CREA Discussion Paper

2009-07

Center for Research in Economic Analysis
University of Luxembourg

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available online : http://fdef.uni.lu/index.php/fdef_FR/economie/crea/discussion_papers/2008

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April 7, 2009

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Under-investment strategies in public goods: The influence of state dependent investment costs*

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Abstract

In this paper we determine and analyze symmetric open loop and Markovian Nash equilibria of a standard capital accumulation differential game which is extended by a state dependent cost function. As an application of this kind of model we do consider knowledge accumulation, lobbying of firms or pollution-abatement with connected objectives. By using Pontryagin maximum principle and Hamilton-Jacob-Bellman equation to the defied "almost" linear-quadratic game, we obtain two different types of Markovian Nash equilibria and, then, we investigate under which conditions or which kind of strategies the under-investment arise stronger or weaker.

Keywords: Differential game, Markovian Nash equilibrium, under-investment.

JEL classification: C71, C72, H41.

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*The authors acknowledge financial support from the German Science Foundation (DFG): EBIM (under GRK1134/1).

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1 Introduction

Under-investment in public goods arises from the fact that the consumption of these goods is non-exclusive, but the provision and the maintenance of such goods is costly. Assuming that all agents benefit independently of their investment in a store of this public good, it is some kind of obvious that agents would invest less while they are expecting that all the others are making enough investment. Hence, the aggregate investment is lower compared to the case where agents are cooperating.[1] This idea is close to the traditional free riding while the investigation of the classical free-riding problem was established in some static frameworks, see, for example, Olson (1965), Chamberlin (1974) and McGuire (1974) among the others. In nutshells, they showed that the equilibrium investment level is lower than the Pareto optimal investment level; and those others run different investigation in one-shot-games. However, in real world there are many voluntary processes which are dynamic, e.g., the investment in a knowledge stock, in environmental improvement, or in social infrastructure and so on. Fershtman and Nitzan (1991) and Wirl (1996) analyze the impact of *open loop* and *closed loop* strategies on agents investment contribution. They find that dynamic considerations, in particular closed loop strategies, lower the ultimate private provision of public goods. They argue that the strategies, by relying on state contingencies, drive the result of under-supply.

The present paper is a reexamination of the under-investment where agents voluntary contribute to build up a common stock. More general than Fershtman and Nitzan (1991) and Wirl (1996), we take a state dependent investment cost function. As an example, we can consider knowledge accumulation, lobbying of firms (or countries) with common aims or shared pollution-abatement issues. If there is already a high state of knowledge an increase of the knowledge stock is more costly, since it requires to use different research methods or may be combining with the hiring of costly experts.

Considering firms (or states) who are lobbying [2] for a common goal, it is obvious

[1] See Cornes and Sandler (1996).

[2] There are currently around 20,000 lobbyists (comparing to 15,000 Commission and European Par-

that the additional impact of theirs actions on governmental decisions is more costly when the influence is already high [3]. This is due to the fact that government is trying to balance between the investing firms and the "rest of society", which is assumed to have conflictive interests.[4] Moreover, government is trying to avoid a dominant position which may go in line with a loss of social welfare. As to the pollution problem–public bad, if the pollution situation is already very bad, the cleaning up process is more costly, which in deed the case that the cost is state-dependent.[5]

Nevertheless, in this present framework, we are interested in the effect of these state dependent costs on the investment behavior of firms. Moreover, we are investigating under which conditions or which kind of strategies the under-investment arise stronger or weaker. The game itself is not standard linear-quadratic or linear state game[6] due to the state dependent cost term, however, we found two different types of explicit Markovian Nash equilibria: one via Hamilton-Jacob-Bellman equation and another one by Pontryagin's maximum principle.

This paper is organized as follows. In section 2, we present our differential game. In section 3 the solutions of the cooperative solution, non-cooperative symmetric open-loop and two types of Markovian Nash equilibria of the defined game are determined. In section 4, we compare the three cases and investigate the influence of state dependent costs on the players' (under-)investment strategies and on the steady states of the differential game. Section 5 concludes.

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liamentary officials) on a daily basis in Brussels (consultants, lawyers, associations, corporations, NGOs etc.) seeking to influence the EUs legislative process, see Coen [2] and references therein.

