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On Spatial Equilibria in a Social Interaction Model

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Abstract

Social interactions are at the essence of societies and explain the gathering of individuals in villages, agglomerations, or cities. We study the emergence of multiple agglomerations as resulting from the interplay between spatial interaction externalities and competition in the land market. We show that the geographical nature of the residential space tremendously affects the properties of spatial equilibria. In particular, when agents locate on an open land strip (line segment), a single city emerges in equilibrium. In contrast, when the spatial economy extends along a closed land strip (circumference), multiple equilibria with odd numbers of cities arise. Spatial equilibrium configurations involve a high degree of spatial symmetry in terms of city size and location, and can be Pareto-ranked.

Keywords: social interaction, multiple agglomerations, spatial economy.

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1 Introduction

A major source of spatial heterogeneity stems from non-market interactions. Social interactions through face-to-face contacts are at the essence of our societies and explain the gathering of individuals in villages, agglomerations, or cities. They translate a psychological need for maintaining relationships with one another, and favor a constant exchange of ideas; see Krugman (1991), Glaeser and Scheinkman (2003), and Fujita and Thisse (2002). In this paper we address the issue of the emergence of multiple agglomerations as the result of the interplay between social interactions and competition in the land market.

The present paper builds on Beckmann's (1976) model. This model provides a simple rationale for the spatial agglomeration of agents as the result of spatial interaction externalities. Agents are distributed along some geographical space and benefit from social interactions with other agents. These social interactions provide individuals with a social interaction benefit while entailing an individual cost as each one must access to distant agents. Moreover the return of spatial interactions is also balanced by a cost of residence as agents compete for land space. When the benefit of social interactions is larger than the commuting and residence costs, agents prefer to locate close to each other, which leads to the formation of agglomerations. In his original work, Beckmann considered the case of a one-dimensional spatial economy extending along an open land strip (line segment). The resulting equilibrium consists in a uni-modal symmetric - bell-shaped - spatial distribution, where agents agglomerate around the city centre, see Fujita and Thisse (2002).

In this paper, first we revisit Beckmann (1976)'s model and then extend it to the case of a spatial economy extending along a closed land strip (circumference). While the modelling along an open land strip seems appropriate to describe the internal structure of cities, the formulation along a closed land strip provides a natural framework to analyze the interaction between multiple agglomerations. Circular spatial frameworks have been studied in 'racetrack economy' models in the context of the New Economic Geography literature, see e.g. Fujita *et al.* (1999), Mossay (2003), or Picard and Tabuchi (2009). Yet, because of the complexity of market interactions, this strand of literature has been able to characterize only a small subset of equilibrium distributions (e.g. the uniform spatial distribution of agents, often referred to as the 'flat-earth' distribution, or constant-access equilibria). As a consequence, this strand of literature has left unresolved issues dealing with the nature and structure of other equilibrium distributions. Among these issues, are the multiplicity of those other equilibria, their possible spatial symmetry, and the allocation of land between inhabited areas and empty hinterlands. Because our social interaction model has a much simpler structure than that involved in market interaction models, we are able to address the above issues and characterize spatial equilibria.

Our results are the following. First we determine the equilibrium and first-best spatial distributions of agents along an open land strip. Social interactions generate the emergence of a single city, meaning that multiple cities can't be sustained in equilibrium along a line. This result is similar to that obtained by Berliant *et al.* (2002) who also showed the emergence of a unique centre in the case of production externalities. In accordance to Fujita and Thisse (2002), the first-best distribution is more concentrated than the equilibrium one; see also Tabuchi (1986). At equilibrium agents choose a too large lot size because they do not internalize other agents' preferences when making their own lot choice. Second we provide the characterization of spatial equilibria emerging along a closed land strip (circular geographical space). In equilibrium, cities are identical and evenly spaced: cities share the same spatial structure and successive firms along the circumference are equidistant. We also show that equilibrium configurations involve an odd number of cities. Furthermore spatial equilibria can be Pareto-ranked. The total welfare of the spatial economy decreases with the number of cities so that the one-city configuration Pareto dominates all the other configurations. Like in the open land strip framework, the first-best distribution corresponds to a single city structure which is more concentrated than the equilibrium distribution.

Our paper deals with the endogenous formation of multiple-centre configurations. A contribution of Berliant *et al.* (2002) is that the nature of spatial externalities matters and affects the properties of spatial equilibria. In Fujita and Ogawa (1980, 1982), multiplicity of equilibria arises because of a fixed factor in the production process. In contrast, the spatial production externalities analyzed by Berliant *et al.* (2002) lead to the formation of a single centre in equilibrium. Our paper identifies another factor affecting the properties of spatial equilibria, namely the nature of the geographical space itself. Our results suggest that loops within a network favour the emergence of multiple cities as opposed to the unique centre emerging along a line segment. To our knowledge, this paper constitutes the very first step toward the characterization of interacting economies extending along spatial (road) networks that involve a loop. In constrast to Fujita and Ogawa (1980, 1982) and Berliant *et al.* (2002), our model is analytically tractable because of the linear structure of the model. For that reason we are able to perform a full general equilibrium analysis, without assuming inelastic land consumption or relying on simulations.

Our paper also contributes to the literature on city structure in a particular way. Since von Thunen, most theoretical works about city structures make the assumption of revolution symmetry around the city centre in order to reduce the spatial dimension from 2 to 1. Recent works on endogenous city formation still make that convenient assumption (e.g. Lucas and Rossi-Hansberg, 2002). As such, it is important to check whether revolution symmetry can be derived as an equilibrium property of a spatial economy where agents can locate freely. Since revolution symmetry does not obtain in our model, our paper sheds some doubt on the use of the revolution symmetry assumption made in the existing literature.

We present the model of social interactions in Section 2. We derive the equilibrium and the first-best spatial distributions of the model along an open line in Section 3. Section 4 characterizes spatial equilibria along a circumference. Section 5 ranks these various equilibria and compares them with the socially optimal distribution.

2 The Model

In this Section we present the economic environment. A unit-mass of agents is distributed along a one-dimensional geographical space according to the density $\lambda(x)$ with $\int \lambda(x)dx = 1$. Agents travel along the one-dimensional road and benefit from social contacts with other agents. The social utility that an agent in location x derives from interacting with other agents is given by

$$S(x) = A - \int \lambda(y) T(x-y) dy \tag{1}$$

The first term A denotes the total return from interacting with other agents. The second term reflects the cost of trips of accessing to distant agents.[1] We consider the case of a linear cost function $T(x-y) = 2\tau |x-y|$, where τ measures the intensity of travelling costs. In our model each agent interacts with *all* the other agents meaning that A will be assumed to be large enough to ensure that $S(x) \geq 0$, for any location x. The surplus $S(x)$ can be interpreted in a context of certainty or uncertainty. Indeed, it can be interpreted literally as the utility derived by an individual who plans to interact with all other agents with probability 1. It can also be interpreted as the expected utility of an individual who plans to interact with a subset of agents whom location and identity are not known at the time of the residence choice. Such an interpretation applied to the case of shopkeepers, sellers, as well as workers who expect to hold several jobs at different locations during their lifetime, or employers who do not have a precise idea about future workers' residences.

