Patching in Algebra

by David Harbater

Abstract

Patching methods have been used in algebra to perform constructions in Galois theory and related areas. More recently, this approach has also been used to obtain local-global principles over function fields of arithmetic curves. These in turn have applications to structures such as quadratic forms and central simple algebras. This article surveys these developments as well as giving background and examples.

Contents

1   Introduction and motivation

This article corresponds to the lecture series given by the author at the Luxembourg Winter School, on the topic of patching in algebra.

Those lectures were complementary to the ones delivered by Moshe Jarden. His lectures focused on patching in Galois theory, whereas these focused more on applications of patching in other aspects of algebra, including variants of Galois theory.

This article is also complementary to the author's long manuscript [Hrb03] on patching and Galois theory. That manuscript focused on the approaches of formal and rigid patching. In contrast, the Luxembourg lectures of Moshe Jarden focused on the approach of algebraic patching, and the lectures presented here focused on an approach called patching over fields.

The lectures presented here were informal in tone, emphasizing the ideas and intuition, providing pictures, and sketching proofs in key cases, rather than seeking generality and completeness. A similar approach is taken here. The presentation draws heavily on [HH10], [HHK09], and [HHK11a], co-authored with Julia Hartmann and with Daniel Krashen. The reader is referred to those manuscripts for more detail.

In the lectures, and in this manuscript, we describe patching for vector spaces and related algebraic objects over function fields of curves that are defined over complete discretely valued fields such as $k((t))$. We give applications to inverse Galois theory and related questions. After describing variants on the patching set-up to allow greater generality and flexibility, we consider the problem of patching torsors, which are useful in classifying many other types of algebraic objects. We then build on the results about torsors to study the notion of local-global principles. Such principles, which are analogous to classical results about global fields, can be regarded in a sense as a complement to patching (studying given global objects by looking locally, rather than constructing global objects by using local ones). Applications are then given to quadratic forms and central simple algebras.

First, we begin with a discussion of the historical background and context of patching.

1.1   Patching in analysis

The idea for patching originated in analysis in the nineteenth century, to construct global objects from local objects; i.e. from objects that are defined on subsets and agree on overlaps. A typical situation is represented by this picture, where objects over S_1 and S_2 that agree over $S_0 = S_1 \cap S_2$ are patched to yield an object over $S = S_1 \cup S_2$:

$$S = S_1 \cup S_2$$
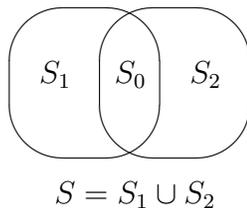
In the analytic situation, the sets S_i could be open subsets of a complex manifold. The objects to be patched might be vector bundles, or perhaps finite extensions of the fields of meromorphic functions on the sets S_i.

An early application of this approach concerned Hilbert's 21st problem, on the existence of a linear linear differential equation having a prescribed monodromy group. See Section 3.2 below for a further discussion of this, including more recent algebraic generalizations.

1.2   From analysis to algebra

Serre's paper GAGA [Ser56] made it possible to pass between complex algebraic geometry (in which one uses classical metric open sets) and algebraic geometry with respect to the Zariski topology. In particular, it showed that the cohomology of a coherent sheaf on a complex projective algebraic variety agrees with the cohomology of the induced sheaf in the analytic topology. As a result, one can, for example, cite results in the text by Griffiths-Harris ([GH78]) in the context of the text by Hartshorne ([Hts77]).

Serre's result can be viewed as a form of patching that passes from analysis to algebra. Namely, one can cover a complex algebraic variety V by metric open sets S_i. GAGA says, for example, that giving holomorphic differential forms ω_i on S_i for all i, which agree on the overlaps $S_i \cap S_j$, is equivalent to giving a *regular* (algebraic) differential form ω on V.

For a more detailed discussion of GAGA from the perspective of patching, see Section 2 of [Hrb03].

1.3   Patching in algebra

The main difficulty in carrying over the patching approach to a more purely algebraic situation is that non-empty Zariski open subsets are very large — in fact, in an irreducible variety, they are dense, with the complement being of lower dimension. Thus, for example, if the objects to be patched are finite extensions of the fields F_i of rational functions on the sets $S_i \subset S$, then the fields F_i are just isomorphic to the field F of rational functions on the variety S itself, and the procedure would not yield anything new. In order to be able to construct new extensions of the field F (e.g. for purposes of Galois theory), one would need smaller sets S_i to consider, whose function fields would strictly contain F.