[3]Cost of lobbying in the U.S.A can be found online [13], which mentioned that "In addition to campaign contributions to elected officials and candidates, companies, labor unions, and other organizations spend billions of dollars each year to lobby Congress and federal agencies. Some special interests retain lobbying firms, many of them located along Washington's legendary K Street; others have lobbyists working in-house". Total costs increases from \$1.44 billion in 1998 to \$3.30 billion in 2008, though there is some decrease in 2009, but still up to \$2.51billion.

[4]Note that the "rest of society" is just exogenously captured by the cost function.

[5]Some examples of cost of pollution cleaning up can be find in [14].

[6]Definition of linear-quadratic and linear-state games see Dockner et al (2000)

2 The model

We consider a symmetric two player differential game where at time t, $t = 0, \cdots, \infty$, player i, $i = 1, 2$, can make an effort $u_i(t)$, which influencs the state $w(t)$ of the public good(or bad). This stock follows a standard capital accumulation rule, given by:[7]

$$\dot{w}(t) = u_1(t) + u_2(t) - \delta w(t) \tag{1}$$

with $w(0) = w_0$ given and $\delta(> 0)$ a constant depreciation rate. Suppose in the following that the state space is finite $w \in [0, \overline{W}]$ with $0 < \overline{W} < \infty$.

Following Fershman and Nitzan (1991), we assume that the two firms have the following benefit function

$$B(w(t)) = w(t)(a_1 - a_2 w(t)),$$

with $a_1, a_2 > 0$, and increasing and concave in w, that is, $\overline{W} \leq \frac{a_1}{a_2}$. Here, a_1 measures marginal investment and a_2 presents marginal adjustment cost.

Of course taking an action (e.g. research, lobbying or abatement) is costly. The costs, which firms have to bear for theirs actions, are denoted by $C_i(t)$ and given by

$$C_i(t) = \frac{u_i^2(t)}{2} + \beta u_i w(t). \tag{2}$$

The coefficient β describes the effect of the state level on the investment costs and it may differs in different applications of the defined model. For example, considering lobbying the state can be interpreted as a goodwill credit. In this case, β is rather expected to be positive, since the state dependent costs are reflecting the existence of an exogenously given opponent lobbying group. Hence, β describes the power of this group. Considering the accumulation of knowledge on the other side, β describes the effect of accumulated knowledge on the costs of acquiring additive knowledge. Here, β does not need to be necessarily strictly bigger than zero. Examples for this effect on

[7] First introduced by Spence (1979).

knowledge stock are easily imaginable. But the following of this paper, for simplicity, we assume that $\beta \geq 0$.

Since the cost function is convex in the control variable u_i, it is more costly to make one big change in the state than many small ones. In our examples this reflects the fact that lobbying is a permanent work as real world seems to show. Considering the example of knowledge accumulation, convexity of cost function reflects that acquiring knowledge takes time and costly. As to the situation of pollution, it suggests that it is better to clean up during the developing process, rather than polluting first and then cleaning up.

Putting all these ingredients together and denoting the interest rate by r, player i chooses u_i and maximizes

$$J_i(u_i(t)) = \int_0^\infty e^{-rt} \left[w(t)(a_1 - a_2 w(t)) - C_i(t) \right] dt, \tag{3}$$

with respect to (1) and the initial value.[8]

3 Solutions

In this section, we determine the players' different symmetric optimal strategies. First, *open-loop* and linear-quadratic *feedback*[9] strategies via Hamilton-Jacob-Bellman equations are showed. Then another explicit feedback strategy via Pontryagin maximum principle. Since we are interested in under-investment, we also determined the solution of the cooperative case, which will serve as a benchmark later.

Before starting to look for different symmetric optimal strategies[10], we review some of the basic definitions. A Nash Equilibrium is a strategy, such that, given the opponents' equilibrium strategies, no player has an incentive to change his own strategy.

[8] From this utility function, we can see when considering knowledge accumulation, $\beta < 0$ may increase benefit of the player.

[9] For more detailed information on open-loop and feedback strategies, see Dockner et al. (2000).

[10] Non-symmetric (optimal) strategies may exist.

The open-loop Nash equilibrium is defined as Nash strategy which is committed at the beginning of the game and which is only time dependent and independent of state of the world; while feedback (or Markovian, or closed-loop) Nash equilibrium is defined as Nash strategy which is changing over time and state of the world.

