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Marriage Formation with Assortative Meeting as a Two-Sided Optimal Stopping Problem[*]

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Abstract

In this paper we study marriage formation through a two-sided secretary problem approach. We consider individuals with nontransferable utility and two different dimensions of heterogeneity, a characteristic evaluated according to the idiosyncratic preferences of potential partners, and a universally-rankable characteristic. There are two possible states of the world, one in which people meet their partner randomly, and one in which the meeting occurs between individuals with similar characteristics. We show that individuals with higher universal characteristic tend to be more picky in their marriage hunting. This does not necessarily mean that they marry later than other individuals, since the higher expected quality of their potential partners in the assortative meeting state can make them marry earlier than individuals with a lower universal characteristic.

JEL codes: C73, C78

Keywords: secretary problem, random meeting, assortative meeting.

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1 Introduction

In economic theory, marriage formation has been studied according to different approaches. A first way is the theory, pioneered by Gale and Shapley (1962) and surveyed and extended by Roth and Sotomayor (1990), who proved the existence of stable matchings.[1] A different perspective is based on the assignment problem (Shapley and Shubik, 1972 and Becker, 1973). In the developments of the assignment problem, the literature borrowed the standard Diamond-Mortensen-Pissarides search framework (see Shimer and Smith, 2000, and Atakan, 2006, *inter alia*).

The underlying assumption of these approaches is that an agent is able to meet all the potential partners in the population, and obtain the best match from that. However, there are many situations where this cannot be the case. For example, an individual may base his or her choice on the potential partner whom he or she meets only. Also, an individual would not be able to meet all the potential partners if they live in another city or country, and even assuming that this may be possible with the new technologies (such as social networks, or dating online services), time is a scarce resource and some individuals will never have the chance of meeting each other, even though they would lead to a stable match. In these types of situations where only few meetings occur compared to the number of potential partners, investigating the best overall matching may not be the most relevant analysis.

In this paper we study marriage formation by assuming that individuals optimise their search strategies (i.e., optimal stopping) rather than waiting for the best partner in absolute terms. We consider heterogeneous agents with nontransferable utility,[2] and we try to generalise marriage formation by taking into account two different dimensions of heterogeneity in the characteristics of an individual. Each individual has a characteristic whose evaluation by potential partners depends on the specific idiosyncratic preference of the partner ("specific" characteristic) and another characteristic ("universal" characteristic) that can be ranked in the same way by all individuals, such as income, beauty, social status, and so on.[3]

[1] Matchings are stable if no pair of individuals would prefer to leave their current partners for each other.

[2] The literature of marriage formation consider a family output, which is endogenously shared through a Nash bargaining process. This assumption is called "transferable utility" (see Sattinger, 1995, Lu and McAfee, 1996, Bloch and Ryder, 2000, Shimer and Smith, 2000 and Atakan, 2006). The alternative "non-trasferable utility" indicates that the family output is exogenously shared (see Morgan, 1995, Burdett and Coles, 1997, Chade 2001).

[3] Caldarelli and Capocci (2001) consider a stable matching problem *à la* Gale and Shapley (1962) where

The framework considered is a refined version of the “secretary problem” (Chow *et al.*, 1964), which can be explained as follows. Imagine an administrator who wants to hire the best secretary out of rankable applicants for a position. Applicants are interviewed one-by-one in random order. A decision about each particular applicant is to be taken immediately after the interview. Once rejected, an applicant cannot be recalled. During the interview, the administrator can rank the applicant among all applicants interviewed so far, but is unaware of the quality of yet unseen applicants. The question is about the optimal strategy (stopping rule) to maximise the probability of selecting the best applicant.

Our model differs from the traditional secretary problem for two main elements. First, in order to represent marriage formation, the secretary problem needs to be considered “two-sided”, i.e., each partner needs to determine his or her stopping rule (Eriksson *et al.*, 2007).[4] Second, like in the literature on the economics of marriage, we consider the possibility that individuals are assortatively matched (see below).

Unlike the stable matching theory, in which each agent is matched with the best partner's possible (that is, the best partner who in turn would like to be matched to that agent), in the secretary problem the quality of the potential partners met is only revealed slowly during the game. On the other hand, in the secretary problem we have no guarantees that the marriages will be stable. Compared to the search models, in which potential partners are met with a Poisson arrival (Diamond, 1982, Mortensen, 1982 and Pissarides, 1990), in our approach a meeting with a potential partner takes place in each period.

Another important difference between the present analysis and the standard literature of marriage formation is the way we deal with the relationship between the partners' characteristics. A common element in the marriage formation analysis is the presence of “assortative matching” (Becker, 1973), which alludes to a relationship between the characteristics of partners.[5] In the matching literature, assortative matching is assumed to occur in equilibrium (Becker, 1973 in the seminal paper and Shimer and Smith, 2000, in the search literature, *inter alia*) according to the characteristics of the utility function. In particular, a positive (negative) assortative matching is optimal in equilibrium whenever the utility function is “supermodular” (submodular) in the partners' characteristics,

partners can be ranked according to a universally classifiable characteristic of an individual.

[4]In addition, in the classical secretary problem a player ranks a potential partner by a natural number from $[0, N]$, and two potential partners cannot have the same ranks, while we suppose that the partner's rank is a real number from $[0, 1]$, and a player can meet partners with the same ranks during the game.

[5]In particular according to the Becker's model, in equilibrium matching is positively assortative if partners are complements.

meaning that the transferable utility function is higher if partners have similar (different) characteristics.[6] Conversely in our model, we assume that individuals with similar universal characteristics have a specific probability of meeting due to facts of life (i.e., attending similar social environments, obtaining the same level of education, etc...), even though this does not necessarily lead to marriage formation. In order to distinguish our approach, we will refer to this type of meeting as "assortative meeting".

The main novelty of our framework is that two different types of meeting are together considered, occurring in alternative states of the world. A meeting can be random (the partner is randomly drawn by the population) or assortative (the potential partner belongs to the same universal rank of the individual) in each period, according to an exogenous and constant probability. From this perspective, the paper offers a comparison of different types of meeting and how these affect the individuals' behaviour.

The results depend on the state of the world in which an individual stands. In assortative meeting, individuals with a high universal characteristic are less demanding if the probability of having assortative meetings in the future is low, and *vice versa*. This result is due to the fact that, given a low probability of being in another assortative meeting state, the quality of the expected future partners is low for individuals with high universal characteristic. Therefore they are less fussy with the choice of a potential partner met today of the same universal rank. In random meeting, individuals with high universal characteristic are harder to please compared to other individuals, and they are more demanding the higher the weight of the universal characteristic. The reason is that an individual with a high universal rank knows that the chance of being in assortative meeting state in the future ensures a high expectation about future meetings, at least from the universal characteristic perspective. This does not necessarily mean that individuals with a high universal characteristic marry later than other individuals. Indeed, individuals with a high universal characteristic expect partners of better quality, which increases the chance of an early marriage.

The remainder of the paper is structured as follows. Section 2 presents the model, and Section 3 shows the baseline results. Section 4 examines the expected ranks, while Section 5 illustrates the expected time necessary to marry. Section 6 and 7 investigate the cases with infinite horizon and state-independent strategies, respectively. Conclusions are

[6] From a technical perspective, supermodularity (submodularity) implies that, denoting as x and y the partners' characteristics, and utility being a function of them, $f(x,y)$, then $f(x,y)$ has the following feature: $\frac{\partial^2 f(x,y)}{\partial x \partial y} \propto \frac{\partial^2 f(x,y)}{\partial y \partial x} > (<)\, 0$.

in Section 8. All the results are formally derived in the appendix.

2 The model

We study a finite, large universe of U men and U' women. Time is discrete and horizon is finite (Damiano *et al.*, 2005), the game starts at period $t = 1$ and lasts for N periods, where $U \gg N$ and $U' \gg N$.[7] In each period a meeting takes place. During the meeting, each player ranks the person of the opposite sex using two characteristics. The first characteristic, denoted by η, reflects the specific, idiosyncratic and universally unrankable trait of an individual. Some individuals like caring and attentive partners, some others prefer independent persons. This is totally subjective and cannot be compared between different individuals. The second characteristic, denoted by I, represents a universally rankable aspect of the individual, such as income, education, social class and so forth.

We assume that each individual evaluates potential mates according to the linear combination of these characteristics, which we will refer as "rank":

$$R = (1 - \alpha)\eta + \alpha I, \tag{1}$$

where $\alpha \in (0, 1)$ weights the importance of the universal characteristic compared to the individual characteristic. We assume α to be public information and identical for all players. The level of α reflects the role played in the romantic choice by universally estimable characteristics (social class, income, education) compared to personal preferences for specific aspects of a partner. For instance, it can be imagined that in a conservative society individuals put more weight on aspects such as the social status or income when they evaluate a partner.

The meeting can be of two types. We denote the set of types as $S = \{r, \bar{r}\}$, where type $s = r$ is called "random" meeting while $s = \bar{r}$ is called "assortative" meeting. A random meeting occurs when an individual meets the partner by chance. This happens anytime the rankable characteristic of an individual (social status, income, education, and so forth) does not influence the occurring meeting. For example, two individuals running into each other at the grocery store, both going to the football stadium or to a public party. Therefore, with random meeting any two people from the universe can meet. Assortative meeting occurs when an individual meets the partner in a contest in which his or her rankable

[7] In Section 6 we extend the analysis to the case with infinite horizon.

characteristic is relevant in determining the meeting. All the encounters at school, at the university, in a family or a private party are examples of assortative meeting. We assume that, with assortative meeting, the universal rank of the potential partner will be the same as the individual's. This assumption is made for simplicity, as considering an imperfect correlation (for instance, implemented with a noise) would complicate the analysis without altering the qualitative features of our results.