In each location x, the residential area is longitudinal to the main road. It is a strip of land space with unit width, which is connected by its edge to the main road. Agents in location x consume a composite good $z(x)$ and some land space $s(x)$. Their utility U is given by

$$U(x) = S(x) + z(x) - \frac{\beta}{2\ s(x)}$$

where $S(x)$ is the social utility as given by relation (1) and β reflects the preference for land. The budget constraint faced by agents is

$$z(x) + R(x)s(x) = Y$$

where Y is the income and $R(x)$ denotes the land rent in x.[2] By using this budget constraint, the utility derived in location x can be rewritten as

$$U(x) = S(x) - \frac{\beta}{2\ s(x)} - s(x)R(x) + Y \tag{2}$$

[1] Note that the term $(\int \lambda(y)T(x-y)dy)$ could be formulated as a monetary cost. Anyway such a reformulation would have no incidence on our analysis.

[2] In the context of a general equilibrium, the variable Y should be interpreted as the valuation of an initial endowment of the composite good z.

This formulation of the utility function differs from Beckmann's formulation in one respect only: we consider an hyperbolic preference for land instead of the logarithmic preference used in Beckmann (1976) and Fujita and Thisse (2002). This will allow us to simplify considerably the characterization of equilibria.

Landlords raise the land rent until no worker moves. Let U^* be an equilibrium utility of workers. The bid rent function is given by

$$\Psi(x) = \max_s \frac{S(x) - \beta/(2s) + Y - U^*}{s}$$

which yields the optimal land consumption $s^*(x)$ as determined by $\beta/[2s^*(x)] = s^*\Psi^*(x) = [S^*(x) + Y - U^*]/2$. At the residential equilibrium, land rents are equal to the bid rents so that $R^*(x) = \Psi^*(x)$, which implies that $s^*R^*(x) = \beta/(2s^*(x))$. The indirect utility can then be written as $U(x) = S(x) - \beta/s^*(x) + Y$. Since the land strip has a unit width, the individual land consumption $s^*(x)$ corresponds to $1/\lambda(x)$, so that the indirect utility in location x can be written in terms of the population density $\lambda(x)$ as

$$V(x) = S(x) - \beta\lambda(x) + Y \tag{3}$$

This means that the residents' utility at location x *linearly* increases with the social return $S(x)$ and *linearly* decreases with the residential density $\lambda(x)$. Utility decreases with the residential density because agents compete for land space and thus face higher land prices in more populated areas. We will take advantage of this linear structure to characterize spatial equilibria and the optimal spatial distribution. The characterization of these spatial configurations constitutes the major contribution of our paper to the existing literature. In what follows we assume without much loss of generality that land has no other use than residence so that the opportunity cost of land is zero.

3 Spatial Equilibrium along a Line Segment

In this Section we formulate our social interaction model along a line segment as studied in Beckmann (1976) and Fujita and Thisse (2002, Chapter 6).

3.1 Spatial Equilibrium

A distribution of agents $\lambda(.)$ constitutes a spatial equilibrium if agents have no incentive to relocate. In other words, $\lambda(x)$ is a spatial equilibrium if $V(x) = \overline{V}$ when $\lambda(x) > 0$ and $V(x) \leq \overline{V}$ when $\lambda(x) = 0$. Because agents reach the same utility over all inhabited locations, we have that $V'(x) = V''(x) = 0$ for all x where $\lambda(x) > 0$. In this paper, any area with a positive population is referred to as a city. We now characterize the spatial distribution along a line segment and show that spatial equilibrium implies the emergence of a single city.

First let us consider a single city located along the interval $[-b, b]$, $b > 0$. By differentiating the social utility expression (1) with respect to x, we have that

$$S'(x) = 2\tau \int_x^{-b} \lambda(y)dy - 2\tau \int_b^x \lambda(y)dy$$
$$S''(x) = -2\tau\ \lambda(x) - 2\tau\ \lambda(x) = -4\tau\ \lambda(x)$$

Because of linear travel costs, $S''(x)$ reduces to a linear function of λ. Hence, by using relation (3), a necessary condition for equilibrium is $V''(x) = S''(x) - \beta\lambda''(x) = 0$, which leads to

$$\lambda''(x) + \delta^2\lambda(x) = 0 \quad \text{where} \quad \delta^2 \equiv 4\tau/\beta \tag{4}$$

The solution to this differential equation is given by

$$\lambda(x) = C\cos[\delta\,(x - x_o)] \tag{5}$$

where C and x_o are constants to be determined. Because $R^*(\pm b) = [\lambda(\pm b)]^2\beta/2 = 0$ and $\int \lambda(x)dx = 1$, the equilibrium spatial structure and the boundary of the city are given by

$$\lambda^*(x) = \frac{\delta}{2}\cos(\delta x) \quad \text{and} \quad b^* = \frac{\pi}{2\delta} \tag{6}$$

This describes the spatial structure of a single city. The density is a concave function of x. We must also ensure that each agent is willing to interact with all the other agents so that $S(x) > 0$ for all x in $(-b^*, b^*)$. We need to assume that $A > \int_{-b}^{b} \lambda^*(y)T(y-b)dy$ or equivalently that $A > \tau\pi/\delta = \pi\sqrt{\beta\tau}/2$. Note that the equilibrium utility level is given by

$$V^* = V^*(x=0) = A - \int_{-b}^{b} \lambda^*(y)T(y)dy - \beta\lambda^*(0) + Y$$
$$= A - \frac{\pi - 2}{\delta}\tau - \frac{1}{2}\beta\delta + Y = A - \frac{\pi}{2}\sqrt{\beta\tau} + Y$$

An important issue is whether multiple cities can co-exist in equilibrium. The answer turns out to be negative. To show this, consider a set of cities possibly separated by empty hinterlands. Let the supports of the $M \geq 2$ cities be the closed intervals $[a_m, b_m]$, $m = 1, 2, 3, ..., M$, with $b_m < a_{m+1}$. Indeed, within each city m, Equation (4) holds and $R^*(a_m) = R^*(b_m) = 0$, so that $\lambda(x)$ can be written as $\lambda_m(x) = C_m \cos[\delta(x - x_m)]$, $\forall x \in [a_m, b_m]$, with $x_m = (b_m - a_m)/2$. From Relations (1) and (3), we get

$$V(x) = A - 2\tau \sum_{m=1}^{M} \int_{a_m}^{b_m} \lambda(y)\,|x - y|\,dy - \beta\lambda(x) + Y$$

