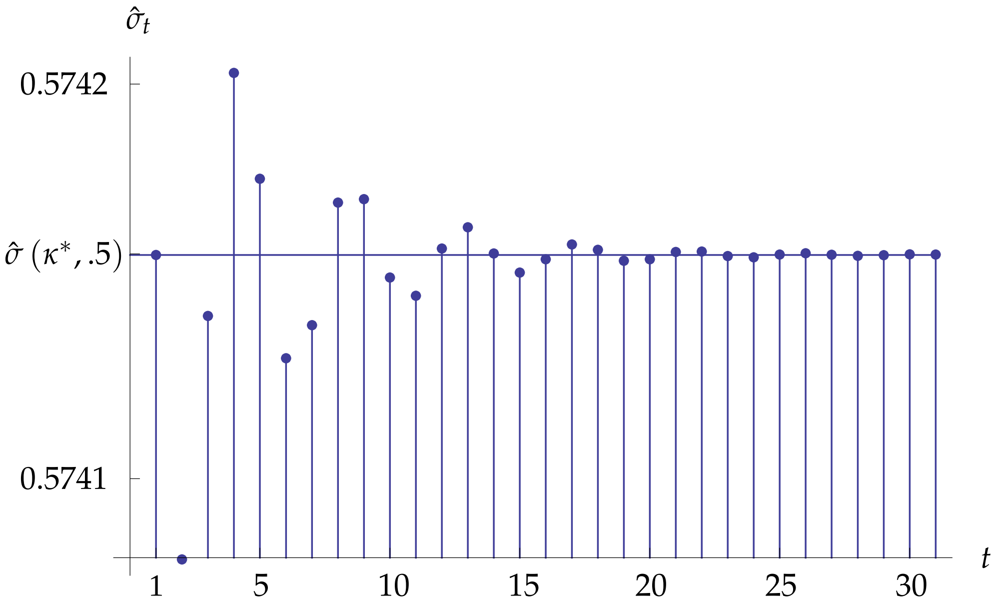
Table 1: Key Results For The Example.

σ - PET	$\hat{\sigma}(\kappa^*(\sigma), \sigma)$ - TES	g^*	Roots	Modulus
0.1	0.23	0.15%	$0.17 \pm 0.55i$	0.57
0.25	0.36	0.43%	$0.13 \pm 0.71i$	0.73
0.5	0.57	0.83%	$0.1 \pm 0.8i$	0.8
0.75	0.79	1.17%	$0.1 \pm 0.83i$	0.83
1	1	1.47%	$0.1 \pm 0.84i$	0.84
2	1.85	2.38%	$0.13 \pm 0.81i$	0.82
5	4.41	3.72%	$0.29 \pm 0.56i$	0.63
8	6.97	4.3%	$0.48, 0.33$	n.a.
10	8.67	4.52%	$0.81, 0.1$	n.a.

Next, I turn to the analysis of the transitional dynamics of the TES in the neighborhood of the steady state. To cover the cases of spiral and monotonic convergence, I focus on $\sigma = 0.5$ and $\sigma = 8$. I assume that these economies are in their steady state at $t = 1$. Then, they experience a one-time shock such that $\kappa_2 > \kappa^*$, for instance due to a one-time increase of the savings rate. This shock is assumed to be so small that the transition to the steady state may be studied using linear approximations.

Figures 1 and 2 depict the adjustment paths to the TES following the shock. In both cases, the evolution of $\hat{\sigma}_t$ inherits the properties of the adjustment of κ_t, i. e., spiral convergence in the case of $\sigma = 0.5$ and monotonic convergence if $\sigma = 8$. Hence, the TES varies over time. Another difference occurs as to the

Figure 1: The Evolution of the TES, $\hat{\sigma}_t = \hat{\sigma}(\kappa_t, \sigma)$, for $\sigma = 0.5$.

