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Differential Game with (A)symmetric Players and Heterogeneous Strategies

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Abstract

This paper presents one kind of heterogeneous strategies in some differential games where one player plays open-loop strategy and the other one plays Markovian strategy. On top of the stationary path, this kind of strategies enable the study of trajectory dynamics, even for asymmetric players' with non-linear-quadratic games.

Keywords: Differential game, Markovian Nash Equilibrium, Heterogeneous strategy.

JEL classification: C73, C72.

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1 Introduction

The aim of this paper is to present heterogeneous strategies for a class of differential games where the players can be asymmetric in terms of their information set, their different objectives, abilities or attitudes; furthermore, players can be symmetric in their characteristics but play asymmetric strategies. We will introduce *heterogeneous strategies* by allowing some players to adopt open-loop strategies, i.e., commit; while the others play Markovian strategies.

Our study draws upon Dockener et al.(2000, p.87), where they note: "$\cdots$ consider equilibria in which some of the players represent their optimal control paths in open-loop form while others choose non-degenerate Markovian strategies. " and further "$\cdots$ the choice to solve a differential game $\cdots$ (for equilibria in which some players use open-loop strategies while others employ non-degenerate Markovian strategies) is part of the modelling stage and one should try to analyze that equilibrium which describes best the situation at hand."

Following this statement, Dockener et al.(2000; Example 4.1) develop an example, where there are two asymmetric players with different objective functions though sharing the same state equation. They first present the open-loop strategies of both players (hence degenerate Markovian strategy). Then based on this open-loop information, they construct another non-degenerate Markovian Nash Equilibrium in which one player still plays open-loop strategy and the other one plays Markovian strategy, *by conjecturing* each other's strategy.

The advantage of this kind of heterogeneous strategy via conjecturing is that it enables us to determine a stationary solution, but also the trajectory dynamics, which is not always the case in most of the Markovian strategies especially for non-linear-quadratic differential games.

Thus far, the literature provides only limited applications of this kind of heterogeneous strategy. In the following section, first, we will present some situations where open-loop and Markovian strategies are played at the same time. Second, we extend Dockener et al.(2000)s example 4.1 to a more general setting and develop the strategy in a more complete way. We conclude in the last section.

2 Differential games and heterogeneous strategies

In this section, we first provide some examples where asymmetric situations appear and different strategies are adopted by the players. Then, we introduce the definition of heterogeneous strategy and the setting of this strategy which can be applied to these examples.

2.1 Applications of heterogeneous strategies

The first example relates to international CO_2 emission control problem. Since the seminal paper of Dockner and Sorger (1996), there have been various contributions using differential games to study transboundary pollution control problems. However, most of them have simply ignored open-loop strategies. The reason mainly is that it was thought that players are too naive to play open-loop as they do not use any information acquired during the game and consequently do not respond to changes in the current pollution stock. In the context of climate change and policies that have been implemented to its mitigation, it is important to distinguish between countries which have taken binding commitments to stabilize/reduce their Greenhouse Gas (GHG) emissions, such as in the Kyoto Protocol, and countries which have proposed more flexible approaches, based on regular updates of the targets to reach, according to current states of the world.

For this kind of problem, the difference between the open-loop and the Markovian Nash strategies can be illustrated as follows: the former (like the EU), under the Kyoto Protocol, take commitments about emission reductions from which they cannot deviate; while the latter (like the USA and China), on the contrary, can deviate from their commitments (or no commitment at all) and regularly revise their targets and policies.

In the first case, we refer to open-loop strategies and in the second case to Markovian strategies.

In this kind of examples, the choices of different players are the abatement efforts or CO_2 reduction and the common state will be the environmental quality or CO_2 stock.