In each period t, the meeting is assortative with exogenous probability $\beta \in (0,1)$ and random with probability $1-\beta$, β being constant, equal to all the players and known by them. The value of β depends on the customs of the society we have in mind. For instance in a traditional society, it is more likely that individual with common background are matched together (β high). To the best of our knowledge, this is the first contribution in the literature in which two types of meeting may alternatively take place.

After each meeting, a man m and a woman w decide whether to propose a marriage to each other. If both propose, the process ends. If at least one of them does not propose, then the game transits to the next period. For simplicity, we assume that being not married is always worse than being married. This assumption implies that, at period N, all the remaining unmatched players are willing to marry.

Since the characteristics of potential partners are not known at the beginning of the game, suppose that an individual i, $i \in \{m, w\}$ in each period $t = 1, \ldots, N$ meets a partner j, $j \in \{m, w\}$ and $j \neq i$ in state s with the following rank:

$$R_i^s(t) = \begin{cases} (1-\alpha)\,\eta_j^t + \alpha I_j^t, & \text{if } s = r \text{ (with prob. } 1-\beta \text{)} \\ (1-\alpha)\,\eta_j^t + \alpha I_i, & \text{if } s = \bar{r} \text{ (with prob. } \beta \text{)} \end{cases} \tag{2}$$

where

- η_j^t is a random variable with continuous uniform distribution in $[0,1]$ for all $t = 1, \ldots, N$, reflecting the idiosyncratic preference of an individual i for a potential partner j. Let η_j^t be independent variables for $t = 1, \ldots, N$.
- I_j^t is a random variable with continuous uniform distribution in $[0,1]$ for all $t = 1, \ldots, N$, representing the universal rank of a potential partner j. Let I_j^t be independent variables for $t = 1, \ldots, N$.
- $I_i = I \in [0,1]$ is the universal rank of the partner with assortative meeting, which is the same as individual i who evaluates the partner j. The personal rank is known

to the individual and does not change throughout the game. This of course is a simplification, as characteristics may change over time, altering I_i. For instance, income generally increases over time, whereas beauty decreases over time.

Considering η_j^t and I_j^t be independent variables means that potential partners met in the past do not influence future meetings. We assume that men and women rank potential partners symmetrically. This assumption is for the sake of simplicity and does not correspond exactly to what happens in the real world. For instance, in many societies beauty is more evaluated by men, whereas income is more evaluated by women (See Coles and Francesconi, 2011). The assumption that men and women rank potential partners symmetrically implies that the universal rank in assortative meeting state I is equal for man m and woman w.

Consider the following noncooperative game. Each player wants to maximise the expected absolute rank of the chosen mate. The strategy of player i is the rule $a = a(t, s)$ that says whether the marriage must be proposed to a potential partner with absolute rank $R_j^s(t)$ in period t and in state s for every $t = 1, \dots, N$. A player's strategy is a set of thresholds such that the player must propose a marriage in period t and in state s if and only if the observed rank is greater than the strategy in t, i.e., $R_j^s(t) > a(t, s)$. Therefore a high a implies that a player is more likely to delay marriage, since he or she needs to meet a potential partner with a high rank R in order to agree to marry. Suppose that all players in the game use this type of strategies.

Definition 1 *The N-period process is a N-period meeting game where all players use the same type of threshold strategies, i.e. player i's strategy in period $t = 1, \dots, N$ and in state s is $a = a(t, s)$.*

We formulate the problem as a dynamic game and use the concept of subgame perfect equilibrium (Selten, 1975).

3 Baseline results

3.1 Bellman equation

The following Bellman equation represents the expected partner's rank, for every s using strategy $a = a(t, s)$, either if a player i marries at t or if he/she waits for the next periods:

$$
\begin{aligned}
E^s(t) = & \max_a \{\Pr^s[\text{marry}](a) E[R_i^s|\text{marry}](a) \\
& + \delta(1 - \Pr^s[\text{marry}](a)) \left(\beta E^{\bar{r}}(t+1) + (1-\beta)E^r(t+1)\right)\}
\end{aligned} \tag{3}
$$

with boundary conditions for $t = N$ and states $s = \bar{r}$ and $s = r$:

$$
E^{\bar{r}}(N) = E[(1-\alpha)\eta_j^N + \alpha I] = \frac{1-\alpha}{2} + \alpha I, \tag{4}
$$

$$
E^r(N) = E[(1-\alpha)\eta_j^N + \alpha I_j^N] = \frac{1}{2}. \tag{5}
$$

Here the Bellman equation $E(t)$ is the maximal expected rank of a partner. $\Pr^s[\text{marry}](a)$ is the probability of marriage in period t when the state is s and player i's strategy is a, $E[R_i^s(t)|\text{marry}](a)$ is the expected rank of a potential partner j met in period t in state s if player i marries using strategy a, and $\delta \in (0, 1]$ is the discount factor. Expression $\beta E^{\bar{r}}(t+1) + (1-\beta)E^r(t+1)$ is the expected payoff of an individual i (or the absolute rank of j) if they chose to not marry in period t and the game transits to the next period. Notice that player i's strategy $a(t, s)$ is within the interval $[0, 1]$ if $s = r$, but from the interval $[\alpha I_i, \alpha I_i + 1 - \alpha]$ if $s = \bar{r}$. The latter is the interval of possible values of the random variable $R_i^r(t)$. The optimal strategies for the last period N are $a^*(N, r) = 0$ for random meeting and $a^*(N, \bar{r}) = \alpha I_i$ for assortative meeting, since a player prefers being married than remaining single.

In order to solve the Bellman equation, we begin by deriving the conditional probability of marrying according to the occurring state at time t. The result is summarised in the following proposition.

Proposition 1 *The conditional probability to marry in the assortative meeting state for any period $t = 1, \ldots, N-1$ is given by*

$$
\Pr^{\bar{r}}[\textit{marry}](a) = \left(1 - \frac{a - \alpha I}{1-\alpha}\right)^2, \tag{6}
$$

while the conditional probability to marry in the random meeting state is given by

1. *For* $\alpha \geq \frac{1}{2}$:

$$\Pr^r[marry](a) = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, 1-\alpha) \\ \left(1 - \dfrac{2a - (1-\alpha)}{2\alpha}\right)^2, & \text{if } a \in [1-\alpha, \alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1] \end{cases} \tag{7}$$

2. *For* $\alpha < \frac{1}{2}$:

$$\Pr^r[marry](a) = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, \alpha) \\ \left(1 - \dfrac{2a - \alpha}{2(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1-\alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{8}$$

The last step for deriving the Bellman equation is to determine the conditional expectation of the expected rank of a person if player marries. This is summarised in the following proposition.

Proposition 2 *The expected rank of a potential partner in the assortative meeting state for any period* $t = 1, \ldots, N-1$ *is given by*

$$E[R_i^{\bar{r}}(t)|marry](a) = \frac{\alpha I + 1 - \alpha + a}{2}, \tag{9}$$

whereas the expected rank of a potential partner in the random meeting state is given by

1. *For* $\alpha \geq \frac{1}{2}$*:*

$$E[R_i^r(t)|marry](a) = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, 1-\alpha) \\ \dfrac{3a^2 - (1+\alpha+\alpha^2)}{6a - 3(1+\alpha)}, & \text{if } a \in [1-\alpha, \alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [\alpha, 1] \end{cases} \tag{10}$$

2. For $\alpha < \frac{1}{2}$:

$$E[R_i^r(t)|marry](a) = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, \alpha) \\ \dfrac{3a^2 - (3 - 3\alpha + \alpha^2)}{6a - 3(2-\alpha)}, & \text{if } a \in [\alpha, 1-\alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [1-\alpha, 1] \end{cases} \cdot \tag{11}$$

3.2 Players' optimal strategies

We are now in a position to determine a player's optimal strategy through the analysis of the Bellman equation (3). First, we examine separately the two states of the world $\bar{r}$ and r for each period $t = 1, ..., N-1$. From now on, we will omit the label i for brevity.

3.2.1 Assortative meeting

First, consider the assortative meeting state $s = \bar{r}$. The Bellman equation (3) is:

$$E^{\bar{r}}(t) = \max_a \left\{ \left(1 - \frac{a - \alpha I}{1-\alpha}\right)^2 \frac{\alpha I + 1 - \alpha + a}{2} + \left(1 - \left(1 - \frac{a - \alpha I}{1-\alpha}\right)^2\right) \delta E(t+1) \right\}, \tag{12}$$

where $E(t+1) = \beta E^{\bar{r}}(t+1) + (1-\beta) E^r(t+1)$ and with boundary conditions (4) and (5). All multipliers in the right hand side part of (12) are nonnegative, so, for each period t from 1 to $N-1$ we investigate $a(t, \bar{r})$ that yields $E^{\bar{r}}(t)$. The following proposition shows the optimal strategy with assortative meeting.