We show that some residents living in city M have an incentive to relocate. By differentiating previous expression, we get

$$V'(x) = -2\tau \left[1 - \int_{a_M}^{b_M} \lambda_M(y)dy\right] - 2\tau \int_{a_M}^{x} \lambda_M(y)dy + 2\tau \int_{x}^{b_M} \lambda_M(y)dy - \beta\lambda'_M(x),\ \forall x \in [a_M, b_M]$$

When a resident relocates to his right, he looses access to the residents to his left, who live either in other cities (first term) or in his own city (second term). He also gains a better access to the residents to his right within his own city (third term) and faces an increase in land rent (resp. a decrease in land rent) if $x \in [a_M, x_M]$, (resp. $x \in [x_M, b_M]$), (last term). In particular, at the centre of city M, $\lambda'_M(x_M) = 0$ and

$$V'(x_M) = -2\tau \left[1 - \int_{a_M}^{b_M} \lambda_M(y)dy\right] = -2\tau(1 - L_M) < 0$$

where L_M denotes the population in city M. Therefore, residents living at the centre of city M have always an incentive to move leftward, and no spatial configuration with $M \geq 2$ cities can't be sustained in equilibrium.

We summarize our results in the following Proposition.

Proposition 1 If $A > \pi\sqrt{\beta\tau}/2$, the spatial equilibrium along a line segment is unique and involves a single unimodal city. The *equilibrium utility level decreases with the travel cost* (τ) and *the preference for residential space* (β).

The *spatial equilibrium distribution is symmetric* with respect to location $x = 0$ and *concave*. This distribution is similar to that obtained by Beckmann except that here the city structure is nowhere convex because of our hyperbolic preference for residential space and the zero opportunity cost of land.

Also the spatial equilibrium distribution involves a unique centre. This result is similar to that obtained by Berliant *et al.* (2002) in the case of spatial knowledge spillovers.

3.2 First-Best Spatial Distribution

In this subsection we determine the first-best distribution of agents as opposed to the equilibrium distribution analyzed so far. A utilitarian planner chooses the best spatial distribution of agents $\lambda(.)$ and the city border b so as to maximize the total welfare denoted by W. The planner's program is therefore

$$\max_{\lambda(.),b} W = \int_{-b}^{b} U(x)\lambda(x)dx = \int_{-b}^{b} [S(x) + z(x) - \frac{\beta}{2\ s(x)}]\lambda(x)dx \tag{7}$$

subject to the budget balance $\int_{-b}^{b}[Y - z(x)]\lambda(x)dx = 0$ and the total population constraint $\int_{-b}^{b}\lambda(x)dx = 1$. Substituting the budget balance into the above expression yields

$$\begin{aligned}\max_{\lambda(.),b} W &= \int_{-b}^{b}[S(x) - \frac{\beta}{2\ s(x)} + Y]\lambda(x)dx \\ &= \int_{-b}^{b}[A - \int_{-b}^{b}\lambda(y)T(x-y)dy - \beta/2\lambda(x) + Y]\lambda(x)dx\end{aligned}$$

By using varational analysis, we show in Appendix A that the first-best distribution satisfies the following relationships

$$\begin{aligned} S(x) - \frac{\beta}{2}\lambda(x) &= \frac{A + \mu - Y}{2} \\ \lambda(b) + \lambda(-b) &= 0 \end{aligned} \tag{8}$$

where $\mu > 0$ is the Lagrange multiplier associated with the total population constraint.

This characterization yields two conclusions. First, it should be that $\lambda(b) = \lambda(-b) = 0$ since $\lambda(x) \geq 0$. Second, the function $S(x) - (\beta/2)\lambda(x)$ is constant so that its first and second derivatives should be nil. Note that this function is similar to the expression of the indirect utility derived in the decentralized equilibrium analysis, see expression (3), except that β is to be replaced by $\beta/2$. Therefore, it should be that at the optimum $\lambda''(x) + (\delta^o)^2\lambda(x)$, where $(\delta^o)^2 = 4\tau/(\beta/2) = 2\delta^2$. This means that $\lambda^o(x) = C^o \cos\delta^o x$

with the city border b^o determined by $\lambda^o(b^o) = 0$, that is $b^o = \pi/(2\delta^o)$, and $C^o = \delta^o/2$ given the total population constraint.

Hence we oberve that *the optimal city has a narrower support than the decentralized city*, $b^o < b^*$. Because the first-best and the equilibrium cities host the same number of residents, the density of residents must be larger at the first-best, $C^o > C^*$.

We summarize our results in the following Proposition.

Proposition 2 The first-best spatial distribution is unimodal and the optimal city is more concentrated than the equilibrium city.

In accordance to Fujita and Thisse (2002), the first-best distribution is more concentrated than the equilibrium one; see also Tabuchi (1986). At equilibrium agents choose a too large lot size because they do not internalize other agents' preferences when making their own lot choice.

4 Spatial Equilibrium along a Circumference

In this Section we consider the robustness of previous results with respect to another form of geographical space. In particular we want to check whether spatial interactions along a line which is closed, such as a circumference, lead to the formation of a single city or to the emergence of multiple centres. The equilibrium characterization is more difficult to obtain along such a geographical space. For the sake of comparison with the equilibrium on a line segment, we therefore focus on the formation of unimodal cities and show that spatial equilibria can involve multiple cities. A major contribution of this paper is to provide the characterization of such multiple agglomerations in equilibrium.

To obtain our results, we proceed in several steps. Like in previous Section, we first derive a necessary equilibrium condition (Lemma 1). This condition expresses the trade-off between the residence cost and the accessing cost to other agents. We then derive another necessary equilibrium condition (Lemma 2) which simply states that an equilibrium distribution is of made of pieces, each of which having the shape of the cosine function as obtained along the line segment in Section 3. We show that in equilibrium antipodal cities can't exist (i.e. cities can't face each other along the circumference) (Lemma 3). This subsequently implies that no equilibrium with an even number of cities can exist (Lemma 4). Finally we show that in equilibrium cities are equally populated and evenly spaced (successive cities are equidistant) along the circumference (Proposition 3). Whereas it may be intuitive that these spatial distributions constitute equilibria, it is far from obvious a priori to exclude other asymmetric distributions in terms of size or location. In this Section we present our results as well as their interpretation.

We discuss spatial configurations involving cities separated by empty hinterlands along a circumference of unit perimeter. M denotes the total number of cities and $[a_m, b_m]$ the support of city m so that the support of the spatial distribution λ can be written as $supp\ \lambda = \bigcup_{m=1}^{M} [a_m, b_m]$. Let H be the set of empty hinterlands that is the set of 'empty' locations so that $H = [0, 1]/supp\ \lambda$. At equilibrium we must have that $V(x) = V^*$, $\forall x \in$suppλ and $V(x) \leq V^*$, $\forall x \in H$.