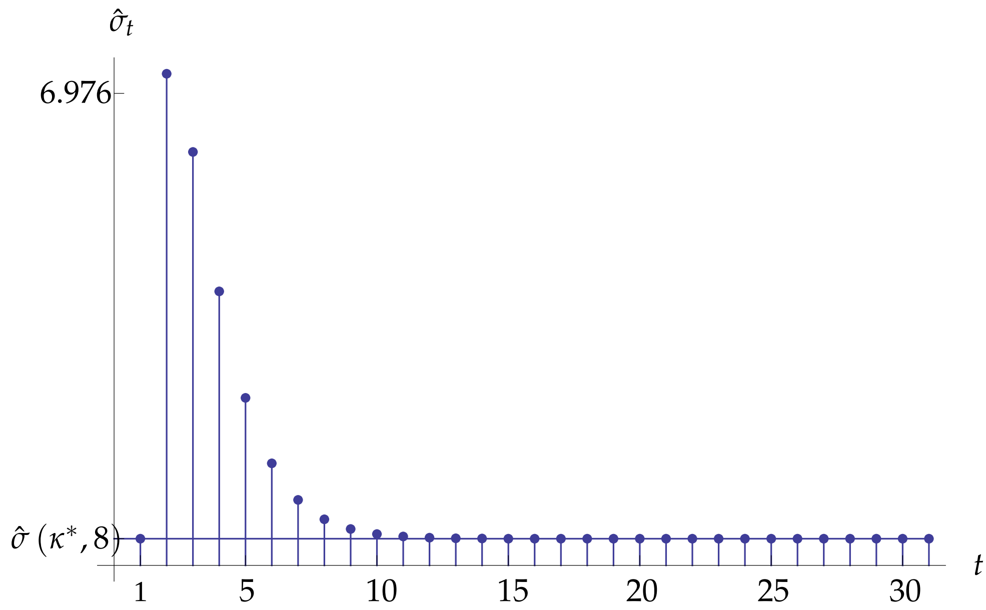


initial response of the TES to an increase in κ above its steady state level. For $\sigma = 0.5$, Figure 1 reveals that $\hat{\sigma}_2 < \hat{\sigma}(\kappa^*)$ whereas Figure 2 shows the opposite for $\sigma = 8$. This may be traced back to the differential response of $\hat{\sigma}(\kappa, \sigma)$ to changes in κ. Figure 3 shows this relationship to be negative for $\sigma = 0.5$. However, it is positive in the case of $\sigma = 8$ as shown in Figure 4.[14]

Summing up, these results suggest that the TES may vary throughout the process of development as conjectured by Arrow, Chenery, Minhas, and Solow (1961). However, extending the findings of Miyagiwa and Papageorgiou (2007), its evolution may not be monotonic.

[14]Figures 3 and 4 are representative in the sense that for all considered values $\sigma < 1$ there is a neighborhood of κ^* such that $\hat{\sigma}(\kappa, \sigma)$ is decreasing whereas for $\sigma > 1$ it is increasing.

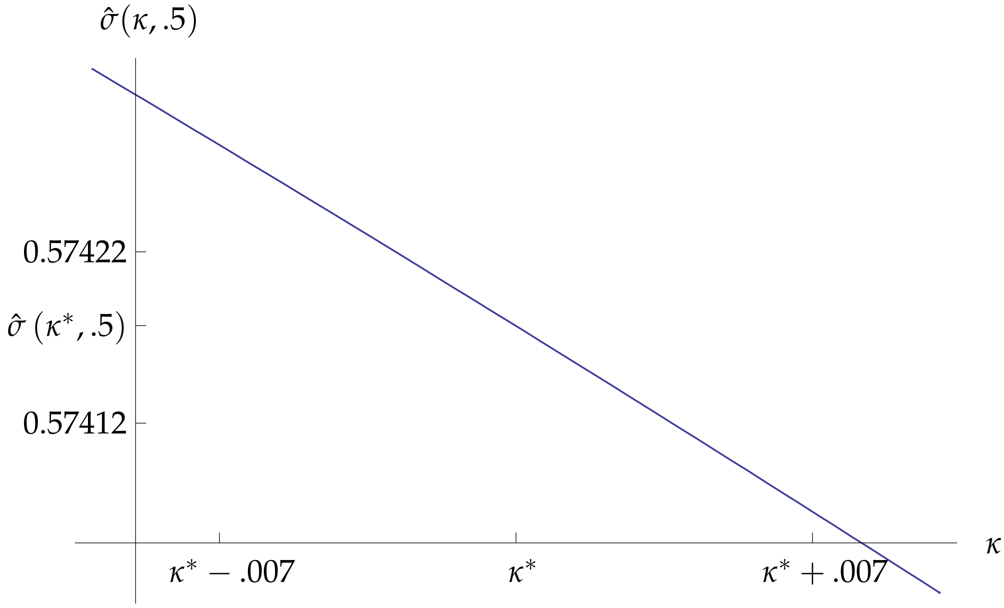
Figure 2: The Evolution of the TES, $\hat{\sigma}_t = \hat{\sigma}(\kappa_t, \sigma)$, for $\sigma = 8$.