Proposition 3 *For each $t = 1, \ldots, N-1$, the optimal strategy $a^*(t, \bar{r})$ in the assortative meeting state $s = \bar{r}$ is:*

$$a^*(t, \bar{r}) = \begin{cases} \alpha I, & \text{if } E(t+1) < \dfrac{4\alpha I + 1 - \alpha}{4\delta}, \\ \dfrac{4\delta E(t+1) - (\alpha I + 1 - \alpha)}{3}, & \text{if } \dfrac{4\alpha I + 1 - \alpha}{4\delta} \leq E(t+1) < \dfrac{\alpha I + 1 - \alpha}{\delta}, \\ \alpha I + 1 - \alpha, & \text{if } E(t+1) \geq \dfrac{\alpha I + 1 - \alpha}{\delta}. \end{cases} \tag{13}$$

In Proposition 3, the optimal strategy is higher the higher an individual's universal rank I in cases when $E(t+1) < \frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta}$ and $E(t+1) \geq \frac{\alpha I + 1 - \alpha}{\delta}$. In other words, it is less likely that an individual would accept to marry if he/she is from a high universal rank. Corollary 1 follows from Proposition 3.

Corollary 1 *In assortative meetings, an individual does not marry anyone before period N if and only if the expected rank $E(t+1)$ at $t+1$ satisfies:*

$$E(t+1) \geq \frac{\alpha I + 1 - \alpha}{\delta} \tag{14}$$

for every $t = 1, \ldots, N-1$.

Condition (14) can be satisfied when the universal rank is very high and the intensity of assortative meeting is also very high. Accordingly, players wait for potential partners with a higher rank in the following meetings. And if inequality (14) is satisfied for every $t = 1, \ldots, N-1$, then a player does not marry until period N in the assortative meeting states.

3.2.2 Random meeting

We turn now to the case of random meeting. In the case in which $\alpha \geq \frac{1}{2}$, the Bellman equation (3) is:

$$E^r(t) = \begin{cases} \max\limits_a \left\{ \left(1 - \frac{a^2}{2\alpha(1-\alpha)}\right)^2 \frac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)} \right. \\ \qquad \left. + \left(1 - \left(1 - \frac{a^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [0, 1-\alpha), \\ \max\limits_a \left\{ \left(1 - \frac{2a - 1 + \alpha}{2\alpha}\right)^2 \frac{3a^2 - (1 + \alpha + \alpha^2)}{6a - 3(1+\alpha)} \right. \\ \qquad \left. + \left(1 - \left(1 - \frac{2a - (1-\alpha)}{2\alpha}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [1-\alpha, \alpha), \\ \max\limits_a \left\{ \left(\frac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2 \frac{2a+1}{3} \right. \\ \qquad \left. + \left(1 - \left(\frac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [\alpha, 1] \end{cases} \tag{15}$$

Conversely if $\alpha < \frac{1}{2}$, then the Bellman equation (3) becomes:

$$E^r(t) = \begin{cases} \max_a \left\{ \left(1 - \frac{a^2}{2\alpha(1-\alpha)}\right)^2 \frac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)} \right. \\ \qquad \left. + \left(1 - \left(1 - \frac{a^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [0,\alpha), \\ \max_a \left\{ \left(1 - \frac{2a-\alpha}{2(1-\alpha)}\right)^2 \frac{3a^2 - (3 - 3\alpha + \alpha^2)}{6a - 3(2-\alpha)} \right. \\ \qquad \left. + \left(1 - \left(1 - \frac{2a-\alpha}{2(1-\alpha)}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [\alpha, 1-\alpha), \\ \max_a \left\{ \left(\frac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2 \frac{2a+1}{3} \right. \\ \qquad \left. + \left(1 - \left(\frac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2\right) \delta E(t+1) \right\}, & \text{if } a \in [1-\alpha, 1]. \end{cases} \tag{16}$$

with boundary conditions (4) and (5). Proposition 4 describes the optimal strategy with random meeting for each period $t = 1, ..., N-1$.

Proposition 4 *For each $t = 1, \ldots, N-1$, the optimal strategy $a^*(t,s)$ in the random meeting state $s = r$ is:*

Case $\alpha \geq \frac{1}{2}$**.**

$$a^*(t,r) = \begin{cases} 0, & \text{if } E(t+1) < \frac{1}{4\delta} \\ 1-\alpha, & \text{if } \frac{1}{4\delta} \leq E(t+1) < \frac{5 - 19\alpha + 11\alpha^2}{6(1-3\alpha)\delta} \\ \frac{1+\alpha}{6} + \frac{2\delta}{3}E(t+1) - \gamma_1, & \text{if } \frac{5 - 19\alpha + 11\alpha^2}{6(1-3\alpha)\delta} \leq E(t+1) < \frac{5\alpha+1}{6\delta} \\ \frac{6\delta E(t+1) - 1}{5}, & \text{if } E(t+1) \geq \frac{5\alpha+1}{6\delta}, \end{cases} \tag{17}$$

where

$$\gamma_1 = \frac{\sqrt{16\delta^2 \left(E(t+1)\right)^2 - 16\delta\left(1-\alpha\right)E(t+1) + 5\alpha^2 + 6\alpha + 5}}{6}. \tag{18}$$

Case $\alpha < \frac{1}{2}$**.**

$$a^*(t,r) = \begin{cases} 0, & \textit{if } E(t+1) < \dfrac{1}{4\delta} \\ \alpha, & \textit{if } \dfrac{1}{4\delta} \le E(t+1) < \dfrac{11\alpha^2 - 3\alpha - 3}{6(3\alpha-2)\delta} \\ \dfrac{2-\alpha}{6} + \dfrac{2}{3}\delta E(t+1) - \gamma_2, & \textit{if } \dfrac{11\alpha^2 - 3\alpha - 3}{6(3\alpha-2)\delta} \le E(t+1) < \dfrac{6-5\alpha}{6\delta} \\ \dfrac{6\delta E(t+1) - 1}{5}, & \textit{if } E(t+1) \ge \dfrac{6-5\alpha}{6\delta}, \end{cases} \tag{19}$$

where

$$\gamma_2 = \frac{\sqrt{16\delta^2(E(t+1))^2 - 16\delta(2-\alpha)E(t+1) + 5\alpha^2 - 16\alpha + 16}}{6}. \tag{20}$$

3.3 Existence and uniqueness of the equilibrium

Given the assumptions on the Bellman equation considered, the following result holds.

Proposition 5 *In a N-period meeting game there exists a unique subgame perfect equilibrium.*

Proposition 5 can be explained as follows. The N-period meeting game is a finite extensive game. In the model we assume that each player participating in the game wants to maximise the rank (1) of the partner that the player will marry. So, the optimal strategy derived by maximising the expected rank for N-period meeting game is optimal for all players participating in the game. The existence of equilibrium in N-period meeting game is straightforward and follows from Selten (1975).

The uniqueness of the subgame perfect equilibrium when all players use optimal strategies $a^* = a^*(t,s)$, $t = 1,\dots,N$, $s = r, \bar{r}$ yielding Bellman function (3) follows from the functional forms used in the right part of (3). In the case of assortative meeting $s = \bar{r}$, then (3) is a continuous function of a with a unique maximum on the interval of possible strategy values $[\alpha I, \alpha I + 1 - \alpha]$ for every $t = 1,\dots,N$. Therefore, each player i has a unique optimal strategy in every period in which assortative meeting takes place. Random meetings can be considered in a similar manner. The functions in the right part of Bellman equations (15) and (16) are continuous in a for both $\alpha \ge \frac{1}{2}$ and $\alpha < \frac{1}{2}$ and has a unique maximum within the interval of possible strategies $[0,1]$ for every $t = 1,\dots,N$. Hence, a player has a unique optimal strategy in every period in which random meeting occurs.

3.4 Analysis of equilibrium

In this section we consider the properties of optimal strategies in both random and assortative meetings.

Proposition 6 *The equilibrium payoff in the assortative (random) meeting state is an increasing (non decreasing) function of a player's individual rank I.*

Proposition 6 intuitively says that players with higher universal rank obtain a higher payoff in equilibrium.

Proposition 7 *For:*

$$\beta > \frac{1}{4\delta^{N-1}}, \tag{21}$$

the optimal strategy $a^*(t, \bar{r})$ in the assortative meeting states $s = \bar{r}$ is a non-decreasing function of a player's universal rank I for any $t = 1, \ldots, N$.

Condition (21) is sufficient but not necessary. Indeed the necessary condition for non-decreasing function $a^*(t, \bar{r})$ of I cannot be obtained explicitly, this due to the recurrent form of optimal strategies. For example, for $N = 2$ the condition $\beta > \frac{1}{4\delta}$ is also necessary. For $N = 3$ the necessary condition is:

$$\beta > \frac{1 - \alpha + \alpha I - \frac{1-\alpha}{4}\sqrt{\frac{3(1-\delta)}{\delta}}}{\delta\alpha}, \tag{22}$$

and so on.

Proposition 7 shows that, in the assortative meeting state, the optimal strategy changes with a player's universal rank according to the intensity of assortative meeting. If β is high, then players with high universal rank are more "demanding", because the future chance of being in the assortative meeting state (and thus to meet high ranked partners) will be higher. Therefore they can wait for a better idiosyncratic match. Conversely, if β is low, then players are more picky if they have a low I, since a low β implies a relatively higher future expectation for low-I types. Indeed, low universal rank players expect a relatively higher payoff from a random meeting.