3.1 Open-loop Strategies

Basing on the definition of open-loop strategy, at the beginning of the game, agents can commit on some strategies for the whole period of game that they can not change at $t > 0$. Hence, the current value Hamiltonian of player i is

$$H_i(w, u_i, \lambda_i) = a_1 w - a_2 w^2 - \frac{u_i^2}{2} - \beta u_i w + \lambda_i [u_i + u_j - \delta w] \tag{4}$$

where λ_i is co-state variable and measures the shadow value of the state variable on the current Hamiltonian. Given this Hamiltonian function is concave in term of state variable, the solution from the first order condition of (4) is not only necessary but also sufficient for the optimization problem. Hence, it is straight forward to get the optimal investment

$$u_i = u_j = \lambda - \beta w.$$

Due to the symmetric assumption, we have the following state and co-state equations system[11]

$$\begin{cases} \dot{w} = -(2\beta + \delta)w + 2\lambda, \\ \dot{\lambda} = (2a_2 - \beta^2)w + (r + \delta + \beta)\lambda - a_1 \end{cases} \tag{5}$$

with transversality condition $\lim_{t\to\infty} e^{-rt}\lambda(t)w(t) = 0$.

The above first order differential equation system gives us explicit solution which shows the transitory path. Furthermore, notice that the coefficient matrix of the above system has two eigenvalues, one positive and one negative. Therefore, the system has

[11] For further information on simplification which goes in line with the assumption of a symmetric differential game, see Dockner et al. (2000).

a saddle point steady state. The negative eigenvalue, which is also the adjustment speed, is

$$\mu_o = \frac{1}{2}\left[(r-\beta) - \sqrt{(r-\beta)^2 + 4[(2\beta+\delta)(r+\delta+\beta) + 2(2a_2-\beta^2)]}\right] (< 0). \tag{6}$$

The saddle point can be obtained from (5)

$$\begin{cases} \overline{w_o} = \frac{a_1}{(r+3\delta/2)\beta+(r+\delta)\delta/2+2a_2} (> 0), \\ \\ \overline{\lambda_o} = \left(\beta + \frac{\delta}{2}\right)\overline{w_o} \end{cases} \tag{7}$$

and the trajectory path is $w_o(t) = (w_0 - \overline{w_o})e^{\mu_o t} + \overline{w_o}$. Hence, the stationary open-loop strategy is

$$\overline{u^o} = \frac{\delta}{2}\overline{w_o}. \tag{8}$$

3.2 Markovian Nash equilibrium via HJB equation

In this subsection, we will look for symmetric stationary Markovian Nash equilibrium and denote the equilibrium strategies by $u_i = u_i(t, w)$ ($i = 1, 2$) which is a function of state variable w by definition. Recall that in the above problem, except for the discounting term, there is no explicit time factor. Therefore, the control functions and Bellman value functions are defined in the state space only. We conjecture a linear-quadratic value function

$$V_i(w) = d_i + e_i w + \frac{f_i}{2}w^2 \tag{9}$$

solves the following Hamilton-Jacob-Bellman (HJB) equation

$$rV_i(w) = \max_{u_i}\left\{\left(a_1 w - a_2 w^2 - \frac{u_i^2}{2} - \beta u_i w\right) + V_i'(w)\left[u_i + u_j - \delta w\right]\right\} \tag{10}$$

where $i, j = 1, 2, i \neq j$. The right hand side maximization yields a unique Markovian strategy for player i

$$u_i + \beta w = V_i'(w) \ \ , \ \ i = 1, 2. \tag{11}$$

Equation (11) elementarily says that marginal costs and marginal benefits of making an effort have to be equal. By construction, marginal costs depend on the state and the effort itself. In other words, the higher the state is, the higher marginal costs are. As already mentioned, this reflects the fact that in lobbying, for instance, it is more costly to change a political attitude, when it is already not balanced. This lies in the nature of such processes, where lobbyists are lobbying against each other.[12] Again, the forces of the lobbying group with contrary aims is assumed to be exogenous. The power of this group is expressed by β.

Combining (9), (10) and (11), we easily get

$$u_i = e_i + (f_i - \beta)w \tag{12}$$

which can be interpreted as a dynamic reaction function expressing agents' optimal contribution as a function of the collected stock.