A second illustration stems from the dynamic game competition between big economies and small ones, such as tax/infrastructre competition among different jurisdictions. Most of the tax competition literature is static. Though the need for dynamic studies of tax competition has been called for by Zissimos and Wooders (2008), Han et al. (2012) are the very few exceptions who study tax competition under a dynamic strategic setting. They assume that a small and a large economy enter a dynamic tax competition game where the size asymmetry of these two economies play an essential role. Regardless of the natural disadvantage of small economies, size sometimes can be considered an asset (Kuznets, 1960; Easterly and Kray, 2000) given the economic success of many micro-states. Han et al (2012) argue that small states are more flexible in their political decision making than much larger countries (Streeten, 1993). Streeten (1993) suggests that problems related to collective action can be solved more easily in small countries. These attributes facilitate greater single-mindedness and focus on economic policy-making and a more rapid and effective response to exogenous change (Armstrong and Read, 1995).

Based on these arguments, Han et al. (2012) consider a situation where the small economy plays Markovian strategy while the large country adopts an open-loop strategy. They take the share of number of firms in each country as the state variable while tax policies as the choice variables.

The last example is based on Dawid and Feichtinger (1996) where a dynamic drug control problem with two players, i.e. drug dealers and the government, is investigated. These players are asymmetric and their game is not linear-quadratic. The authors develop an explicit solution of the stationary feedback strategies. Further developments could involve government control based on law and/or regulations, while the drug dealers would be more flexible and strategic in order to avoid government control.

Heterogeneous strategies could be applied as well in such a context.

This list of examples is not exhaustive. Further applications could be considered. In the following, we introduce heterogeneous strategies which could be applied to this kind of games.

2.2 Heterogeneous strategies

For simplicity, consider a two-player differential game. Each player $i(=1,2)$ chooses $u_i \in U_i$ (where U_i is the choice space for player i) to maximize their objective function Π_i

$$\max_{u_i} \Pi_i(u_i, u_j) = \max_{u_i} \int_0^{\infty} e^{-rt} f_i(t, u_i, u_j, x(t))dt,\ i,j = 1,2, i \neq j$$

where $u_j \in U_j$ is the strategy of player j. For simplicity, we assume $x(t) \in X$ is the shared common state of the system [1] (with X, the state space) and r is a positive constant denoting time preference, and is the same for both players. The state of the system is given by the following differential equation

$$\dot{x}(t) = g(t, u_1(t), u_2(t), x(t)), \forall t \geq 0, \tag{1}$$

with initial condition $x(0)$ given.

For this differential game, we define the heterogeneous strategy as:

Definition 1 ***(Heterogeneous strategies)*** *A 2-tuple* (Ψ_1, Ψ_2) *of functions* $\Psi_1 : X \times [0, +\infty) \rightarrow \mathbb{R}_+$ *and* $\Psi_2 : [0, +\infty) \rightarrow \mathbb{R}_+$*, with* $\Psi_1 = \Psi_1(x,t), \forall (x,t) \in X \times [0, +\infty)$ *and* $\Psi_2 = \Psi(t), \forall t \in [0, +\infty)$*, is called a Heterogeneous Strategic Nash Equilibrium if, for each* $i = 1,2$*, an optimal control path* u_i *of player* i *exists and is given by the Markovian Strategy for player 1:* $u_1(t) = \Psi_1(x(t), t)$*, and open-loop strategy for player 2:* $u_2(t) = \Psi_2(t)$*.*

In other words, player 1 adopts a Markovian strategy and its optimal choice u_1 depends not only on time t, but also on the current state of the system $x(t)$; while player 2's optimal strategy u_2 depends only on time, i.e. there is an irrevocable commitment.

[1] The concept and method would be analogous if there were different states for different players.

In the sequel, we build on the *conjecturing* technique from Example 4.1 of Dockner et al. (2000). There are two stages: stage one is an imaginary stage and no game is really played. However, at this stage both players assume that both of them play open-loop strategy, providing information for choosing their real strategies in the next stage and conjecturing the other players strategy. At stage two, the game starts, players play strategies based on their previous calculated information.