The next proposition shows how the optimal strategy changes according to the universal rank of a player in the random meeting state. In the random meeting state, the high-I types generally are more patient, as their future potential partners generally have a higher expected rank, due to the chance of being in the assortative meeting state.

Proposition 8 *For the random meeting state $s = r$, the optimal strategy $a^*(t, r)$ is a non-decreasing function of a player's individual rank I.*

Finally, we examine the effects of a variation of the intensity of assortative meeting β. Notice that $E(t+1)$ is a function of parameter β, in particular

$$\frac{\partial E(t+1)}{\partial \beta} \lesseqqgtr 0 \text{ as } E^{\bar{r}}(t+1) \lesseqqgtr E^r(t+1) \tag{23}$$

Therefore the effect of β on the optimal strategy in both assortative and random meeting state depends on which future conditional expectation is higher. Since the value of $E^{\bar{r}}(t+1)$ strictly depends on the individual's universal rank, then in turn it is more likely that $E^{\bar{r}}(t+1) > E^r(t+1)$ (and in turn $\frac{\partial E(t+1)}{\partial \beta} > 0$) for higher levels of I.

4 Expected ranks

In this section we compute the approximated player's payoff in the N-period meeting game when players use optimal strategies. This analysis of optimal payoffs aims to find an easy formula to calculate the approximated payoff for any duration N using the parameters α, β, I.[8]

For $\beta \in [0, 1)$ it is not possible to obtain analytical solutions, so that we focus on the case in which players can attend assortative meetings only, $\beta = 1$, and we refer to this case as "assortative game". Along this section we omit superscripts s and $\bar{r}$ for brevity, and we assume $\delta = 1$ for simplicity. The result is summarised in the following proposition.

Proposition 9 *The expected rank of an accepted partner in an assortative game is*

$$E(1) \approx \alpha I + (1-\alpha)\left(1 - \left(\frac{32N+76}{27}\right)^{-\frac{1}{2}}\right). \tag{24}$$

Begin to notice that when N tends to infinity, then in Proposition 9, $E(1)$ approximately tends to $\alpha I + 1 - \alpha$, which is the maximum payoff possible for a player in the assortative game. Intuitively, an infinite-periods game ensures that, at some point, a player will meet a partner with preferred individual characteristics.

[8] The approach of this section is similar to Eriksson *et al.* (2007).

Expression (24) helps to approximately obtain the payoff in an assortative game without solving Bellman equation (12) recurrently. There are the results of comparison of the exact and approximated payoffs for different game duration N. In Table 1 and 2 we consider the optimal payoff for different N and $\alpha = 0.25$ ($\alpha = 0.80$). In particular, they show the optimal payoff calculated by solving the Bellman equation recurrently (column 2), the approximated optimal payoff calculated by formula (24) (column 3) and the mistake of approximation in percentage (column 4) by the formula $|\widetilde{E}(1) - E(1)|/E(1)$. According to Table 1 and 2, the simple formula obtained by Proposition 9 is a good approximation of the optimal payoff.

N	$E(1)$	$\widetilde{E}(1)$	Mistake (in %)
10	0.7574	0.7417	2.08
50	0.8514	0.8423	1.07
100	0.8786	0.8694	1.05
200	0.8989	0.8891	1.09
500	0.9174	0.9068	1.16

Table 1: The exact and approximated payoffs for $\alpha = 0.25$, $I = 0.75$

N	$E(1)$	$\widetilde{E}(1)$	Mistake (in %)
10	0.7519	0.7478	0.55
50	0.7754	0.7746	0.10
100	0.7822	0.7818	0.05
200	0.7872	0.7871	0.01
500	0.7918	0.7918	0.00

Table 2: The exact and approximated payoffs for $\alpha = 0.80$, $I = 0.75$

5 Expected time to marry

In this section we examine how long a player expects to remain unmarried. We denote T as a discrete random variable representing the number of the periods in which a player expects to marry, where $T = 1, 2, ..., N$. In order to calculate the mathematical expectation of the number of periods needed to marry we need to find the probability that a player marries in each particular period t. Denote this probability as P_t, $\forall\ t = 1, \ldots, N$. For period 1, this probability can be defined by the following expression:

$$P_1 = (1-\beta)\,\mathrm{Pr}^r[\mathrm{marry}](a(1,r)) + \beta\,\mathrm{Pr}^{\bar{r}}[\mathrm{marry}](a(1,\bar{r})) \equiv M_1. \tag{25}$$

For period 2, the probability of marrying is as follows:

$$P_2 = (1-M_1)\big((1-\beta)\,\mathrm{Pr}^r[\mathrm{marry}](a(2,r)) + \beta\,\mathrm{Pr}^{\bar{r}}[\mathrm{marry}](a(2,\bar{r}))\big) = (1-M_1)M_2. \tag{26}$$

Hence for period k, the probability can be obtained by the expression:

$$\begin{aligned} P_k =& (1-M_1)\ldots(1-M_{k-1})\,((1-\beta)\,\mathrm{Pr}^r[\mathrm{marry}](a(k,r)) \\ &+ \beta\,\mathrm{Pr}^{\bar{r}}[\mathrm{marry}](a(k,\bar{r}))\big) = (1-M_1)\ldots(1-M_{k-1})M_k. \end{aligned} \tag{27}$$

If a player does not marry in the first $N-1$ periods of the game and participates in the last N^{th} period he marries in this period with probability 1 (given the assumption that a player always prefers to marry than to remain single), i.e.,

$$P_N = (1-M_1)\ldots(1-M_{N-1}). \tag{28}$$

The expectation of T is determined as follows.

Proposition 10 *The expected time to marry is given by:*

$$ET = P_1 + 2P_2 + \ldots + NP_N = \sum_{i=1}^{N} i\left\{\prod_{k=1}^{i-1}(1-M_k)\right\}M_i. \tag{29}$$

The expected number of periods before to marry is a function of the player's strategy a and all parameters α, β, I. We examine the properties of Proposition 10 using a numerical simulation. First we consider it for different universal ranks. We appoint the following parameters values: $\beta = 0.7$, $N = 100$, $\delta = 1$, $I = 0.01$, 0.33, 0.66, and 0.99. Consider first $\alpha = 0.25$ (Table 3). In this case, the higher the rank of a player, the less the expected time of marrying. This result can be explained as follows. A player with high universal rank tends to have higher expectations about future meetings. This is due to the chance of being in the assortative meeting state. Indeed, this increases the likelihood that the player meets a potential partner with the same universal rank. Therefore a player with high rank is generally more "demanding" about a partner type. Nonetheless, the chance of being in the assortative meeting state for a high-universal rank individual also has the effect of increasing the quality of each meeting. As a consequence, a player with high universal rank may in fact marry sooner than other individuals as the second effect can offset the first one.

For $\alpha = 0.80$, the relationship between universal rank and time to marry is non-monotone: the time to marry is low for individuals with low levels of universal rank, it increases for medium levels of universal rank and it decreases again for high universal rank. Two factors contribute to obtain this. First, the higher importance of α makes individuals with a high universal characteristic more picky in their partner choice, thus delaying the marriage. The first effect prevails on the second effect when the universal characteristic is not so high, but for very high universal characteristic the second effect more than offsets the first effect, so that the time expected of marrying is lower. Thus individuals with a very high universal characteristic tend to marry sooner than other individuals. Alternatively, individuals with medium-high rank tend to marry later because they are choosy and the quality of individuals they meet is more likely to be lower.

I	$\alpha = 0.25$	$\alpha = 0.80$
0.01	46.38	42.36
0.33	44.60	42.82
0.66	42.48	50.53
0.99	42.27	41.55

Table 3: Expected number of periods needed to marry for different I

Finally, we consider the change in the expected number of periods before marrying for different β. We appoint the following parameters values: $I = 0.9$, $N = 100$, $\delta = 1$, $\beta = 0.01$, 0.33, 0.66, and 0.99. As in the previous example, we assume either $\alpha = 0.25$ or $\alpha = 0.80$ As shown by Table 2, the effect of a variation of β is qualitatively similar to the effect of a variation of I. This seems intuitive, considering that an increase of β relatively increases the importance of I in an individual's payoff.

β	$\alpha = 0.25$	$\alpha = 0.80$
0.01	44.70	43.24
0.33	43.66	41.90
0.66	42.42	42.01
0.99	41.81	41.80

Table 4: Expected number of periods needed to marry for different β

6 Infinite horizon

In this section we consider the same problem with infinite periods. We show that the findings are qualitatively similar to the results obtained in the finite case. Suppose that the universes U and U' are infinite, and the game last an infinite number of periods. In this case, the backward Bellman approach considered above cannot be applied. In order to have boundary conditions to solve the Bellman equation, given the infinite periods, we consider the stationary behaviour of the players. In other words, a player's strategy $a = a(s)$ does not depend on the time period. The functional equation for the player's payoff which is the expected partner's rank in the game beginning from state s is now:

$$\begin{aligned} E^s(a(\bar{r}), a(r)) = & \Pr^s[\text{marry}](a(s))E[R_i^s|\text{marry}](a(s)) \\ & +\delta(1 - \Pr^s[\text{marry}](a(s)))\left(\beta E^{\bar{r}}(a(\bar{r}), a(r)) + (1-\beta)E^r(a(\bar{r}), a(r))\right). \end{aligned} \quad (30)$$

Equation (30) is the equivalent of equation (3) for the case with infinite periods. Denote vector $(E^{\bar{r}}(a(\bar{r}), a(r)), E^r(a(\bar{r}), a(r)))^T$ as $\mathbb{E}(a(\bar{r}), a(r))$ and rewrite equation (30) in the vectorial form:

$$\mathbb{E}(a(\bar{r}), a(r)) = \mathbb{A}_1 + \delta \mathbb{A}_2(\beta, 1-\beta)\mathbb{E}(a(\bar{r}), a(r)), \quad (31)$$

where

$$\begin{aligned} \mathbb{A}_1 = (\mathbb{A}_{11}, \mathbb{A}_{12}) = & \left(\Pr^{\bar{r}}[\text{marry}](a)E[R_i^{\bar{r}}|\text{marry}](a),\ \Pr^r[\text{marry}](a)E[R_i^r|\text{marry}](a)\right)^T, \\ \mathbb{A}_2 = (\mathbb{A}_{21}, \mathbb{A}_{22}) = & \left(1 - \Pr^{\bar{r}}[\text{marry}](a),\ 1 - \Pr^r[\text{marry}](a)\right)^T. \end{aligned} \quad (32)$$

Given equation (31) we obtain the following result.