Substituting (12) into (10), rearranging the terms and comparing the coefficients, we obtain

$$\begin{cases} rd_i = \dfrac{e_i^2}{2} + e_i e_j, \\ re_i = a_1 - \delta e_i + (e_i f_j + e_j f_i) + (e_i(f_i - 2\beta)), \\ \dfrac{rf_i}{2} = -a_2 - \delta f_i + \dfrac{(f_i - \beta)^2}{2} + f_i\,(f_j - \beta)\,. \end{cases} \tag{13}$$

By symmetric assumption, it yields

$$f = f_i = f_j = \frac{1}{6}\left[(4\beta + r + 2\delta) \pm \sqrt{(4\beta + r + 2\delta)^2 + 12(2a_2 - \beta^2)}\right], \tag{14}$$

which [13] is real and checks that f is increasing with respect to β . From (14) and (13), we can get the other coefficients

$$e = e_i = \frac{a_1}{(r + \delta) + 2\beta - 3f}, \quad d = d_i = \frac{3e}{2r}.$$

Therefore, the state dynamics becomes

$$\dot{w} = u_1 + u_2 - \delta w = 2e + [2(f - \beta) - \delta]\,w$$

[12] For a differential game with competing lobbies and contrary objectives, see Wirl (1996).

[13] See appendix

which is globally asymptotically stable if $2(f-\beta)-\delta<0$, granted when taking the negative root of (14)[14] and which correspondences to the case of strategic substitutes. While if positive root is taken, $f>\beta+\frac{2\delta}{3}+\frac{r}{3}>\beta+\frac{\delta}{2}>\beta$, which leads to the unstable path of the saddle point.

The unique positive steady state is

$$\overline{w_m}=-\frac{2e}{2(f-\beta)-\delta}$$

and the state variable in the Markovian Nash equilibrium[15] evolves according to

$$w_m(t)=(w(0)-\overline{w_m})e^{[2(f-\beta)-\delta]t}+\overline{w_m}$$

where the adjustment speed is $s(\beta)=2(f-\beta)-\delta<0$.

Notice that with the same argument as Dockner and Long(1993) or another different argument as Wirl(1996), we can prove that there are infinite solutions due to the infinite time horizontal problem with incomplete boundary conditions to the HJB equation. Furthermore, as mentioned by Wirl(1996), as well Dockner and Long(1993), the above linear-quadratic strategy obtained by HJB equation is the worse scenario comparing to the other strategies.

3.3 Feedback strategy via Pontryagin Maximum principle

However, as mentioned by Dawid and Feichtinger (1996), "in order to carry out the sensitivity analysis..., we have to restrict our attention to the analytically computable feedback equilibrium". In the following, by Pontryagin's maximum principle, we show another example of explicit feedback strategy.

Player i's Hamiltonian function is defined as

$$H_i(w,u_i,\lambda_i,t)=\left[a_1w-a_2w^2-\frac{u_i^2}{2}-\beta u_iw\right]+\lambda_i\left[u_i+u_j^*(w)-\delta w\right]$$

[14]Proof see appendix.

[15]In this way, the given Markovian Nash equilibrium is not sub-game perfect since the strategy depends on the initial state condition explicitly.

where λ_i is co-state variable and $u_j^*(w)$ is optimal strategy of player j. Pontryagin's maximum principle[16] shows that

$$\begin{aligned} u_i &= \lambda_i - \beta w, \\ \dot{\lambda}_i &= r\lambda - \tfrac{\partial H_i}{\partial w} = (r+\delta)\lambda_i - \lambda_i \tfrac{\partial u_j^*(w)}{\partial w} + 2a_2 w + \beta u_i - a_1, \end{aligned} \tag{15}$$

with transversality condition $\lim_{t\to\infty} e^{-rt}\lambda(t)w(t) = 0$.

With symmetric assumption, costate variables $\lambda_i = \lambda_j = \lambda$ and optimal strategies $u_i^* = u_j^* = u^* = \lambda - \beta w$. Hence, the state and costate equations are

$$\begin{cases} \dot{w} = -(2\beta+\delta)w + 2\lambda, \\ \dot{\lambda} = (2a_2 - \beta^2)w + (r+\delta+2\beta)\lambda - a_1 \end{cases} \tag{16}$$

with initial and transversality conditions given. The above system is almost the same as the one of open-loop strategy (5) except there is one more term, $\beta\lambda$, in the second equation, which comes from the feedback information $\lambda_i \dfrac{\partial u_j^*(w)}{\partial w}$.