5 Concluding Remarks

This paper suggests that there are new effects linking the predicted growth performance of an economy to the elasticity of substitution once I leave the setting of the one-sector growth models of de La Grandville (1989) and Klump and de La Grandville (2000). In the three-sector economy under scrutiny here the direction of technical change, i. e., the economy's choice between capital- and labor-augmenting technical change, is endogenous. I find that an economy with a greater PES between capital and labor has a greater steady-state growth rate of output per worker. This is due to the efficiency effect of the elasticity of substitution of Klump and de La Grandville (2000), i. e., for a given efficient capital intensity, output per efficient labor increases in the elasticity of substitution. In the present context, innovation investments that raise the productivity of labor become more profitable due to the efficiency effect. Therefore, the steady-state growth rate is higher.

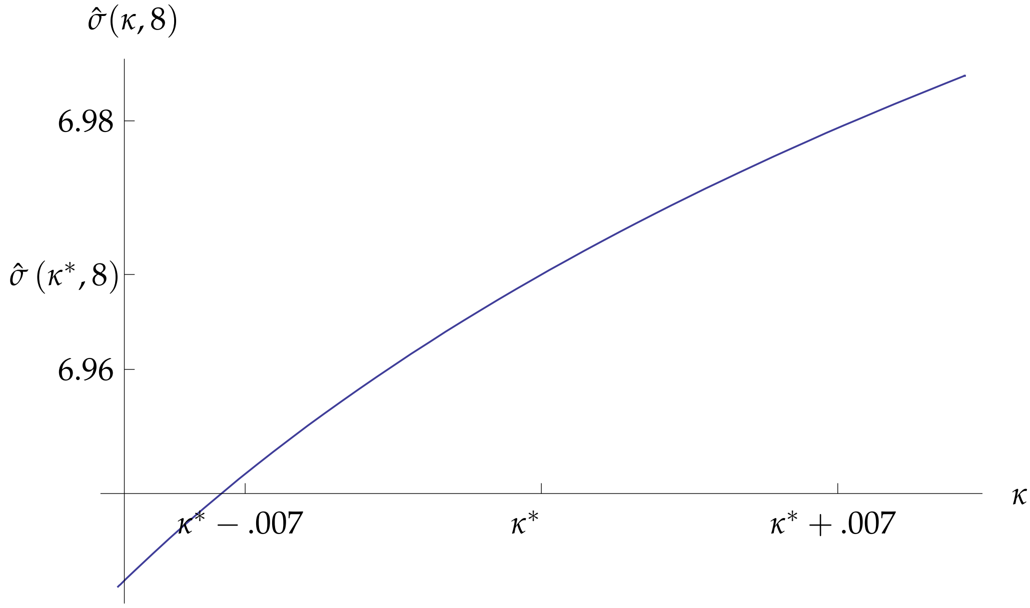
Figure 3: The Dependence of the TES, $\hat{\sigma}(\kappa, \sigma)$, on κ in a neighborhood of κ^* for $\sigma = 0.5$.