Proposition 11 *Suppose $\delta \neq 1$ and that players do not use their highest strategies (*

$a(\bar{r}) = \alpha I + 1 - \alpha$ and $a(r) = 1$). Then a player's payoff is:

$$\mathbb{E}(a(\bar{r}), a(r)) = \frac{1}{1 - \delta(\beta \mathbb{A}_{21} + (1-\beta)\mathbb{A}_{22})} \begin{pmatrix} 1 - \delta(1-\beta)\mathbb{A}_{22} & \delta(1-\beta)\mathbb{A}_{21} \\ \delta\beta\mathbb{A}_{22} & 1 - \delta\beta\mathbb{A}_{21} \end{pmatrix} \mathbb{A}_1. \tag{33}$$

When $\delta = 1$ and players use their highest strategies in any period and for any type of meeting we obtain the following system of Bellman equations:

$$\begin{aligned} E^{\bar{r}}(a_{\max}(\bar{r}), a_{\max}(r)) &= \beta E^{\bar{r}}(a_{\max}(\bar{r}), a_{\max}(r)) + (1-\beta) E^{r}(a_{\max}(\bar{r}), a_{\max}(r)), \\ E^{r}(a_{\max}(\bar{r}), a_{\max}(r)) &= \beta E^{\bar{r}}(a_{\max}(\bar{r}), a_{\max}(r)) + (1-\beta) E^{r}(a_{\max}(\bar{r}), a_{\max}(r)), \end{aligned} \tag{34}$$

which amounts to $E^{\bar{r}}(a_{\max}(\bar{r}), a_{\max}(r)) = E^{\bar{r}}(a_{\max}(\bar{r}), a_{\max}(r))$. In this case a player never proposes a marriage in the game, and the game never stops.

As in the finite case, the player's optimal strategy maximises the expected rank of the expected partner:

$$\beta E^{\bar{r}}(a(\bar{r}), a(r)) + (1-\beta) E^{r}(a(\bar{r}), a(r)) = \frac{\beta \mathbb{A}_{11} + (1-\beta)\mathbb{A}_{12}}{1 - \delta(\beta \mathbb{A}_{21} + (1-\beta)\mathbb{A}_{22})}, \tag{35}$$

$$\text{s.t. } a(\bar{r}) \in [\alpha I, \alpha I + 1 - \alpha] \text{ and } a(r) \in [0, 1].$$

We analyse the optimal strategies and player's payoff in equilibrium in a numerical example. In particular, we compare the equilibrium payoff with infinite horizon with the results in the finite case when $N = 100$. We appoint the following parameters values: $\delta = 0.95$, and either $\alpha = 0.25$ or $\alpha = 0.80$. In Table 5, we keep $\beta = 0.7$ and we examine the equilibrium payoff for $I = 0.01$, $I = 0.33$, $I = 0.66$ and $I = 0.99$. In Table 6, we keep $I = 0.9$ and we show the results for $\beta = 0.01$, $\beta = 0.33$, $\beta = 0.66$ and $\beta = 0.99$. The tables show that the equilibrium payoffs in finite and infinite case are the very close. These results show that the analytical results obtained in the finite case are robust by assuming an infinite horizon.[9]

[9] Upon request, we can provide additional numerical examples in which the similarities of results between the finite and the infinite case are confirmed.

	$\alpha = 0.25$		$\alpha = 0.80$	
I	Infinite game	Finite game	Infinite game	Finite game
0.01	0.5359	0.5359	0.4838	0.4838
0.33	0.5762	0.5359	0.4838	0.4838
0.66	0.6272	0.5359	0.6283	0.6283
0.99	0.6864	0.5359	0.8735	0.8735

Table 5: Equilibrium payoff for finite ($N = 100$) and infinite game with $\beta = 0.7$ and different I.

	$\alpha = 0.25$		$\alpha = 0.80$	
β	Infinite game	Finite game	Infinite game	Finite game
0.01	0.6065	0.6066	0.6199	0.6198
0.33	0.6392	0.6392	0.7559	0.7559
0.66	0.6666	0.6666	0.8010	0.8009
0.99	0.6890	0.6890	0.8202	0.8201

Table 6: Equilibrium payoff for finite ($N = 100$) and infinite game with $I = 0.9$ and different β.

7 State-independent strategies

In this section we modify the N-period meeting game as follows. Suppose that, for every $t = 1, \ldots, N$, a player i uses the same strategy $a(t)$ in assortative and random meetings, so that $a = a(t) = a(t, \bar{r}) = a(t, r)$. This situation reflects the situations in which an individual does not know exactly which type of meeting (state) takes place in every period.

In this modified meeting game we consider the payoff of a player in the N-period game, as the linear combination of the player's expected payoffs in the games beginning with particular meetings (assortative and random):

$$E(1) = \beta E^{\bar{r}}(1) + (1 - \beta) E^{r}(1). \tag{36}$$

The Bellman equation for the maximal expected rank $E(t)$ in period t takes the form of:

$$\begin{aligned} E(t) = &\max_a \left\{ \beta \Pr^{\bar{r}}[\text{marry}](a) E[R^{\bar{r}}(t)|\text{marry}](a) \right. \\ &+(1-\beta) \Pr^{r}[\text{marry}](a) E[R^{r}(t)|\text{marry}](a) \\ &\left. + \delta \left\{ \beta(1-\Pr^{\bar{r}}[\text{marry}](a)) + (1-\beta)(1-\Pr^{r}[\text{marry}](a)) \right\} E(t+1) \right\}, \end{aligned} \tag{37}$$

with boundary condition:

$$E(N) = \beta \left(\frac{1-\alpha}{2} + \alpha I \right) + \frac{1-\beta}{2}. \tag{38}$$

With state-independent strategies, a player uses the same strategies for random and assortative meetings in the same period. Then, the set of possible strategies are in the set $[0,1]$ for all states. The probability to marry is given by:

$$\Pr^{\bar{r}}[\text{marry}](a) = \begin{cases} 1, & \text{if } a \in [0, \alpha I), \\ \left(1 - \dfrac{a - \alpha I}{1-\alpha}\right)^2, & \text{if } a \in [\alpha I, \alpha I + 1 - \alpha), \\ 0, & \text{if } a \in [\alpha I + 1 - \alpha, 1]. \end{cases} \tag{39}$$

Moreover in the assortative meeting state, the conditional expectation of the absolute rank of the chosen j under the condition that the marriage takes place in period t is:

$$E[R_i^{\bar{r}}(t)|\text{marry}](a) = \begin{cases} \alpha I + \dfrac{1-\alpha}{2}, & \text{if } a \in [0, \alpha I), \\ \dfrac{\alpha I + 1 - \alpha + a}{2}, & \text{if } a \in [\alpha I, \alpha I + 1 - \alpha), \\ 0, & \text{if } a \in [\alpha I + 1 - \alpha, 1]. \end{cases} \tag{40}$$

With state-independent strategies, the player's optimal strategy is implicitly defined. Notice that the player's payoff in the N-period meeting game with state-independent strategies, i.e. the expected rank of the potential partner, is not larger than the payoff in the game with state-dependent strategies.

8 Concluding remarks

We have studied marriage formation through a two-sided secretary problem approach, where individuals have two different dimensions of heterogeneity, and two possible types of meetings, a random and an assortative one, may occur over time. We show that individuals with a high universal characteristic tend to be more picky in their marriage hunting. This does not necessarily mean that they marry later than other individuals, since the higher expected quality of their potential partners in the assortative meeting state can make them marry earlier than individuals with lower universal characteristic.

The analysis carried out did not consider divorce explicitly, but this indeed can be easily implemented. Once assumed that divorce occurs with exogenous probability, then there is no reason to expect that this probability may change according to whether two individuals decide to marry or not in a certain period, apart from the fact that of course the probability of divorcing increases with the length of a relationship.

A further development may take into account different universal characteristics for men and women. According to the customs considered, these may change according to gender. For example in western societies, men appoint a higher value to beauty compared to women, whereas women appoint a higher value to financial security (See Coles and Francesconi, 2011). These developments of the current model are left for future work.

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Appendix

Proof of Proposition 1

In order to determine the conditional probability to marry, it is necessary first to obtain the probability distribution of the potential partner's rank.