The same argument as in the previous section the above firs order conditions (16) are not only necessary, but also sufficient for the optimization problem. Furthermore, the steady state is

$$w_p^* = \frac{2a_1}{(r+\delta)(2\beta+\delta) + 2(\delta\beta+\beta^2+2a_2)}, \quad \lambda_p^* = \frac{2\beta+\delta}{2} w_p^*, u_p^* = \frac{\delta w_p^*}{2}, \tag{17}$$

which is saddle path stable.

The above system is linear, so explicit solution can be obtained[17] and is given by

$$w_p(t) = \left(w(0) - w_p^*\right) e^{s_1 t} + w_p^*, \quad \lambda(t) = \frac{(s_1 + 2\beta+\delta)}{2}\left(w(0) - w_p^*\right) e^{s_1 t} + \frac{w_p^*(2\beta+\delta)}{2} \tag{18}$$

where $s_1 = \frac{1}{2}\left[r - \sqrt{r^2 + 4(2\beta+\delta)(r+\delta) + 2(2a_2 + \delta\beta + \beta^2)}\right]$ (< 0) and the transversality condition holds automatically by taking this convergent path.

In order to state the under-investment, we close this section by the benchmark strategy of the cooperative case.

[16]See, for example, Theorem 4.2, Dockner et al (2000), or Page 35, Cachon and Netessine(2004)

[17]See detail in appendix.

3.4 Cooperative solution

Different from the above non-cooperative games, in the following we assume that the agents behave cooperatively, that is, firms maximize aggregate profit. The solution of this problem will be used in the next section in order to measure the under-investment. The common profit function the two firms are maximizing is

$$J_c = \int_0^{\infty} e^{-rt} \left[2(a_1 w - a_2 w^2) - C_i(t) - C_j(t)\right] dt,$$

subject to

$$\dot{w} = u_i + u_j - \delta w$$

for given initial condition w_0. The current value Hamiltonian is defined as

$$H_i(w, u_i, u_j, \nu) = 2(a_1 w - a_2 w^2) - \left[\frac{u_i^2}{2} + \beta u_i w\right] - \left[\frac{u_j^2}{2} + \beta u_j w\right] + \nu[u_i + u_j - \delta w]$$

where ν is co-state variable. Again, it is straight forward to determine the optimal control from the first order condition which is

$$u_i = u_j = \nu - \beta w.$$

Analogously to the competitive open-loop case, it yields the state and co-state system

$$\begin{cases} \dot{w} = -(2\beta + \delta)w + 2\nu, \\ \dot{\nu} = 2(2a_2 - \beta^2)w + (r + \delta + 2\beta)\nu - 2a_1. \end{cases} \tag{19}$$

From Jacobian matrix of (19), there are two eigenvalues: one positive and one negative. Hence, the system is also saddle point stable. The negative eigenvalues is given by

$$\mu_e = \frac{1}{2}\left[r - \sqrt{r^2 + 4[2\beta(r + 2\delta) + \delta(r + \delta) + 8a_2]}\right] (< 0). \tag{20}$$

From (19), the saddle point is

$$\begin{cases} \overline{w_e} = \dfrac{a_1}{(r + 2\delta)\beta/2 + (r + \delta)\delta/4 + 2a_2} (> 0), \\ \overline{\lambda_e} = \left(\beta + \dfrac{\delta}{2}\right) \overline{w_e}. \end{cases} \tag{21}$$

The convergent trajectory path is, then,

$$w_e(t) = (w_0 - \overline{w_e})e^{\mu_e t} + \overline{w_e}$$

and the stationary strategy in the cooperative case is

$$\overline{u^e} = \frac{\delta}{2}\overline{w_e}. \tag{22}$$

4 The influence of state dependent costs

In this section we analyze the effects of the state dependent cost function. In doing so, we compare investments taken by the firms under different strategies. Then, we investigate the effect of $\beta > 0$ on the steady state values of our three scenarios.

It is straightforward that $\overline{w_0} < \overline{w_e}$, hence, there is under-investment in open-loop Nash equilibrium. By definition of f and e, the closed-loop steady state can be rewritten as

$$\overline{w_m} = \frac{-2e}{2(f-\beta)-\delta} = \frac{a_1}{-(f-\beta-\delta/2)(r+\delta+2\beta-3f)}.$$

Denote $x = (r+2\delta)\beta/2 + (r+\delta)\delta/4 + 2a_2$ and $y = -(f-\beta-\delta/2)(r+\delta+2\beta-2f)$. In order to compare the firms' investments taken under closed- and open-loop strategies, we only need to compare the x and y, due to the fact the two numerators in $\overline{w_m}$ and $\overline{w_e}$ are the same. If $y > x$, there is under-investment, otherwise, there is no.