Unlike other channels linking the elasticity of substitution to a country's growth performance, the central result of this paper does not depend on particular assumptions on the household side of the economy. All relevant steady-state conditions on growth rates follow from Uzawa's Steady-State Growth Theorem. Theorem 1 derives a great deal of its generality from this fact. The price, however, is that it does not include predictions concerning transitional dynamics.

To make a first step in the analysis of the role of the PES for transitional dynamics, I consider a household sector with a constant savings rate. My findings suggest that the PES may determine whether the adjustment to the steady state is characterized by spiral or monotonic convergence. The transitional evolution of the TES inherits these properties. Hence, as conjectured by Arrow, Chenery, Minhas, and Solow (1961), the TES changes during the process of development. However, my results suggest that the relationship between the state of the economy and its TES may be complex. Untangling this complexity remains an open

Figure 4: The Dependence of the TES, $\hat{\sigma}(\kappa, \sigma)$, on κ in a neighborhood of κ^* for $\sigma = 8$.



question for future research. The present paper points to an issue a possible answer has to take into account: endogenous technical change is a reason for why an economy's TES is endogenous.

6 Appendix

6.1 Proofs

6.1.1 Proof of Proposition 1

Upon substitution of (2.6) and (2.7) in the respective zero-profit condition of (2.15) gives

$$f_\sigma(\kappa_t) - \kappa_t f'_\sigma(\kappa_t) = (1 - \delta + \hat{q}_t^A) i'(\hat{q}_t^A) + i(\hat{q}_t^A), \quad (6.1)$$

$$f'_\sigma(\kappa_t) = (1 - \delta + \hat{q}_t^B) i'(\hat{q}_t^B) + i(\hat{q}_t^B). \quad (6.2)$$

Denote the right-hand side of both conditions by $RHS\left(\hat{q}^j\right)$, $j = A, B$. Due to the properties of the function i defined in (2.10), the range of $RHS\left(\hat{q}^j\right)$ is $\mathbb{R}_+$. Moreover, $RHS'\left(\hat{q}^j\right) > 0$ on $\mathbb{R}_+$.

Similarly, denote by $LHS^j(\kappa_t, \sigma)$, $j = A, B$, the left-hand sides of these conditions. Due to the properties of the CES function, $LHS^j(\kappa_t, \sigma)$ is continuous and strictly positive on $\mathbb{R}_{++}$. Hence, for any pair $(\kappa_t, \sigma) \in \mathbb{R}^2_{++}$ there is a unique $\hat{q}_t^A = g^A(\kappa_t, \sigma) > 0$ that satisfies (6.1) and a unique $\hat{q}_t^B = g^B(\kappa_t, \sigma) > 0$ that satisfies (6.2). Total differentiation of (6.1) and (6.2) delivers $d\hat{q}_t^A / d\kappa_t \equiv g_\kappa^A(\kappa_t, \sigma) > 0$ and $d\hat{q}_t^B / d\kappa_t \equiv g_\kappa^B(\kappa_t, \sigma) < 0$, respectively. ■

6.1.2 Proof of Proposition 2

In total there are 19 variables and 19 equations. By (2.6) and (2.7) the prices $p_{L,t}$ and $p_{K,t}$ depend on κ_t in a unique way. Proposition 1 shows that $\hat{q}_t^A$ and $\hat{q}_t^B$ depend on κ_t through single-valued functions. Then, an application of the equilibrium conditions of Definition 1 reveals that there are unique values l_t, k_t, w_t, R_t, a_t, b_t, A_t, B_t, n_t, m_t, $Y_{K,t}$, $Y_{L,t}$, Y_t depending on $\kappa_t \in \mathbb{R}_{++}$ through either $\hat{q}_t^A$ or $\hat{q}_t^B$. Recall that $y_{l,t} = y_{k,t} = 1$ for all $t \geq 1$ in accordance with Proposition 1. ■

6.1.3 Proof of Corollary 1

Immediate from Proposition 1 and equation (2.26). ■

6.1.4 Proof of Proposition 3

For a steady state, the evolution of A_t and B_t as given by (2.17) requires $q_t^A = q^A$ and $q_t^B = q^B$ for all $t \geq \tau$. Since $I_t^j > 0$ I have $q^j > 0$, $j = A, B$. With this in mind, the neoclassical economy with endogenous capital- and labor-augmenting technical change is isomorphic to the environment to which the Steady-State Theorem of Uzawa (1961) applies.