Probability density and cumulative distribution functions

Assortative meeting: $s = \bar{r}$**.** We find the probability density distribution function $f_{R_i^{\bar{r}(t)}}(x)$ of $R_i^{\bar{r}}(t) = (1-\alpha)\,\eta_j^t + \alpha I$ by using the consolidation formula of independent random variables:

$$f_{R_i^{\bar{r}}(t)}(x) = \frac{1}{1-\alpha} f_{\eta_j^t}\left(\frac{x-\alpha I}{1-\alpha}\right) = \begin{cases} \dfrac{1}{1-\alpha}, & \text{if } x \in [\alpha I, \alpha I + 1 - \alpha] \\ 0, & \text{if } x \notin [\alpha I, \alpha I + 1 - \alpha], \end{cases} \tag{41}$$

where $f_{\eta_j^t}(x)$ is a probability density function of the variable η_j^t. Thus the cumulative distribution function $F_{R_i^{\bar{r}}(t)}(x) = \Pr\{R_i^{\bar{r}}(t) \le x\} = \int_{-\infty}^{x} f_{R_i^{\bar{r}}(t)}(u)du$ of the random variable $R_i^{\bar{r}}(t)$ is as follows:

$$F_{R_i^{\bar{r}}(t)}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, \alpha I) \\ \dfrac{x-\alpha I}{1-\alpha}, & \text{if } x \in [\alpha I, \alpha I + 1 - \alpha) \\ 1, & \text{if } x \in [\alpha I + 1 - \alpha, \infty) \end{cases} \tag{42}$$

Therefore, the linear transformation of η_j^t keeps the same distribution type but changes the interval of possible values, i.e. the distribution of rank $R_i^{\bar{r}}(t)$ is a continuous uniform in the interval $[\alpha I, \alpha I + 1 - \alpha]$.

Random meeting: $s = r$**.** A player i ranks a potential partner j as follows: $R_i^r(t) = (1-\alpha)\eta_j^t + \alpha I_j^t$. Here the random variables η_j^t and I_j^t, $t = 1, \ldots, N$ are independent and have the same uniform continuous distribution on the interval $[0, 1]$. The expression for the probability density distribution function $f_{R_i^r(t)}(x)$ of a random variable $R_i^r(t)$ can be found using the formula of consolidation of two continuous independent variables:

- Case $\alpha \geq \frac{1}{2}$:

$$f_{R_i^r(t)}(x) = \int_{-\infty}^{\infty} f_{(1-\alpha)\eta_t}(u) f_{\alpha I_j^t}(x-u) du = \tag{43}$$

$$= \begin{cases} \dfrac{x}{\alpha(1-\alpha)}, & \text{if } x \in [0, 1-\alpha) \\ \dfrac{1}{\alpha}, & \text{if } x \in [1-\alpha, \alpha) \\ \dfrac{1-x}{\alpha(1-\alpha)}, & \text{if } x \in [\alpha, 1] \\ 0, & \text{if } x \notin [0, 1] \end{cases}$$

- Case $\alpha < \frac{1}{2}$:

$$f_{R_i^r(t)}(x) = \begin{cases} \dfrac{x}{\alpha(1-\alpha)}, & \text{if } x \in [0, \alpha) \\ \dfrac{1}{1-\alpha}, & \text{if } x \in [\alpha, 1-\alpha) \\ \dfrac{1-x}{\alpha(1-\alpha)}, & \text{if } x \in [1-\alpha, 1] \\ 0, & \text{if } x \notin [0, 1] \end{cases} \tag{44}$$

For $s = r$, the cumulative distribution function $F_{R_i^r(t)}(x)$ of the random variable $R_i^r(t)$ according to α is:

- Case $\alpha \geq \frac{1}{2}$:

$$F_{R_i^r(t)}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ \dfrac{x^2}{2\alpha(1-\alpha)}, & \text{if } x \in [0, 1-\alpha) \\ \dfrac{2x-(1-\alpha)}{2\alpha}, & \text{if } x \in [1-\alpha, \alpha) \\ 1 - \dfrac{(1-x)^2}{2\alpha(1-\alpha)}, & \text{if } x \in [\alpha, 1) \\ 1, & \text{if } x \in [1, \infty) \end{cases} \tag{45}$$

- Case $\alpha < \frac{1}{2}$:

$$F_{R_i^r(t)}(x) = \begin{cases} 0, & \text{if } x \in (-\infty, 0) \\ \dfrac{x^2}{2\alpha(1-\alpha)}, & \text{if } x \in [0, \alpha) \\ \dfrac{2x-\alpha}{2(1-\alpha)}, & \text{if } x \in [\alpha, 1-\alpha) \\ 1 - \dfrac{(1-x)^2}{2\alpha(1-\alpha)}, & \text{if } x \in [1-\alpha, 1) \\ 1, & \text{if } x \in [1, \infty) \end{cases} \tag{46}$$

Notice that in the case of random meeting $s = r$ the distribution of rank $R_i^r(t)$ is not uniform.

Conditional probability

Given the probability density and the cumulative distribution functions, we are now able to determine the conditional probabilities to marry. We consider the two cases according to $\alpha \geq \frac{1}{2}$, $\alpha < \frac{1}{2}$, and we find the expressions of probability to marry $\Pr^s[\text{marry}](a)$ under the condition that the state is s and a player i uses strategy a. This is the probability that both players i and j who met in period t accept to marry under the condition that their choices are independent and they both use the same type of strategies.

If the meeting is assortative ($s = \bar{r}$), the conditional probability to marry is as follows:

$$\Pr^{\bar{r}}[\text{marry}](a) = \Pr\left\{\left(R_i^{\bar{r}}(t) > a(t,\bar{r})\right) \cap \left(R_j^{\bar{r}}(t) > a(t,\bar{r})\right)\right\}, \tag{47}$$

where the events $R_i^{\bar{r}}(t) > a(t,\bar{r})$ and $R_j^{\bar{r}}(t) > a(t,\bar{r})$ are independent, so that:

$$\Pr^{\bar{r}}[\text{marry}](a) = \left(\Pr\left\{R_i^r(t) > a(t,r)\right\}\right)^2 = \left(1 - \frac{a - \alpha I}{1-\alpha}\right)^2, \tag{48}$$

where $a = a(t,\bar{r}) \in [\alpha I, \alpha I + 1 - \alpha]$. In the case of random meeting ($s = r$), this probability is given by:

$$\Pr^r[\text{marry}](a) = \left(\Pr\left\{R_i^r(t) > a(t,r)\right\}\right)^2 = \left(1 - F_{R_i^r(t)}(a(t,r))\right)^2 \tag{49}$$

- For $\alpha \geq \frac{1}{2}$:

$$\Pr^r[\text{marry}](a) = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, 1-\alpha) \\ \left(1 - \dfrac{2a - (1-\alpha)}{2\alpha}\right)^2, & \text{if } a \in [1-\alpha, \alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1] \end{cases} \tag{50}$$

- For $\alpha < \frac{1}{2}$:

$$\Pr^r[\text{marry}](a) = \begin{cases} \left(1 - \dfrac{a^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [0, \alpha) \\ \left(1 - \dfrac{2a - \alpha}{2(1-\alpha)}\right)^2, & \text{if } a \in [\alpha, 1-\alpha) \\ \left(\dfrac{(1-a)^2}{2\alpha(1-\alpha)}\right)^2, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{51}$$

Proof of Proposition 2

We denote as $E[R_i^s(t)|\text{marry}](a)$ the expectation of absolute rank of the potential partner j chosen by a player' i, under the condition that the marriage takes place in period t and $E[R_i^s(t)|\text{marry}](a)$ is a function of a player i's strategy a. For $s = \bar{r}$, the conditional expectation is given by:

$$\begin{aligned} E[R_i^{\bar{r}}(t)|\text{marry}](a) =& \frac{E[R_i^{\bar{r}}(t)|R_i^{\bar{r}}(t) > a] \Pr\{R_j^{\bar{r}}(t) > a\}}{\Pr^{\bar{r}}[\text{marry}](a)} \\ =& \frac{E[R_i^{\bar{r}}(t)|R_i^{\bar{r}}(t) > a] \Pr\{R_j^{\bar{r}}(t) > a\}}{\Pr\{R_i^{\bar{r}}(t) > a\} \Pr\{R_j^{\bar{r}}(t) > a\}} = \frac{E[R_i^{\bar{r}}(t)|R_i^{\bar{r}}(t) > a]}{\Pr\{R_i^{\bar{r}}(t) > a\}} \\ =& \frac{\int_a^\infty u f_{R_i^{\bar{r}}(t)}(u) du}{\int_a^\infty f_{R_i^{\bar{r}}(t)}(u) du} = \frac{\alpha I + 1 - \alpha + a}{2}, \end{aligned} \tag{52}$$

where $a = a(t, \bar{r}) \in [\alpha I, \alpha I + 1 - \alpha]$.