Consider some agent located in city m so that $x \in [a_m, b_m]$. We define $P^+(x)$ (resp. $P^-(x)$) as the population that is located at a clockwise (resp. counterclockwise) distance from x smaller than 1/2. This means that $P^+(x)$ and $P^-(x)$ divide the total population into that at the right and that at the left of x. A first order differentiation of the indirect utitity $V(x)$ yields the following lemma.

Lemma 1 In equilibrium $P^+(x) - P^-(x) = \beta\lambda'(x)/(2\tau)$, $\forall x \in supp\ \lambda$.

Proof. See Appendix B.1. ■

This condition expresses the trade-off between the residence cost and the accessing cost: in equilibrium an increase in residence cost must be compensated by a better access to distant agents.[3] So as to illustrate Lemma 1, suppose that $\lambda'(x) < 0$, so that an agent located at x enjoys a lower residence cost by moving clockwise. Lemma 1 says that this gain in terms of residence cost should be balanced by a larger accessing cost. This means that the population that the agent gets closer to (i.e. the population at his right, P^+) should be less numerous that the population he gets further away from (i.e. the

[3] This condition between the residence and accessing costs is similar to Muth's (1969) gradient condition on land rents and commuting costs.

population at his left P^-). *The marginal residential cost of moving to the right or to the left corresponds to the marginal gain of accessing to other agents.*

Because agents may access to other agents by travelling to the right or to the left, they will be sensitive to the fact that other agents may be located in the opposite location along the circumference. For this reason, it is useful to rely on the concept of *antipodal cities* which are defined as cities 'facing' each other along the circumference. More precisely, the spatial distribution $\lambda(.)$ is said to admit antipodal cities if there exist two inhabited locations x and $x+1/2$, i.e. $\lambda(x)\lambda(x+1/2) > 0$. By contrast, there are no antipodal cities if $\lambda(x)\lambda(x+1/2) = 0$ for every location $x \in [a_m, b_m]$, $m = 1, ..., M$.

Differentiating once more the spatial indirect utility $V(x)$ yields another necessary condition, namely $V''(x) = 0$. First we consider the case in the absence of antipodal cities and show that each piece of the equilibrium distribution is determined by a spatial structure similar to that given by expression (5). This is summarized in the following Lemma.

Lemma 2 Consider a spatial structure involving M cities of support $[a_m, b_m]$, $m = 1, ..., M$. Suppose that there are no antipodal cities. Then, an equilibrium distribution is given by $\lambda(x) = C_m \cos[\delta(x - x_m)]$, $\forall x \in \text{supp}\lambda$, where $\delta^2 = 4\tau/\beta$, $\delta(b_m - a_m) = \pi$, $x_m = (b_m - a_m)/2$, and C_m is a positive constant.

Proof. See Appendix B.2. ■

The structure of cities is given by the *same cosine function* as in the case of an economy extending along a line segment, see expression (6). Note that Lemma 2 applies to spatial economies with no antipodal cities only. In what follows we show that in fact no antipodal cities can exist. In order to illustrate this, we show that a spatial configuration consisting of 2 symmetric antipodal cities located at the North and the South of the circumference ($x = 0$ and $x = 1/2$) can not be sustained in equilibrium. The supports of these cities are denoted by $[-b, b]$ and $[1/2 - b, 1/2 + b]$, see left panel in Figure 1. By applying Lemma 1 in locations $x \in [-b, b]$ and $x + 1/2$, we get

$$\lambda'(x) + \lambda'(x+1/2) = \frac{2\tau}{\beta}\left[P^+(x) - P^-(x) + P^+(x+1/2) - P^-(x+1/2)\right]$$

Given that $P^+(x) = P^-(x + 1/2)$ and $P^-(x) = P^+(x + 1/2)$, the RHS of the above relation is equal to 0, which leads to an inconsistency given that in our example, $\lambda'(x) = \lambda'(x + 1/2) \neq 0$ if $x \neq 0$. The above condition actually implies that if $\lambda'(x) > 0$ then $\lambda'(x + 1/2) < 0$: if equilibrium land rents increase in location x, they should necessarily decrease in location $x + 1/2$. As a consequence, if $\lambda'(-b) > 0$, then $\lambda'(1/2 - b)$ should be negative which would imply some negative population levels. This situation is illustrated in the left panel of Figure 1.

INSERT FIGURE 1 HERE.

The following Lemma generalizes the argument made above and rules out any spatial distribution involving antipodal cities in equilibrium.

Lemma 3 There exists no spatial equilibrium with antipodal cities, except the uniform distribution.

A first implication of Lemma 3 is that, except the uniform equilibrium, any spatial equilibrium distribution involves *empty hinterlands.* Such spatial equilibria result from the natural tension between the supply and the residents' self-organized use of space. Indeed, in equilibrium, when residents in a particular city have no arbitrage opportunity, the population density follows the law given in Lemma 2. This law determines not only the use of space in each city but also the city support (i.e. $[a_m, b_m]$). As a consequence, there is no reason for which the union of city supports, $\bigcup_m [a_m, b_m]$, should fit the available space on the circumference. This is what explains the existence of empty hinterlands between cities. The present characterization of equilibria contrasts with that obtained in the existing racetrack models of spatial agglomeration that faced a sharp difficulty to identify equilibria other than the uniform spatial distribution, called 'flat earth', and constant-access equilibria (see for instance Fujita *et al.* (1999), Mossay (2003), and Picard and Tabuchi (2009)). The present paper identifies a new class of spatial equilibria ignored so far in the existing literature.

A second implication of Lemma 3 lies in the impossibility to get equilibria with an even number of cities. As an illustration, let us explain the argument for a configuration that would involve an even number of identical cities. By Lemma 3, we know that these cities can't be antipodal. Suppose that cities are located at asymmetric distances. The case of 2 such cities is depicted in the right panel of Figure 1. By applying Lemma 1 at the centre x_m of a city, we get that $P^+(x_m) = P^-(x_m)$ because the land rent gradient is nil at the city centre ($\lambda'(x_m) = 0$). This means that the populations at the right and at the left of the city centre x_m should be equal, which is inconsistent with our example since one side of the city will involve an even number of cities while the other side will involve an odd number of cities, given that the total number of cities is even. In this example, the argument applies because cities are of equal size. The following Lemma extends the argument to the case of spatial distributions involving cities of different size.

Lemma 4 Any non-uniform spatial equilibrium displays an odd number of cities.

Proof. See Appendix B.3. ■

What is now left to be determined is the size of cities and their location along the circumference. In Appendix B.4, we apply an argument used in Lemma 4 to pairs of cities located on the circumference. Then we show that such pairs of cities have an identical population size in equilibrium. By inference, all cities must have the same size. Furthermore, once cities have an identical size, it is easy to understand why they should be evemly spaced along the circumference. This is because any asymmetry in the location of these cities would necessarily confer an advantage to residents of some city and a disavantage to residents of some other city, thus precluding equilibrium. We summarize our results in the following Proposition.