Simple calculation shows that

$$\begin{aligned} y &= (r+2\delta)\beta + (r+\delta)\frac{\delta}{2} + 2a_2 + \beta^2 - f(2\beta + \frac{\delta}{2}) \\ &= x + (\beta+\delta/2)(r/4 - f) + \delta^2/4 + \beta(\beta+\delta) > x. \end{aligned}$$

As a conclusion, we have the following result.

Proposition 1 *For the above different strategies, for any fixed β, there is*

$$\overline{w_m} < \overline{w_o} < \overline{w_e},$$

or put it differently compared to the cooperative case the under-investment under linear-quadratic closed loop is higher than under open loop.

The use of linear-quadratic feedback strategies makes the firms more aggressive (lower contribution) which lowers profits.[18]

However, as mentioned by Wirl(1996, Page 559) "*the linear-Markov-strategy provides the worst outcome,...* "[19] we notice that the feedback strategy obtained through Pontryagin's maximum principle (though still linear-Markov strategy in this present model by chance) in deed some time could be even worse. We present this finding in the following proposition, which proof is given in the appendix.

Proposition 2 *(a) If $\beta = 0$, the steady state level of state variable checks $\overline{w_o} = w_p^* < \overline{w_e}$, and $\overline{w_o} > w_p^*$ for any $\beta > 0$.*

(b) $\overline{w_m} \geq w_p^*$, if and only if $\beta \geq \beta_0$; while $\overline{w_m} \leq w_p^*$, if and only if $\beta \leq \beta_0$.

The above two propositions can be shown in the following Figure 1, which is plotted by Mathematica by taking parameters $r = 0.05$, $\delta = 0.04$,$a_1 = 2$, $a_2 = 0.2$, and these parameters are robust.

The first result in Proposition 2 reads that in case of $\beta = 0$, which is the case studied by Fershtman and Nitzan (1991) and Wirl (1996), the feedback strategy (obtained by Pontryagin maximum principle) could be as good as open-loop strategy which confirms the statement of Wirl(1996) that the finding of Fershtman and Nitzan(1991) *hinges upon 'linearity' rather than on the Markov-requirement*. However, in the case of β positive, or cost is state dependent, things could change. Using the same Pontryagin maximum

[18]See Fershman and Nitzan (1991), Fershtman and Kamien (1987), Maskin and Tirole (1987).

[19]Wirl(1996) compares among the sub-game perfect Markovian Nash equilibria which is not the case here.

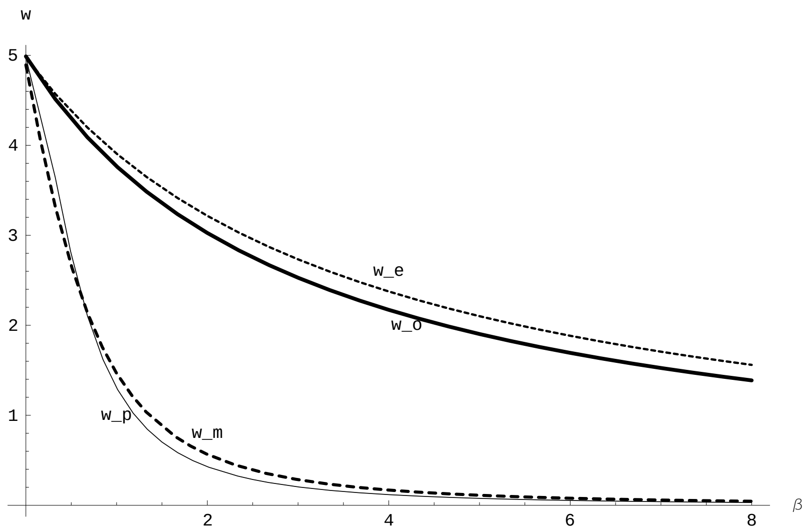

Figure 1: The effect of β on steady state.

principle to obtain the strategies, we notice that feedback strategy is worse than the open-loop one, and the *gap* between closed loop and open loop is not monotone with respect to β. More than that, with the increases of β we also notice that the feedback strategy via Pontryagin maximum principle could lead to even more under-investment than the linear-quadratic feedback strategy obtained through HJB equation. This, on the other side, confirms the statement of Fershtman and Nitzan(1991) (page 1063): *This anticipation reinforces the free riding effect.* If not because of feedback information, by the same Pontryagin maximum principle we should have $w_p^* = \overline{w_o}$, however, that is not the case. Therefore, there is mixed effect between feedback information and linearity.