For $s = r$, we make use of the analysis carried out for determining the conditional expectation for $s = \bar{r}$ using equations (43), (44), (50), (51):

$$\begin{aligned} E[R_i^r(t)|\text{marry}](a) =& \frac{E[R_i^r(t)|R_i^r(t) > a]\,\Pr\{R_j^r(t) > a\}}{\Pr^r[\text{marry}](a)} \\ =& \frac{E[R_i^r(t)|R_i^r(t) > a]}{\Pr\{R_i^r(t) > a\}} = \frac{\int_a^\infty u f_{R_i^r(t)}(u)du}{\int_a^\infty f_{R_i^r(t)}(u)du}. \end{aligned} \tag{53}$$

For $\alpha \geq \frac{1}{2}$, equation (53) becomes:

$$E[R_i^r(t)|\text{marry}](a) = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, 1-\alpha) \\ \dfrac{3a^2 - (1+\alpha+\alpha^2)}{6a - 3(1+\alpha)}, & \text{if } a \in [1-\alpha, \alpha) \\ \dfrac{2a+1}{3}, & \text{if } a \in [\alpha, 1] \end{cases} \tag{54}$$

whereas for $\alpha < \frac{1}{2}$, equation (53) becomes:

$$E[R_i^r(t)|\text{marry}](a) = \begin{cases} \dfrac{2a^3 - 3\alpha(1-\alpha)}{3a^2 - 6\alpha(1-\alpha)}, & \text{if } a \in [0, \alpha) \\ \dfrac{3a^2 - (3-3\alpha+\alpha^2)}{6a - 3(2-\alpha)}, & \text{if } a \in [\alpha, 1-\alpha) \cdot \\ \dfrac{2a+1}{3}, & \text{if } a \in [1-\alpha, 1] \end{cases} \tag{55}$$

Proof of Proposition 3

To find the optimal strategy for period t and state $s = \bar{r}$ we first differentiate the expression in the right part of (12) with respect to a, then we equate the differential with zero and solve it for a. We denote the solution as $b^{t,\bar{r}}$. There are two solutions:

$$\begin{aligned} b_1^{t,\bar{r}} &= \tfrac{4}{3}\delta E(t+1) - \tfrac{1}{3}\left[\alpha I + 1 - \alpha\right], \\ b_2^{t,\bar{r}} &= \alpha I + 1 - \alpha. \end{aligned} \tag{56}$$

Consider two possible cases for the value of expected rank $E(t+1)$: $E(t+1) < \frac{\alpha I+1-\alpha}{\delta}$ and $E(t+1) \geq \frac{\alpha I+1-\alpha}{\delta}$:

1. Let $E(t+1) < \frac{\alpha I+1-\alpha}{\delta}$, so that $b_1^{t,\bar{r}} < b_2^{t,\bar{r}}$. In this case the second derivative of the right

part of (12) with respect to a calculated in $b_1^{t,\bar{r}}$ ($b_2^{t,\bar{r}}$) equals to $\frac{-2(\alpha I+1-\alpha-\delta E(t+1))}{(1-\alpha)^2}$ $\left(\frac{2(\alpha I+1-\alpha-\delta E(t+1))}{(1-\alpha)^2}\right)$. Thus the strategy $a = b_1^{t,\bar{r}}$ maximises the right part of (12) whereas $a = b_2^{t,\bar{r}}$ minimizes it. Hence the function in the right part of (12) decreases in $[b_1^{t,\bar{r}}, b_2^{t,\bar{r}}]$. If additionally $b_1^{t,\bar{r}} < \alpha I$, then the optimal strategy is the minimum possible value for the strategy, i.e. $a^*(N, \bar{r}) = \alpha I$. For $b_1^{t,\bar{r}} \geq \alpha I$, the strategy $a^*(N, \bar{r}) = b_2^{t,\bar{r}}$ maximises the right part of (12).

2. Let $E(t+1) \geq \frac{\alpha I+1-\alpha}{\delta}$. In this case $b_2^{t,\bar{r}} < b_1^{t,\bar{r}}$ and $a = b_1^{t,\bar{r}}$ minimizes the right part of (12) while $a = b_2^{t,\bar{r}}$ maximises it. Function in the right part of (12) increases from $a = \alpha I$ to $a = b_2^{t,\bar{r}}$ where obtains the maximum value.

Proof of Proposition 4

For brevity, we will consider the case $\alpha \geq \frac{1}{2}$ and omit the case $\alpha < \frac{1}{2}$ as it is very similar.[10] The problem is to find the maximum of the piecewise function in the right part of (15) with respect to the strategy $a = a(t, r)$. This function is continuous with respect to a. When $a \in [0, 1-\alpha)$, then it has a unique maximum at $a = 0$. The second derivative of the function in the right part of (15) calculated in $a = 0$ equals $\frac{4\delta E(t+1)-1}{2\alpha(1-\alpha)}$. If $E(t+1) < \frac{1}{4\delta}$, then the strategy $a^*(N, r) = 0$ maximises the right part of (15). Also, the right part of (15) is a decreasing function with respect to a in the interval of possible strategy values $[0, 1]$. This implies that the optimal strategy is $a^*(N, r) = 0$.

For $E(t+1) \geq \frac{1}{4\delta}$, the right part of (15) increases in the interval $a \in [0, 1-\alpha)$. Consider the case in which $a \in [1-\alpha, \alpha)$. Differentiation of the function in the right part of (15) yields:

$$\begin{aligned} b_1^{t,r} &= \tfrac{1}{6}(1+\alpha) + \tfrac{2}{3}\delta E(t+1) - \tfrac{1}{6}\sqrt{16\delta^2(E(t+1))^2 - 16(1+\alpha)\delta E(t+1) + 5\alpha^2 + 6\alpha + 5}, \\ b_2^{t,r} &= \tfrac{1}{6}(1+\alpha) + \tfrac{2}{3}\delta E(t+1) + \tfrac{1}{6}\sqrt{16\delta^2(E(t+1))^2 - 16(1+\alpha)\delta E(t+1) + 5\alpha^2 + 6\alpha + 5}, \end{aligned} \tag{57}$$

where $b_1^{t,r} < b_2^{t,r}$. The second derivative of the function in the right part of (15) in $b_1^{t,r}$ is negative, while the second derivative of it in $b_2^{t,r}$ is positive. Hence $b_1^{t,r}$ maximises the function in the right part of (15) and $b_2^{t,r}$ minimizes it, and function in the right part of (15) decreases from $b_1^{t,r}$ to $b_2^{t,r}$. Here we should consider three cases:

[10] The complete proof can be provided upon request.

1. For $b_1^{t,r} < 1-\alpha$ $\left(\Leftrightarrow \frac{1}{4\delta} \leq E(t+1) < \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta}\right)$, the function in the right part of (15) decreases on the interval $[1-\alpha, 1]$. Thus, the optimal strategy is $a^*(N,r) = 1-\alpha$.

2. For $1-\alpha \leq b_1^{t,r} < \alpha$ $\left(\Leftrightarrow \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E(t+1) < \frac{5\alpha+1}{6\delta}\right)$, the function in the right part of (15) increases on $[0, b_1^{t,r})$ and decreases on $(b_1^{t,r}, 1]$, so that the optimal strategy is $a = b_1^{t,r}$.

3. For $b_1^{t,r} \geq \alpha$, the function in the right part of (15) increases on $[0,\alpha)$. For $[\alpha, 1)$, it has one extreme point $a = b_3^{t,r}$, where

$$b_3^{t,r} = -\frac{1}{5} + \frac{6}{5}\delta E(t+1), \tag{58}$$

and the second derivative shows that it maximises the function in the right part of (15). Hence it increases on $[0, b_3^{t,r})$ and decreases on $[b_3^{t,r}, 1]$, so that the optimal strategy is $a^*(N,r) = b_3^{t,r}$ if and only if $b_3^{t,r} \in [\alpha, 1)$, i.e. $E(t+1) \geq \frac{5\alpha+1}{6\delta}$.

Proof of Proposition 6

First consider the case with assortative meeting, $s = \bar{r}$. For $a^*(t,\bar{r}) = \alpha I$, the payoff in equilibrium $E^{\bar{r}}(t)$ is an increasing function of the universal rank I as

$$\frac{\partial E^{\bar{r}}(t)}{\partial I} = \alpha. \tag{59}$$

For $a^*(t,\bar{r}) = \alpha I + 1 - \alpha$, differentiation of $\partial E^{\bar{r}}(t)$ with respect to I yields:

$$\frac{\partial E^{\bar{r}}(t)}{\partial I} = \frac{\partial E(t+1)}{\partial I}. \tag{60}$$

Given $\frac{\partial E(N)}{\partial I} = \alpha\beta > 0$, we can easily prove the positiveness of $\frac{\partial E(N)}{\partial I}$ for any $t = 1, \ldots, N-1$. Consider next the case $a^*(t,\bar{r}) = \frac{4\delta E(t+1) - (\alpha I + 1 - \alpha)}{3}$ when $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E(t+1) < \frac{\alpha I + 1 - \alpha}{\delta}$, in which:

$$\begin{aligned}\frac{\partial E^{\bar{r}}(t)}{\partial I} =& \frac{16\alpha}{9(1-\alpha)^2}\left(\delta E(t+1) - (1-\alpha+\alpha I)\right)^2 \\ &- \frac{16\delta\left(\delta E(t+1) - (1-\alpha+\alpha I)\right)^2 - 9\delta(1-\alpha)^2}{9(1-\alpha)^2}\frac{\partial E(t+1)}{\partial I}.\end{aligned} \tag{61}$$

The right hand side is positive for any $t = 1, \ldots, N$ because $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E(t+1) < \frac{\alpha I + 1 - \alpha}{\delta}$ and the fact that $\frac{\partial E(N)}{\partial I} = \alpha\beta > 0$. Therefore, in the assortative meeting case, a player's payoff in equilibrium is an increasing function of the universal rank I.