Proposition 3 The set of spatial equilibria consists of the uniform distribution (the 'flat earth' distribution) as well as any odd number of identical and evenly spaced non-antipodal cities, each of which having an internal structure as given by Lemma 2.

Proof. See Appendix B.4. ■

In contrast to Beckmann's result (1976) and Fujita and Thisse (2002), *multiple-city configurations do emerge along a circular geographical space.* The characterization of spatial equilibria is obtained in the context of a general equilibrium analysis due to the linear structure of the model. It implies *the existence of empty hinterlands and a high degree of spatial symmetry in terms of size and location.* According to Lemma 4, *configurations with an even number of cities do not exist*, and antipodal cities can't be sustained, see Lemma 3.

In equilibrium, the support of cities should fit the unit perimeter of the circumference ($2M\pi/\delta < 1$), so that the maximum number of cities is given by $M^{\max}$ =int$(\delta/(2\pi))$. On the other hand, since the total population is 1, we have that $M \int_{-\frac{\pi}{2\delta}}^{\frac{\pi}{2\delta}} C\cos(\delta x)\, dx = 1$, meaning that $C_m = C = \delta/(2M)$, $\forall m$. These two last conditions put an upper bound $M^{\max}$ on the admissible number of cities and relate the amplitude C of each city to the total number of cities M.

The existing literature already emphasized the type of spatial externalities in determining the number of centres emerging in equilibrium, see Berliant *et al.* (2002). Our results contribute heavily to the theoretical understanding of multiple centres. Propositions 1 and 3 identify the geographical space itself (circumference vs. line segment) as affecting the properties of spatial equilibria (multiplicity vs. uniqueness of centres).

Since von Thunen, most theoretical works related to the study of urban agglomerations have assumed (revolution) symmetry around city centres. Recent works on endogenous city formation still make that convientient assumption, e.g. Lucas and Rossi-Hansberg (2002). In our social interaction model, revolution symmetry does not necessarily hold along a circumference. As such, our analysis sheds some doubt on the use of that assumption made in the literature.

5 Pareto-Ranking of Equilibria and Optimum

In this Section we rank the spatial equilibria obtained in Section 4 in the sense of Pareto. Then we compare the Pareto dominating equilibrium with the first-best distribution.

Consider a spatial equilibrium with an odd number M of identical evenly spaced non-antipodal cities as given by Proposition 3. With no loss of generality, we assume that the first city is centered at $x = 0$. In equilibrium the utility is the same for all residents and corresponds to the utility of residents located at $x = 0$, which is given by

$$V^*(M) = A - \beta\lambda(a_1 + \frac{\pi}{2\delta}) - \sum_{m=1}^{M}\int_{a_m}^{a_m+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy + Y$$

where a_m corresponds to the left-border of city m. Developments given in Appendix $C.1$ lead to

$$V^*(M) = A - \tau\frac{\pi - 2}{\delta M} - \tau\frac{M^2 - 1}{2M^2} - \frac{2\tau}{\delta M} + Y$$

The first term represents the benefit of social interactions, the second one the agent's travel cost to other agents in their own city, the third one the travel cost to agents living in other cities, and the next to last one the land rent. It can be shown that $V^*(M)$ is a decreasing function in the admissible interval $[1, \delta/(2\pi)]$. Therefore spatial equilibria can be ranked as sumarized in the following Proposition.

Proposition 4 The smaller the number of cities, the larger the total welfare of the equilibrium distribution. If $\delta > 2\pi$ (resp. $\delta < 2\pi$), then the Pareto-dominating spatial equilibrium configuration corresponds to a single city distribution (resp. the uniform distribution).

Proof. Appendix $C.2$ ■

Of course, when no city as given by Lemma 2 can fit the unit perimeter of the circumference, the only possible equilibrium is the uniform distribution $\lambda(x) = 1$. Now we determine the first-best distribution of residents along the circumference.

Proposition 5 When $\delta > \pi$ (resp. $\delta < \pi$), the optimal spatial configuration corresponds to a single city distribution with welfare $A - \pi\sqrt{\beta\tau}/(2\sqrt{2}) + Y$ (resp. the uniform spatial distribution of agents with welfare $A - \beta - \tau/2 + Y$).

Proof. Appendix $C.3$ ■

Like in Beckmann's framework in Section 3.2, *the social optimum involves a single city which is more concentrated than the equilibrium distribution.* Of course, this occurs provided that the optimal city can fit the unit perimeter. Otherwise, the first-best distribution corresponds to the uniform distribution of agents. While an increase of the travel cost (τ) favours the optimal agglomeration, an increase of the preference for residential space (β) favours the optimal uniform distribution of residents.

6 Conclusion

We have studied a spatial model of social interactions. We have shown that only a single city can emerge along a line segment. On the other hand, along a circumference, multiple equilibria can emerge. We have shown that in equilibrium, cities are identical, in odd numbers, and evenly spaced along the circumference. The smaller the number of cities, the larger the total welfare of the spatial economy. The first-best distribution corresponds to a single city which is more concentrated than the equilibrium city. Our paper constitutes a very first analysis toward the characterization of spatial equilibria along spatial (road) networks that include loops. It identifies the geographical space itself as a very important factor affecting the properties of spatial equilibria (multiplicity, size, spacing).

Appendix

6.1 Appendix A: Proof of $S(x) - \beta/2\lambda(x) = (A + \mu - Y)/2$ and $\lambda(b) + \lambda(-b) = 0$

Proof. The Lagrange functional L of problem (7) can be written as

$$L = \int_{-b}^{b} [S(x) - \frac{\beta}{2}\lambda(x) + Y]\lambda(x)dx - \mu[\int_{-b}^{b} \lambda(x)dx - 1]$$

where μ is the Lagrange multiplier associated with the total population constraint and we used $\lambda(x) = 1/s(x)$.

First, we determine a first-order condition with respect to the city border b. By differentiating the above expression with respect to b, we get

$$[S(b) - \frac{\beta}{2}\lambda(b) + Y - \mu]\lambda(b) + [S(-b) - \frac{\beta}{2}\lambda(-b) + Y - \mu]\lambda(-b) = 0 \tag{9}$$

Second, we determine the first-order condition with respect to the spatial distribution λ. By using Relation (1), the Lagrange functional can be rewritten as

$$L = \int_{-b}^{b} [A - \int_{-b}^{b} \lambda(y)T(x - y)dy - \beta/2\lambda(x) + Y]\lambda(x)dx - \mu[\int_{-b}^{b} \lambda(x)dx - 1]$$