To be more precise, the process is the following:

Stage 2. Player 1 (the Markovian strategic player) takes player 2's (open loop) strategy $\Psi_2(t)$ as given, and hence, faces the following optimization problem:

$$\begin{cases} \max\limits_{u_1} \displaystyle\int_0^\infty e^{-rt} f_1(t, u_1(x,t), \Psi_2(t), x)dt, \\ \text{subject to } \; \dot{x}(t) = g(t, u_1(x(t),t), \Psi_2(t), x(t)). \end{cases} \tag{2}$$

The corresponding current-value Hamiltonian for player 1 is

$$\mathcal{H}_1(x, \lambda_1, u_1, t) = f_1(t, u_1(x,t), \Psi_2(t), x) + \lambda_1(t) g(t, u_1(x,t), \Psi_2(t), x)$$

where λ_1 denotes player 1's costate variable.

Player 2 (open-loop strategy player) takes player 1's Markovian strategy $\Psi_1(x,t)$ as given and faces the following problem:

$$\begin{cases} \max\limits_{u_2} \displaystyle\int_0^\infty e^{-rt} f_2(t, \Psi_1(x,t), u_2(t), x(t))dt, \\ \text{subject to } \dot{x}(t) = g(t, \Psi_1(x(t),t), u_2(t), x(t)). \end{cases} \tag{3}$$

Similarly, the corresponding current-value Hamiltonian for player 2 is

$$\mathcal{H}_2(x, \lambda_2, u_2, t) = f_2(t, \Psi_1(x,t), u_2, x) + \lambda_2(t) g(t, \Psi_1(x,t), u_2, x)$$

where λ_2 is the costate variable for player 2.

In order to conjecture the functional forms of $\Psi_1(x,t)$ and $\Psi_2(t)$, the two players analyse the situation in which both of them play open-loop strategies in stage 1.

Stage 1. Suppose both players play open-loop strategies, then their current value open-loop Hamiltonian functions can be easily written down as well as their first order conditions, respectively. The first order conditions offer the optimal open-loop strategies, provided some sufficient conditions (i.e., concavity of objective functions and state equations) are checked. Suppose those optimal open-loop strategies are: $\Psi_1(t) = \Psi_1(x(t), t)$ and $\Psi_2(t) = \Psi_2(x(t), t)$, $\forall t \geq 0$.

Continue to stage 2 The open-loop strategy player (here, player 2) conjectures player 1s with the following modification: player 2 guesses that strategy $\Psi_1(t) = \Psi_1(x(t), t)$ will be $\Psi_1(x, t)$ with *any state variable* $x \in X$ since player 1 plays Markovian strategy. Therefore, player 2's conjecturing of player 1's strategy is: $\Psi_1(x, t)$, for any $(x, t) \in X \times [0, \infty)$.

Given player 2 will still play open-loop strategy, player 1 will conjecture that player 2's strategy is $\Psi_2(t)$.

The first order condition yields player 1's equilibrium choices $u(x, t)$ by: [2]

$$\frac{\partial \mathcal{H}_1}{\partial u_1} = \frac{\partial f_1}{\partial u_1} + \lambda_1 \frac{\partial g}{\partial u_1} = 0. \tag{4}$$

The costate variable λ_1 verifies equation

$$\dot{\lambda}_1(t) = r\lambda_1 - \frac{d\mathcal{H}_1}{dx} = r\lambda_1 - \left[\left(\frac{\partial f_1}{\partial x} + \lambda_1 \frac{\partial g}{\partial x} \right) + \left(\frac{\partial f_1}{\partial u_1} + \lambda_1 \frac{\partial g}{\partial u_1} \right) \frac{\partial \Psi_1(x, t)}{\partial x} \right],$$

where $\frac{d\mathcal{H}_1}{dx}$ denotes the total derivative of $\mathcal{H}_1$ with respect to x and the last term is equal to zero, by the first order condition (4). Thus, player 1's co-state equation reads

$$\dot{\lambda}_1(t) = r\lambda_1(t) - \left(\frac{\partial f_1}{\partial x} + \lambda_1(t) \frac{\partial g}{\partial x} \right) \tag{5}$$

with transversality condition $\lim_{t \to \infty} e^{-rt} \lambda_1(t) x(t) = 0$.