5 Conclusion

In the present paper we are focusing on the effect of state dependent investment costs in a standard dynamic capital accumulation game. It follows that the steady state investment is always decreasing in the cost parameter due to the fact that marginal costs are increasing in this cost parameter, hence, yielding lower optimal investment. Furthermore, we compare two types *explicit feedback strategies* among infinite many. One is given by HJB equation and one via Pontryagin maximum principle. We notice that the linearity and feedback information have mixed effect on the steady state level of state variable. With the increases of state dependent cost parameter, the strategy from Pontryagin maximum principle dominates the one from HJB in the sense that there is lower stationary level of state variable. That shows when the cost is higher, firm is more aggressive and the feedback information dominates the linearity framework. While, when the state dependent cost parameter is small, the linear HJB strategy is the worse case between these two.

Appendix

A. Proof f is real, increasing in β and $2(f-\beta)-\delta<0$ for negative root in (14)

Step 1. f is real.

Proof. Recall $f=\frac{1}{6}\left[(4\beta+r+2\delta)\pm\sqrt{(4\beta+r+2\delta)^2+12(2a_2-\beta^2)}\right]=f(\beta)$. Denote $\Delta=(4\beta+r+2\delta)^2+12(2a_2-\beta^2)$. It is straightforward

$$\Delta=4\beta^2+8\beta(2\delta+r)+(2\delta+r)^2+24a_2=(2\beta+2\delta+r)^2+4\beta(2\delta+r)+24a_2,$$

which is always positive given the assumption of parameters. Therefore, f is always real.

Step 2. $f(\beta)$ is increasing in term of β and $f(0)<0$.

Proof. Denote $b_1=2\delta+r$. It is easy to see that

$$6f(\beta)=4\beta+b_1-\sqrt{(4\beta+b_1)^2+24a_2-12\beta^2},$$

and $6f(0) = b_1 - \sqrt{b_1^2 + 24a_2} < 0$. Moreover,

$$\frac{d(6f(\beta))}{d\beta} = 4 - \frac{4(4\beta + b_1) - 12\beta}{\sqrt{\Delta}} = 4\left[1 - \frac{\beta + b_1}{\sqrt{\Delta}}\right] > 0,$$

due to the fact $\Delta > (\beta + b_1)^2$ as can be seen from step 1.

Hence $\frac{d(f(\beta))}{d\beta} > 0$, or $f(\beta)$ is increasing in term of β.

Step 3. There is β_0, such that $f(\beta_0) = 0$ and $f(\beta) > 0$ for $\beta > \beta_0$.

Proof. $6f(\beta) = 0$ if and only if $4\beta + b_1 = \sqrt{\Delta} = \sqrt{(4\beta + b_1)^2 + 12(2a_2 - \beta^2)}$. Taking square on both sides, it is equivalent to $2a_2 - \beta^2 = 0$ or $\beta_0 = \sqrt{2a_2}$.

As a result, we have

$$f(\beta) = \begin{cases} < 0, & \beta < \beta_0, \\ = 0, & \beta = \beta_0, \\ > 0, & \beta > \beta_0. \end{cases}$$

Step 4. $2(f - \beta) - \delta < 0$ for all $\beta \geq 0$ by taking negative root in (14).

Proof. $2(f - \beta) - \delta < 0$ if and only if $2f < 2\beta + \delta$ or $6f < 3(2\beta + \delta)$ which is equivalent to

$$4\beta + b_1 - \sqrt{\Delta} < 6\beta + 3\delta,$$

or $-\sqrt{(4\beta + r + 2\delta)^2 + 12(2a_2 - \beta^2)} < 2\beta + \delta - r$. This is always true for any $\beta \geq 0$, $r > 0$ and $\delta > 0$[20].

B. Obtain dynamics solution via Pontryagin maximum principle

Recall the dynamic equation system (16)

$$\begin{cases} \dot{w} = -(2\beta + \delta)w + 2\lambda, \\ \dot{\lambda} = (2a_2 - \beta^2)w + (r + \delta + 2\beta)\lambda - a_1. \end{cases} \tag{23}$$

[20]For $\beta < 0$ see footnote 16.