Now consider some infinitesimally small variation $\widetilde{\lambda}(x)$ around the optimal solution $\lambda(x)$. The variation of L is given by

$$\begin{aligned}\widetilde{L} &= \textstyle\int_{-b}^{b}[(A+Y-\mu)-\beta\lambda(x)]\widetilde{\lambda}(x)dx \\ &- \textstyle\int_{-b}^{b}\int_{-b}^{b}T(x-y)\lambda(y)\widetilde{\lambda}(x)dxdy - \int_{-b}^{b}\int_{-b}^{b}T(x-y)\lambda(x)\widetilde{\lambda}(y)dxdy \\ &= \textstyle\int_{-b}^{b}[(A+Y-\mu)-\beta\lambda(x)-2\int_{-b}^{b}T(x-y)dy\lambda(y)]\widetilde{\lambda}(x)dx\end{aligned}$$

given that $\int_{-b}^{b}\int_{-b}^{b}T(x-y)\lambda(x)\widetilde{\lambda}(y)dxdy = \int_{-b}^{b}\int_{-b}^{b}T(y-x)\lambda(y)\widetilde{\lambda}(x)dxdy = \int_{-b}^{b}\int_{-b}^{b}T(x-y)\lambda(y)\widetilde{\lambda}(x)dxdy$ by using the symmetry of $T(x)$. At the optimum, $\widetilde{L}$ must be equal to zero for any admissible variation $\widetilde{\lambda}(x)$ around the optimal distribution $\lambda(x)$. This implies successively

$$(A+Y-\mu)-\beta\lambda(x)-2\textstyle\int_{-b}^{b}T(x-y)dy\lambda(y) = 0$$

$$2[A-\textstyle\int_{-b}^{b}T(x-y)dy\lambda(y)]-\beta\lambda(x) = A+\mu-Y$$

$$S(x)-\frac{\beta}{2}\lambda(x) = \frac{A+\mu-Y}{2}$$

Therefore the function $S(x)-(\beta/2)\lambda(x)$ is a constant. Finally, by substituting this last expression evaluated in $x=b$ and $x=-b$ into expression (9), we get

$$[\frac{A+\mu-Y}{2}+Y-\mu][\lambda(b)+\lambda(-b)] = 0$$

which yields $\lambda(b)+\lambda(-b)=0$. ■

6.2 Appendix B.1: Proof of Lemma 1

Proof. The set of intervals $\{[a_m,b_m]\}_{m=1,\dots,M}$ denote the city supports. We need to introduce additional definitions to describe the location of cities with respect to each other. We denote by I_m^+ (resp. I_m^-) the set of indices of cities that are located at a clockwise (resp. counterclockwise) distance from interval m inferior to 1/2. We consider an agent located at $x\in[a_m,b_m]$. When $x+1/2\notin H$, we denote by j_m the interval index to which $x+1/2$ belongs to. The utility of an agent located in $x\in[a_m,b_m]$ in city m can then be written as

$$V(x) = A - 2\tau[\sum_{i\in I_m^+}\int_{a_i}^{b_i}(y-x)\lambda(y)dy + \sum_{i\in I_m^-}\int_{a_i}^{b_i}(1-(y-x))\lambda(y)dy]$$
$$- 2\tau[\int_{a_m}^{x}(x-y)\lambda(y)dy + \int_{x}^{b_m}(y-x)\lambda(y)dy] - \beta\lambda(x)$$
$$- 2\tau\chi_{\text{supp}\lambda}(x+1/2)\left[\int_{a_{jm}}^{x+1/2}(y-x)\lambda(y)dy + \int_{x+1/2}^{b_{jm}}(1-(y-x))\lambda(y)dy\right] + Y$$

where $\chi_{\text{supp}\lambda}$ denotes a characteristic function so that $\chi_{\text{supp}\lambda}(x)$ is equal 1, if $x \in$suppλ, and 0 otherwise. By differentiation with respect to x, we get

$$-2\tau\sum_{i\in I_m^+}\int_{a_i}^{b_i}(-1)\lambda(y)dy - 2\tau\sum_{i\in I_m^-}\int_{a_i}^{b_i}(1)\lambda(y)dy - 2\tau\int_{a_m}^{x}(1)\lambda(y)dy - 2\tau\int_{x}^{b_m}(-1)\lambda(y)dy$$
$$+2\tau\chi_{\text{supp}\lambda}(x+1/2)\left[\int_{a_{jm}}^{x+1/2}\lambda(y)dy - \int_{x+1/2}^{b_{jm}}\lambda(y)dy\right] - \beta\lambda'(x) = 0$$

We get the stated result by writing $P^+(x) = \left(\sum_{i\in I_m^+}\int_{a_i}^{b_i} + \int_{x}^{b_m} + \chi_{\text{supp}\lambda}(x+1/2)\int_{a_{jm}}^{x+1/2}\right)\lambda(y)dy$ and $P^-(x) = \left(\sum_{i\in I_m^-}\int_{a_i}^{b_i} + \int_{a_m}^{x} + \chi_{\text{supp}\lambda}(x+1/2)\int_{x+1/2}^{b_{jm}}\right)\lambda(y)dy$. ■

Appendix B.2. : Proof of Lemma 2

Proof. Like in Appendix $B.1$, I_m^+ (resp. I_m^-) denote the set of indices of cities that are located at a clockwise (resp. counterclockwise) distance from interval m inferior to 1/2. Let us consider some agent located in city m at $x \in [a_m, b_m]$. Given that no city is antipodal to city m, we have that $x + 1/2 \in H$. Given this, the First Order Condition (FOC) provided in the Proof of Lemma 1 in Appendix $B.1$, $V'(x) = 0$, can be written now as

$$-2\tau\sum_{i\in I_m^+}\int_{a_i}^{b_i}(-1)\lambda(y)dy - 2\tau\sum_{i\in I_m^-}\int_{a_i}^{b_i}(1)\lambda(y)dy$$
$$-2\tau\int_{a_m}^{x}(1)\lambda(y)dy - 2\tau\int_{x}^{b_m}(-1)\lambda(y)dy - \beta\lambda'(x) = 0$$

By further differentiation with respect to x, we get

$$-2\tau\lambda(x) - 2\tau\lambda(x) - \beta\lambda''(x) = 0$$

$$\lambda''(x) + \delta^2\lambda(x) = 0$$

where $\delta^2 = 4\tau/\beta$. The general solution to this differential equation is given by

$$\lambda(x) = C_m\cos[\delta\,(x - x_m)]$$

where C_m and x_m are constants to be determined. Note that in equilibrium, $\lambda(a_m)$ and $\lambda(b_m)$ can't be strictly positive. For instance, if $\lambda(b_m)$ were strictly positive, then agents in location b_m would have an incentive to move to the hinterland in location $b_m + \varepsilon$ with $\varepsilon > 0$ infinitesimally small. By doing so they would save a finite marginal residence cost while facing only an infinitesimal marginal accessing cost. Therefore $\delta(b_m - a_m) = \pi$ and $x_m = (b_m - a_m)/2$. ■

Appendix B.3: Proof of Lemma 4

Proof. By applying Lemma 1 at the centre x_m of each city m, we get $P^+(x_m) - P^-(x_m) = 0$, $m = 1, 2, ..., M$. These conditions can be written in the following matrix form

$$\underbrace{\begin{bmatrix} 0 & a_{12} & \cdots & a_{1M} \\ -a_{12} & 0 & \cdots & a_{2M} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{1M} & -a_{2M} & \cdots & 0 \end{bmatrix}}_{A} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_M \end{bmatrix} = 0$$

where $a_{ij} \in \{-1, +1\}$ indicates whether $j \in I_i^-$ (city j is a right-neighbor of city i) or $j \in I_i^+$ (city j is a left-neighbor of city i). We refer to matrix A as the 'neighborhood' matrix.