To see this, consider the resource constraint (3.4), which may be written as

$$C_t + I_t^K + A_t L_t i\left(q^A\right) + B_t K_t i\left(q^B\right) = Y_t. \tag{6.3}$$

Define net output as

$$\tilde{Y}_t = \tilde{F}_\sigma\left(B_t K_t, A_t L_t\right) \equiv F_\sigma\left(B_t K_t, A_t L_t\right) - A_t L_t i\left(q^A\right) - B_t K_t i\left(q^B\right). \tag{6.4}$$

One readily verifies that the net production function $\tilde{F}$ has constant returns to scale in K_t and L_t, and, using (6.1), (6.2) and the properties of f_σ and i, positive and diminishing marginal products of K_t and L_t.

Hence, the environment described by (i) $\tilde{Y}_t = \tilde{F}_\sigma\left(B_t K_t, A_t L_t\right)$, (ii) the resource constraint $C_t + I_t^K = \tilde{Y}_t$, (iii) capital accumulation according to (3.2), and (iv) growth of the labor force at a constant rate is the one to which the Steady-State Growth Theorem of Uzawa (1961) applies (see, Schlicht (2006) and Jones and Scrimgeour (2008)). Hence, in a steady-state it must be that $q^B = \delta$ and $g^* = q^A - \delta$. ■

6.1.5 Proof of Theorem 1

From (3.7) a change in σ affects κ^* since such a change must leave g^B unaffected. Denote this relationship by $\kappa^* = \kappa^*(\sigma)$. An application of the implicit function theorem to (6.2) reveals that $\kappa^*(\sigma)$ satisfies

$$\frac{d\kappa^*}{d\sigma} = -\frac{\partial f'_\sigma}{\partial \sigma}\frac{1}{f''_\sigma}, \tag{6.5}$$

where the argument of f_σ is κ^*.

To study the effect of σ on the growth rate of output per worker, write $g^* = g^A(\kappa^*(\sigma),\sigma) - \delta$ such that

$$\frac{dg^*}{d\sigma} = \frac{\partial g^A(\kappa^*,\sigma)}{\partial \kappa^*}\frac{d\kappa^*}{d\sigma} + \frac{\partial g^A(\kappa^*,\sigma)}{\partial \sigma}. \tag{6.6}$$

From (6.1), I derive

$$\frac{\partial g^A}{\partial \kappa} = -\frac{\kappa^* f''_\sigma}{(1-\delta+g^A)\, i''(g^A) + 2i'(g^A)}, \tag{6.7}$$

$$\frac{\partial g^A}{\partial \sigma} = \frac{\frac{\partial f_\sigma}{\partial \sigma} - \kappa^* \frac{\partial f'_\sigma}{\partial \sigma}}{(1-\delta+g^A)\, i''(g^A) + 2i'(g^A)}, \tag{6.8}$$

where the argument of g^A is (κ^*,σ) and the argument of f_σ is κ^*. Substitution of (6.5), (6.7), and (6.8) in (6.6) gives

$$\frac{dg^*}{d\sigma} = \frac{\frac{\partial f_\sigma}{\partial \sigma}}{(1-\delta+g^A)\, i''(g^A) + 2\, i'(g^A)} > 0. \tag{6.9}$$

The sign of $dg^*/d\sigma$ follows since $sign[\partial f_\sigma(\kappa^*)/\partial\sigma] > 0$ for $\kappa^* \neq \bar{\kappa}$ in accordance with the proof of Theorem 1 of Klump and de La Grandville (2000). ■