Finally consider the case with random meeting, $s = r$, and suppose $\alpha \geq 1/2$ (the case where $\alpha < 1/2$ can be considered in the same way and leads to the same results). We show the proof when $a^* \in [0, 1-\alpha)$, and omit the cases in which $a^* \in [1-\alpha, \alpha)$ and $a^* \in [\alpha, 1]$, as the algebra is very similar and leads to the same results. Differentiating $E^r(t)$ w.r.t. I yields:

$$\begin{aligned}\frac{\partial E^r(t)}{\partial I} =& 2\left(1 - \frac{(a^*)^2}{2\alpha(1-\alpha)}\right)\left(-\frac{a^*\frac{\partial a^*}{\partial I}}{\alpha(1-\alpha)}\right)\left(\frac{2(a^*)^3 - 3\alpha(1-\alpha)}{3(a^*)^2 - 6\alpha(1-\alpha)} - \delta E(t+1)\right) \\ &+ \left(1 - \frac{(a^*)^2}{2\alpha(1-\alpha)}\right)^2 \left(\frac{18a^*\frac{\partial a^*}{\partial I}\alpha(1-\alpha)(1-2a^*)}{(3(a^*)^2 - 6\alpha(1-\alpha))^2} - \delta\frac{\partial E(t+1)}{\partial I}\right) + \delta\frac{\partial E(t+1)}{\partial I}.\end{aligned} \tag{62}$$

For $a^* = 0$ and $E(t+1) < \frac{1}{4\delta}$, we obtain $E^r(t) = 1/2$, so that $E^r(t)$ is a non-decreasing function of I. The similar result can be obtained for the case $a^* = 1-\alpha$ and $E(t+1) \in [\frac{1}{4\delta}, \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta})$. For $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E(t+1) < \frac{5\alpha+1}{6\delta}$ and $E(t+1) \geq \frac{5\alpha+1}{6\delta}$ we obtain $\frac{\partial E^r(t)}{\partial I} \gtrless 0$ if $\frac{\partial E(t+1)}{\partial I} \gtrless 0$. Given $\frac{\partial E(N)}{\partial I} = \alpha\beta > 0$, we prove that for any $t = 1, \ldots, N$, the player's payoff in random meeting is an increasing function of I. Therefore, in any possible case, a player's optimal payoff in random meetings is a non-decreasing function of the universal rank.

Proof of Proposition 7

It is straightforward that for $E(t+1) < \frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta}$ and $E(t+1) \geq \frac{\alpha I + 1 - \alpha}{\delta}$, the optimal strategy is a constant, hence $\frac{\partial a^*}{dI} = \alpha$ is a non-decreasing function of the universal rank I.

Consider next $\frac{\alpha I}{\delta} + \frac{1-\alpha}{4\delta} \leq E(t+1) < \frac{\alpha I + 1 - \alpha}{\delta}$, for which the optimal strategy is $a^*(t, \bar{r}) = \frac{4\delta E(t+1) - (\alpha I + 1 - \alpha)}{3}$. Here we obtain $\frac{\partial a^*}{dI} = \frac{4\delta}{3}\frac{\partial E(t+1)}{\partial I} - \frac{\alpha}{3}$, where $\frac{\partial E(t+1)}{\partial I} = \beta\frac{\partial E^{\bar{r}}(t+1)}{\partial I} + (1-\beta)\frac{\partial E^r(t+1)}{\partial I}$. Thus $\frac{\partial a^*}{dI}$ is non-negative during the whole game if and only if $\frac{\partial E(t+1)}{\partial I} > \frac{\alpha}{4\delta}$.

For $t = N-1$, the player's payoff $\frac{\partial E(N)}{\partial I} = \alpha\beta$, hence the condition of non-negativity is $\beta > \frac{1}{4\delta}$. Now consider $t = N-2$. Substituting the optimal strategy $a^*(N-2, \bar{r})$ into expression (62) and writing down the condition $\frac{\partial E(N-1)}{\partial I} > \frac{\alpha}{4\delta}$ yields:

$$[\delta E(t+1) - (1-\alpha+\alpha I)]^2 \frac{16}{9(1-\alpha)^2} (1-\delta\beta)\,\delta + \left(\delta^2\beta - \frac{1}{4}\right) > 0. \tag{63}$$

Given $\beta > \frac{1}{4\delta^2}$ we can easily prove that $\frac{\partial E(N-1)}{\partial I} > \frac{\alpha}{4\delta}$. Therefore $a^*(N-2,\bar{r})$ is a non-decreasing function of universal rank I. By repeating the procedure recurrently for all t we prove the result of the proposition.

Proof of Proposition 8

Consider $\alpha \geq 1/2$.[11] The non-negativity of $\partial a^*(t,r)/\partial I$ is straightforward for $E(t+1) < \frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta}$. The optimal strategy $a^*(t,r)$ is a non-decreasing function for $E(t+1) \geq \frac{5\alpha+1}{6\delta}$ iff $\partial E(t+1)/\partial I$ is non-negative, which is proved by Proposition 6.

Now consider $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E(t+1) < \frac{5\alpha+1}{6\delta}$:

$$\frac{\partial a^*(t,r)}{\partial I} = \frac{2\delta}{3}\frac{\partial E(t+1)}{\partial I} \times \tag{64}$$

$$\left(1 - \frac{4\delta E(t+1) - 2(1-\alpha)}{\sqrt{16\delta^2\left(E(t+1)\right)^2 - 16\,(1-\alpha)\,\delta E(t+1) + 5\alpha^2 + 6\alpha + 5}}\right)$$

We can easily obtain $\frac{\partial a^*(t,r)}{\partial I} \gtreqless 0$ when $\frac{\partial E(t+1)}{\partial I} \gtreqless 0$ since the right hand side is always positive when $\frac{5-19\alpha+11\alpha^2}{6(1-3\alpha)\delta} \leq E(t+1) < \frac{5\alpha+1}{6\delta}$. Therefore, given Proposition 6 we prove Proposition 8.

Proof of Proposition 9

First, substitute the optimal strategy a^* from Proposition 3 into the Bellman equation (12) and obtain the following recurrence:

$$E(t) = E(t+1) - \frac{16}{27(1-\alpha)^2}\left(E(t+1) - (\alpha I + 1 - \alpha)\right)^3. \tag{65}$$

[11] The case with $\alpha < 1/2$ yields the same results and it is omitted.

with boundary condition for $t = N$:

$$E(N) = \frac{1-\alpha}{2} + \alpha I. \tag{66}$$

If we approximate the difference $E(t+1) - E(t)$ by the derivative dE/dt, we get:

$$\frac{dE}{dt} = \frac{16}{27} \left(E - (\alpha I + 1 - \alpha)\right)^3 \tag{67}$$

with boundary condition (66). Equation (67) has an exact solution:

$$E(t) = \alpha I + 1 - \alpha - (1-\alpha) \left(\frac{32(N-t)}{27} + 4 \right)^{-\frac{1}{2}}. \tag{68}$$

If we substitute $t = 1$ to the last expression, we prove the proposition.

Proof of Proposition 11

The Bellman equation in the vectorial form for the game with infinite horizon is:

$$\mathbb{E}(a(\bar{r}), a(r)) = \mathbb{A}_1 + \delta \mathbb{A}_2(\beta, 1-\beta) \mathbb{E}(a(\bar{r}), a(r)), \tag{69}$$

By transforming equation (69) to obtain the explicit form of $\mathbb{E}(a(\bar{r}), a(r))$, we get

$$\left(\mathbb{I} - \delta \mathbb{A}_2(\beta, 1-\beta)\right) \mathbb{E}(a(\bar{r}), a(r)) = \mathbb{A}_1. \tag{70}$$

If the determinant of matrix $(\mathbb{I} - \delta \mathbb{A}_2(\beta, 1-\beta))$ does not equal to zero, then the solution of the last equation is vector $\mathbb{E}(a(\bar{r}), a(r))$ that is determined by the following expression:

$$\mathbb{E}(a(\bar{r}), a(r)) = \left(\mathbb{I} - \delta \mathbb{A}_2(\beta, 1-\beta)\right)^{-1} \mathbb{A}_1. \tag{71}$$

The determinant of matrix $(\mathbb{I} - \delta \mathbb{A}_2(\beta, 1-\beta))$ equals to zero when $\delta = 1$ and, at the same time, both elements of matrix $\mathbb{A}_2$ equal to zero. The elements $\mathbb{A}_{21}$, $\mathbb{A}_{22}$ equal to zero if and only if a player uses his/her highest possible strategy. Finally, we compute the matrix $(\mathbb{I} - \delta \mathbb{A}_2(\beta, 1-\beta))^{-1}$ and obtain:

$$\mathbb{E}(a(\bar{r}), a(r)) = \frac{1}{1 - \delta(\beta \mathbb{A}_{21} + (1-\beta)\mathbb{A}_{22})} \begin{pmatrix} 1 - \delta(1-\beta)\mathbb{A}_{22} & \delta(1-\beta)\mathbb{A}_{21} \\ \delta\beta\mathbb{A}_{22} & 1 - \delta\beta\mathbb{A}_{21} \end{pmatrix} \mathbb{A}_1. \tag{72}$$