It turns out that the determinant of a matrix can be expressed as $\det A = \sum_{\gamma \in \Gamma} \varepsilon(\gamma) \prod_{\gamma_i} a_{i\gamma_i}$, where γ is a permutation of $\{1, 2, ..., M\}$, Γ the set of derangements of $\{1, 2, ..., M\}$, and $\varepsilon : \Gamma \to \{-1, 1\}$. Given that the number of such derangements is odd when M is even and $a_{ij} \in \{-1, 1\}$ for $j \neq i$, $\det A$ corresponds to a sum of an odd number of terms equal to -1 or $+1$. Given this, whenever M is even, $\det A$ is non-zero and the only solution to the linear system $A\ P = 0$, is $P = 0$. Note that when M is odd, $\det A = 0$ because $A = -A^T$. ■

Appendix B.4: Proof of Proposition 3

As is the case in racetrack models of spatial agglomeration, the uniform distribution is a trivial equilibrium. The point of our analysis is to focus on the characterization of other equilibria.

First we prove that cities such as given by Lemma 2 should be equally populated and that they should be evenly spaced along the circumference. Let M be an odd number of cities that are clockwisely indexed. Let P_m be the population of city m and $\#I_m^+$ (resp. $\#I_m^-$) be the number of cities that are located on the right (resp. left) of city m. We define the following symmetry concept in the location of cities.

Definition (Neighborhood Symmetry) A spatial distribution displays the neighborhood symmetry if each city has the same number of cities on its left and on its right: $\#I_m^+ = \#I_m^- = (M-1)/2$, $\forall m = 1, ..., M$.

We also define pairs of cities located on the circumference as follows.

Definition (Paired Cities) Consider the centre x_m of some city m and its symmetric location $x_{m+1/2} \equiv x_m + 1/2$. From that location, move clockwise (resp. counterclockwise) to the next first city, say city j with centre x_j. Then consider the symmetric location of that centre x_j, $x_{j+1/2} \equiv x_j + 1/2$. Cities m and j are said to be clockwisely *paired* (resp. counterclockwisely paired) if there is no other city in the interval $(x_m, x_{j+1/2})$ (resp. $(x_{j+1/2}, x_m)$).

Given these definitions, we establish the three following Lemmas.

Lemma B.4.1 If cities m and j are paired, then $P_m = P_j$.

Proof. Consider 2 paired cities m and j being clockwisely paired. By applying Lemma 1 at the centres of cities m and j, we have that $P^+(x_m) = P^-(x_m)$ and $P^+(x_j) = P^-(x_j)$. This necessarily implies that $P_m = P_j$. ■

Lemma B.4.2 Under neighborhood symmetry, $P_m = \overline{P}$, $\forall m$.

Proof. First we show by contradiction that under neighborhood symmetry, each city can be clockwisely and counterclockwisely paired. Assume that some city m can't be paired. By assumption it has $(M-1)/2$ right- and left- neighbors (i.e. $\#I_m^+ = \#I_m^-$). If it can't be paired, then there is a city, say city h, in the interval $(x_m, x_{j+1/2})$ or in the interval

$(x_{j+1/2}, x_m)$, as described in the pairing construction. This implies that $\#I_h^+ \neq \#I_h^-$ which violates the neighborhood symmetry.

As each city m can be paired to cities $m + \frac{M+1}{2}$ and $m + \frac{M-1}{2}$, $Pm = P_{m+\frac{M+1}{2}} = P_{m+\frac{M-1}{2}}$, meaning that $P_m = \overline{P}$, $\forall m$. ■

Lemma B.4.3 Neighborhood symmetry holds.

Proof. We show that if neighborhood symmetry didn't hold, then there would be a city with a negative population.

Step 1. We show that if neighborhood symmetry does not hold, then there exists some city m that can't be paired clockwisely. By assumption, there is some city m for which the numbers of right- and left-neighbors are different (i.e. $\#I_m^+ \neq \#I_m^-$). Consider the clockwise pairing of cities, but city m. The maximum number of cities that can be paired clockwisely is given by $2\min(\#I_m^+, \#I_m^-)$. This means the number of cities that remain unpaired among the $M-1$ cities is at least given by $\max(\#I_m^+, \#I_m^-)$-$\min(\#I_m^+, \#I_m^-)$. This number is necessarily even. Even when accounting for the clockwise pairing of city m, there will always remain at least a city that can't be paired.

Step 2. Partition cities into cities that can be clockwisely paired and those that cannot be. Consider two successive cities m (that can't be clockwisely paired) and $m+1$ (that can be clockwisely paired). Applying Lemma 1 at the centre of those cities implies that $P_m + P_{m+1} = 0$ so that the population of some city should be negative. ■

From Lemmas $B.4.2$ and $B.4.3$, it naturally follows that cities are equally populated.

Lemma B.4.4 All cities are equally populated, $P_m = \overline{P} = 1/M$, $\forall m$.

Given this Lemma we can show that successive cities along the circumference are equidistant.

Lemma B.4.5 Cities are evenly spaced along the circumference (i.e. $x_m - x_{m-1} = 1/M$, $\forall m$).

Proof. By Lemma B.4.4, we know that $P_m = 1/M$. The interaction costs for agents located in city centres are given by

$$m < (M-1)/2 : IC_m = 2\tau\{\sum_{k=1}^{k=m+(M-1)/2} |x_k - x_m| + \sum_{k=m-(M-1)/2+M}^{M} [1-(x_k - x_m)\}$$
$$m = (M-1)/2 : IC_{(M-1)/2} = 2\tau\{\sum_{k\neq(M-1)/2} |x_k - x_{(M-1)/2}|$$
$$m > (M-1)/2 : IC_m = 2\tau\{\sum_{k=m-(M-1)/2+1}^{k=M} |x_k - x_m| + \sum_{k=1}^{m+(M-1)/2-M} [1-(x_m - x_k)]\}$$

Because of the neighborhood symmetry and Lemma B.4.4, these costs IC_m should be equal - say to C-. We then have

$$Ax = b$$

where $b^T = [C-(M-1)/2, ..., C-1, C, C+1, ..., C+(M-1)/2]/(2\tau)$, and A is the neighborhood matrix introduced in Appendix B.3.