Similarly, the optimal choices of player 2, $u_2(t)$ is given by

$$\frac{\partial \mathcal{H}_2}{\partial u_2} = \frac{\partial f_2}{\partial u_2} + \lambda_2 \frac{\partial g}{\partial u_2} = 0. \tag{6}$$

[2] A similar notation can also be found in Cachon and Netessine (2004).

And the costate equation is

$$\dot{\lambda}_2(t) = r\lambda_1 - \frac{d\mathcal{H}_2}{dx} = r\lambda_2 - \left[\left(\frac{\partial f_2}{\partial x} + \frac{\partial f_2}{\partial u_1}\frac{\partial \Psi_1(x,t)}{\partial x}\right) + \lambda_2\left(\frac{\partial g}{\partial x} + \frac{\partial g}{\partial u_1}\frac{\partial \Psi_1(x,t)}{\partial x}\right)\right]. \quad (7)$$

The associated transversality condition is $\lim_{t\to\infty} e^{-rt}\lambda_2(t)x(t) = 0$.

Denote the solution of (4) and (6) as $u_1^* = \Psi_1^*(x,t), \forall (x,t) \in X \times [0,\infty)$ and $u_2^* = \Psi_2^*(t), \forall t \geq 0$, respectively. To be more precise $\Psi_1^*(x,t)$ is a function of state x, the costate variable evaluated at time t, $\lambda_1(t)$, and $\Psi_2^*(t)$ is a function of state and costate variables both evaluated at t, that is, $\Psi_1^*(x,t) = \Psi_1^*(x, \lambda_1(t), t)$ and $\Psi_2^*(t) = \Psi_2^*(x(t), \lambda_2(t), t)$. Substituting these two into the Hamiltonian, we can readily check that the maximized Hamiltonian $\mathcal{H}_1^*(x, \lambda_1, t)$ and $\mathcal{H}_2^*(x, \lambda_2, t)$ are given by

$$\mathcal{H}_1^*(x,\lambda_1,t) = f_1(t, \Psi_1^*(x,\lambda_1,t), \Psi_2^*(x,\lambda_2,t), x) + \lambda_1\, g(t, \Psi_1^*(x,\lambda_1,t), \Psi_2^*(x,\lambda_2,t), x)$$

and

$$\mathcal{H}_2^*(x,\lambda_2,t) = f_2(t, \Psi_1^*(x,\lambda_1,t), \Psi_2^*(x,\lambda_2,t), x) + \lambda_2\, g(t, \Psi_1^*(x,\lambda_1,t), \Psi_2^*(x,\lambda_2,t), x)$$

If we impose sufficient concavity and smoothness conditions on the objective functions $f_i(t, u_1, u_2, x)$ and state function $g(t, u_1, u_2, x)$, the maximized Hamiltonian is concave with respect to the state variable x. Hence, by Dockner et al (2000)s Theorem 3.2, $u_i^*(t)$ $(i = 1, 2)$ are optimal paths. Thus, the solution $\{u_1^*(x,t), u_2^*(t)\}$ for $x \in X$ and $t \geq 0$ form a pair of non-degenerate Markovian Nash Equilibria.

Finally, substituting $u_1^* = \Psi_1^*(x,t)$ and $u_2^* = \Psi_2^*(t)$ into the canonical system: state equation (1), two costate equations (5) and (7), we can obtain the solution for the whole trajectory path of the differential game.

3 Conclusion

A particular case of heterogeneous strategies are introduced in adifferential game, based on the extension and generalization of Exmpale 4.1 in Dockener et al (2000).

Here, heterogeneous means one player plays open-loop strategy while the other player plays Markovian strategy. The key idea of these kind of strategies is based on the guess of the rival's open-loop strategies. The biggest advantage of this kind of strategy is that it offers solution not only along the stationary path, but also for the whole trajectory path, which is especially useful in the case of asymmetric players' non-linear-quadratic differential game.

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