From the first equation, we have

$$2\lambda = \dot{w} + (2\beta + \delta)w,$$

and

$$\ddot{w} = -(2\beta+\delta)\dot{w} + 2\dot{\lambda} = -(2\beta+\delta)\dot{w} + 2(2\beta+\delta+r)\lambda + 2(2a_2 - \beta^2)w - 2a_1.$$

Substituting λ into the above equation and arranging the terms, it follows

$$\ddot{w} = r\dot{w} + [(2\beta+\delta)(\delta+r) + 2(\beta^2+\beta\delta+2a_2)]w - 2a_1. \tag{24}$$

Its eigenvalues check

$$s^2 - rs - [(2\beta+\delta)(\delta+r) + 2(\beta^2+\beta\delta+2a_2)] = 0,$$

which gives eigenvalues

$$s = \frac{r \pm \sqrt{r^2 + 4[(2\beta+\delta)(\delta+r) + 2(\beta^2+\beta\delta+2a_2)]}}{2}.$$

By taking the negative root, denoted by s_1, we have the negative eigenvalue only, which will promise the convergent path of the transition dynamics.

Due to the fact that in (24), the non-homogenous term is only constant, therefore we can assume that the particular solution of (24) is a constant A, which is undetermined.

Substituting the particular solution A into (24), it yield

$$A = \frac{2a_1}{[(2\beta+\delta)(\delta 1 r) + 2(\beta^2+\beta\delta 1 2a_2)]}\left(= w_p^*\right).$$

Therefore the general solution of (24) is $w = c_1 e^{s_1 t} + A$ where c_1 is any nonzero constant, s_1 and A are given above.

Combining with the analysis of steady state and initial condition, we have that $c_1 = w(0) - w_p^*$.

Furthermore, $\lambda(t) = \frac{(s_1+2\beta+\delta)}{2}\left(w(0) - w_p^*\right)e^{s_1 t} + \frac{w_p^*(2\beta+\delta)}{2}$.

We finish the proof.

C. Proof of Proposition 2

Part (a) is straightforward. So we only study part (b) where

$$w_p^* = \frac{2a_1}{(r+\delta)(2\beta+\delta)+2(\delta\beta+\beta^2+2a_2)}$$

and $\overline{w_m}$ can be rewritten as

$$\overline{w_m} = \frac{2a_1}{(2\beta+\delta-2f)(r+\delta+2\beta-3f)}.$$

Denote z_p and z_m the denominators of w_p^* and $\overline{w_m}$, respectively. In order to prove (b), we only need to check the relationship between z_p and z_m.

It is easy to see

$$\begin{aligned} z_m &= (2\beta+\delta-2f)(r+\delta+2\beta-3f) \\ &= (2\beta+\delta)(r+\delta)+2\beta(2\beta+\delta)-3f(2\beta+\delta)-2f(2\beta+\delta+r)+6f^2 \\ &= (2\beta+\delta)(r+\delta)+4\beta^2+2\beta\delta-f(10\beta+5\delta+2r)+6f^2, \end{aligned}$$

and

$$z_m - z_p = 2\beta^2 - 4a_2 - (10\beta+5\delta+2r)f + 6f^2.$$

By the definition of f, it follows

$$\begin{aligned} 6f^2 &= \frac{1}{6}\left[(4\beta+r+2\delta)^2 - 2(4\beta+r+2\delta)\sqrt{\Delta} + (4\beta+r+2\delta)^2 + 12(2a_2-\beta^2)\right] \\ &= 2(4\beta+r+2\delta)\frac{1}{6}\left[(4\beta+r+2\delta)-\sqrt{\Delta}\right] + 4a_2 - 2\beta^2 \\ &= 2(4\beta+r+2\delta)f + 4a_2 - 2\beta^2. \end{aligned}$$

Hence

$$z_m - z_p = -(2\beta+\delta)f \begin{cases} \geq 0, & \text{if } f \leq 0, \\ \leq 0 & \text{if } f \geq 0. \end{cases}$$

As a result,

$$\begin{cases} \overline{w_m} \leq w_p^*, & \text{if } \beta \leq \beta_0, \\ \overline{w_m} \geq w_p^*, & \text{if } \beta \geq \beta_0, \end{cases}$$

where β_0 is defined, such that, $f(\beta_0 = 0)$. We finish the proof.

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