It turns out that matrix A has rank $M-1$. This is because the minor (m,m) of A is a neighborhood matrix corresponding to a configuration where city m has been removed, and thus is of rank $M-1$ since the determinant of a neighborhood matrix is non zero when the number of cities is even, see Proof of Lemma 4 in Appendix B.3. Then the unique solution to $Ax = b$ is necessarily $x_m - x_{m-1} = M^{-1}$, $\forall m$. ∎

Appendix C.1: Proof of $V^*(M) = A - \tau\frac{\pi-2}{\delta M} - \tau\frac{M^2-1}{2M^2} - \frac{2\tau}{\delta M} + Y$

Consider an equilibrium with an odd number M of identical evenly spaced cities. The equilibrium utility is given by

$$\begin{aligned}
V^*(M) &= A - \beta\lambda(a_1 + \frac{\pi}{2\delta}) - \sum_{m=1}^{M}\int_{a_m}^{a_m+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy + Y \\
&= A - \beta\lambda(a_1 + \frac{\pi}{2\delta}) - \int_{a_1}^{a_1+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy \\
&- \sum_{m=2}^{\frac{M+1}{2}}\int_{a_m}^{a_m+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy - \sum_{m=\frac{M+1}{2}+1}^{M}\int_{a_m}^{a_m+\frac{\pi}{\delta}} T(a_1 + \frac{\pi}{2\delta}, y)\lambda(y)dy + Y \\
&= A - \beta\lambda(a_1 + \frac{\pi}{2\delta}) - 2\tau\int_{a_1}^{a_1+\frac{\pi}{2\delta}} (a_1 + \frac{\pi}{2\delta} - y)\lambda(y)dy - 2\tau\int_{a_1+\frac{\pi}{2\delta}}^{a_1+\frac{\pi}{\delta}} (y - (a_1 + \frac{\pi}{2\delta}))\lambda(y)dy + Y \\
&- 2\tau\sum_{m=2}^{\frac{M+1}{2}}\int_{a_m}^{a_m+\frac{\pi}{\delta}} (y - (a_1 + \frac{\pi}{2\delta}))\lambda(y)dy - 2\tau\sum_{m=\frac{M+1}{2}+1}^{M}\int_{a_m}^{a_m+\frac{\pi}{\delta}} (1 - (y - (a_1 + \frac{\pi}{2\delta})))\lambda(y)dy
\end{aligned}$$

$$= A - \frac{4\tau}{\delta^2}\frac{\delta}{2M} - 2\tau\frac{\pi - 2}{\delta^2}\frac{\delta}{2M} - \tau\frac{M^2 - 1}{2M^2} + Y$$
$$= A - \frac{2\tau}{\delta M} - \tau\frac{\pi - 2}{\delta M} - \tau\frac{M^2 - 1}{2M^2} + Y \tag{10}$$

Appendix C.2: Proof of Proposition 4

Proof. As $\partial_M V^* = \frac{M\pi - \delta}{M^3\delta}\tau$, $\partial_M V^* = 0$ for $M = \delta/\pi > 1$ since $\delta > 2\pi$. This means that $\partial_M V^* < 0$ in the interval $[1, \delta/(2\pi)]$. Thus $V^*(M)$ decreases with M, and the maximum of $V^*(M)$ is reached when $M = 1$. On the other hand, the flat-earth welfare is given by $V(\text{flat earth}) = \int_0^1 \left[A - \beta - \int_0^1 T(x, y)dy + Y\right] dx = A - \beta - \tau/2 + Y$. It is always inferior to $V^*(M = 1)$ when the single city fits the circumference perimeter ($\delta > 2\pi$). ■

6.3 Appendix C.3: Proof of Proposition 5

Now we derive the first best spatial distribution. By assuming that the opportunity cost of land is 0, the first best spatial configuration solves the following problem

$$\max_{\lambda(.)} \int \left[A - \int T(x, y)\lambda(y)dy - \frac{\beta}{2}\lambda(x) + Y\right] \lambda(x)dx$$

$$\text{st.} \int \lambda(x)dx = 1$$

The Lagrange functional L is given by

$$L = \int \left[A - \int T(x, y)\lambda(y)dy - \frac{\beta}{2}\lambda(x) + Y\right] \lambda(x)dx - \mu(\int \lambda(x)dx - 1)$$

where μ is the Lagrange multiplier associated with the total population constraint. The first variation of L is given by

$$\int \left[A - \mu - \beta\lambda(x) - 2\int T(x, y)\lambda(y)dy + Y\right] \widetilde{\lambda}(x)dx$$

Since at the optimum, this variation should be zero, we have

$$\int T(x, y)\lambda(y)dy + \frac{\beta}{2}\lambda(x) = \frac{A - \mu + Y}{2}$$

Given that $S(x) - \frac{\beta}{2}\lambda(x) = A - \int T(x,y)\lambda(y)dy - \frac{\beta}{2}\lambda(x)$, the above expression can rewritten as

$$S(x) - \frac{\beta}{2}\lambda(x) = \frac{A + \mu - Y}{2}$$

It means that at the optimum $V(x) = S(x) - \frac{\beta}{2}\lambda(x)$ is constant. Compared to the decentralized equilibrium, $\beta/2$ appears instead of β. As a consequence the optimal city is more concentrated than the equilibrium city. The optimal welfare is then given by Relation (10),

$$V^*(M = 1, \beta/2) = A - \frac{\tau}{2} + \frac{\tau}{2} - \pi\frac{\tau}{\sqrt{\frac{4\tau}{\beta/2}}} = A - \frac{\pi}{2\sqrt{2}}\sqrt{\tau\beta}$$

provided that the optimal city fits the circumference perimeter ($\delta > \pi$).

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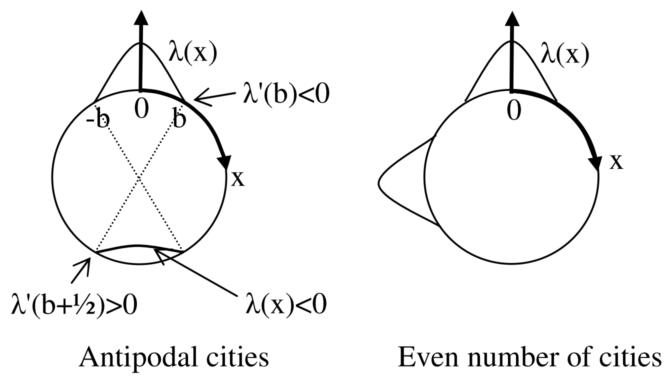

Figure 1: Two examples of spatial distributions that are not equilibria.

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2009-10	September 2009	A Harmonization of First and Second Natures	Pierre M. Picard, Dao-Zhi Zeng
2009-11	September 30, 2009	Local social capital and geographical mobility	Quentin David, Alexandre Janiak, Etienne Wasmer