6.1.6 Proof of Corollary 2

From (2.26) it follows that

$$\frac{d\hat{\sigma}}{d\sigma} = \frac{1}{1+\sigma(\varepsilon^A_\kappa - \varepsilon^B_\kappa)}\left[1 + \varepsilon^A_\kappa - \varepsilon^B_\kappa + \sigma(1-\sigma)\left(\frac{\partial \varepsilon^A_\kappa}{\partial\sigma} - \frac{\partial \varepsilon^B_\kappa}{\partial\sigma}\right)\right]. \tag{6.10}$$

Evaluated at $\sigma = 1$, I have $d\hat{\sigma}/d\sigma = 1$. In view of Theorem 1, Corollary 2 follows immediately.■

6.2 Local Stability of the Dynamical System of Section 4

In this Appendix, I establish that the two difference equations (4.2) and (4.3) may be transformed into a single second-order, autonomous, non-linear difference equation in κ_t. The linear approximation of this equation at κ^* may then be used to analyze the local stability of the steady state of (4.2) and (4.3).

Rewrite (4.2) as

$$B_t = \phi\left(\kappa_{t+1}, \kappa_t\right), \tag{6.11}$$

where $\phi : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$. Upon replacing the latter in (4.3), I find

$$\phi\left(\kappa_{t+2}, \kappa_{t+1}\right) = \phi\left(\kappa_{t+1}, \kappa_t\right)\left(1 - \delta + g^B\left(\kappa_{t+1}, \sigma\right)\right), \tag{6.12}$$

which implicitly defines a second-order, autonomous, non-linear difference equation in κ_t given κ_1 and κ_2. I denote this difference equation by $\kappa_{t+2} = \eta\left(\kappa_{t+1}, \kappa_t\right)$. Its steady state, $\kappa_t = \kappa^*$ for all t, is given by $\delta = g^B\left(\kappa^*, \sigma\right)$.

To analyze the local stability of the steady state I study the linear approximation of $\eta(\cdot, \cdot)$ at $\kappa_{t+1} = \kappa_t = \kappa^*$. Denote ϕ_i, $i = 1, 2$, the partial derivative of ϕ with respect to the *ith* argument. The total differential of (6.12) evaluated at the steady state delivers

$$\frac{d\kappa_{t+2}}{d\kappa_{t+1}} = \frac{\partial \eta}{\partial \kappa_{t+1}} = \frac{\phi_1 + \phi g^B_\kappa\left(\kappa^*\right) - \phi_2}{\phi_1} \equiv a_1, \tag{6.13}$$

$$\frac{d\kappa_{t+2}}{d\kappa_t} = \frac{\partial \eta}{\partial \kappa_t} = \frac{\phi_2}{\phi_1} \equiv a_2, \tag{6.14}$$

where the argument of ϕ and η is (κ^*, κ^*).

With (6.13) and (6.14), I may approximate κ_{t+2} in a sufficiently close neighborhood of the steady state by

$$\kappa_{t+2} = a_1\kappa_{t+1} + a_2\kappa_t + R, \tag{6.15}$$

where $R = \kappa^*\left(1 - a_1 - a_2\right)$. The stability of the linear difference equation (6.15) is determined by its characteristic equation $\mu^2 - a_1\mu - a_2 = 0$. The roots of the latter are given in the forth column of Table 1. In case of distinct complex roots, I add their modulus in Column 5.

Figures 1 and 2 show the evolution of $\hat{\sigma}_t = \hat{\sigma}\left(\kappa_t, \sigma\right)$ for $t = 1, ..., 31$ in the neighborhood of the steady state for the cases $\sigma = 0.5$ and $\sigma = 8$, respectively. To derive these trajectories, I solve (6.15) for $\kappa_1 = \kappa^*(\sigma)$ and $\kappa_2 = \kappa^*(\sigma) + 0.005$, and use the solution $\{\kappa_t\}_{t=1}^{31}$ to compute $\hat{\sigma}_t = \hat{\sigma}\left(\kappa_t, \sigma\right)$. Figures 3 and 4 depict $\hat{\sigma}\left(\kappa, \sigma\right)$ of equation (2.26) for $\kappa \in \left(\kappa^* - 0.007, \kappa^* + 0.007\right)$.

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