Ironically, a way of dealing with the limitation of the Zariski topology is due to Zariski himself. His approach used completions. As a first example, consider the affine (x, t)-plane $\mathbb{A}_k^2 = \text{Spec}(k[x, t])$ over a field k. Viewing formal power series as "functions defined near the origin," we can take the ring $k[[x, t]]$ and its spectrum $\text{Spec}(k[[x, t]])$, viewing this as a small neighborhood of the origin. Here the power series ring $k[[x, t]]$ is the completion of the ring $k[x, t]$ at the maximal ideal $\mathfrak{m} = (x, y)$ corresponding to the origin.

The corresponding picture is:
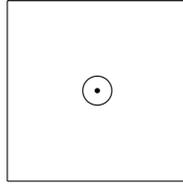
The function field of $\text{Spec}(k[[x, t]])$ is the fraction field $k((x, t))$ of $k[[x, t]]$, called the field of Laurent series in x, t over k. (Note, though, that this fraction field is not complete. Also, the elements in this field are not necessarily series in x and t; e.g. $1/(x + t)$.)

As a more sophisticated example, we can consider a "small neighborhood" of the x-axis in $\mathbb{A}_k^2$. The x-axis is defined by the ideal (t) in $k[x, t]$; and motivated by the previous example we can take the t-adic completion of $k[x, t]$. This is $\lim_{n \to \infty} k[x, t]/(t^n) = k[x][[t]]$. Its spectrum can be viewed as a "tubular neighborhood" of the x-axis in $\mathbb{A}_k^2$, made up by a union of small neighborhoods of all the closed points on the x-axis. Intuitively, this tubular neighborhood "pinches down" near the point at infinity, since that point is not on the (affine) x-axis.
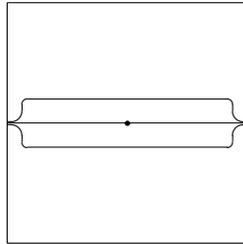
Note, for example, that the element $x - t$ defines a curve in the affine (x, t)-plane that meets this neighborhood non-trivially, intersecting the x-axis at the origin; and correspondingly, $(x - t)$ is a proper ideal in the ring $k[x][[t]]$. On the other hand, the element $1 - xt$ defines a curve in the affine plane that does not meet this tubular neighborhood, since it approaches the x-axis at $x = \infty$; and indeed, $(1 - xt)$ is the unit ideal in $k[x][[t]]$.

1.4   Formal patching

Grothendieck developed the above idea into the theory of formal schemes. Using that theory, one can do *formal patching*. In this situation, one can regard a ring like $k[x][[t]]$ as analogous to the ring of holomorphic functions on a complex metric open set; and by giving objects over such rings one can obtain a more global object.

The key theorem is a analog of Serre's GAGA in the context of formal schemes. It was referred to as GFGA in Grothendieck's paper [Gro59] of the same name, to emphasize the parallel and to explain that it related the formal and algebraic contexts, just as Serre's result related the analytic and algebraic contexts. This result later came to be known as Grothendieck's Existence Theorem. See [Gro61, Corollaire 5.1.6].

Beginning the 1980's, I used this approach in Galois theory, e.g. to realize all finite groups as Galois groups over fields such as $K(x)$ where K is the fraction field of a complete local domain that is itself not a field (e.g. $K = \mathbb{Q}_p$ for some prime p, or $K = k((t))$ for some field k). See [Hrb87]. Other results included the freeness of the absolute Galois group of $k(x)$ for k an arbitrary algebraically closed field ([Hrb95]), and the proof of Abhyankar's Conjecture on the Galois groups of étale covers of affine curves in characteristic p ([Hrb94]). See Section 1.5 below for a further discussion of such results and related work of others. Also see Section 3 of [Hrb03].

1.5   Rigid patching

Another form of patching is based on Tate's theory of rigid analytic spaces (see [Tat71]). This theory is modeled on that of complex analytic spaces, but with differences to account for the fact that the topology induced by an ideal is totally disconnected. These differences allow analytic continuation to be unique, as when working over $\mathbb{C}$, thereby leading to a "rigid" structure. (In contrast, a "floppy" structure that would result from using a more naive construction in this totally disconnected context.)

Rigid analytic spaces are phrased in terms of convergent power series in a non-archimedean metric. For example, for the affine x-line over $K = k((t))$, the ring $K\{x\}$ of power series in $K[[x]]$ that are convergent on the closed t-adic unit disc turns out to be the same as the ring $k[x][[t]][t^{-1}]$, which is a localization of the ring considered above. In fact, Raynaud later reinterpreted rigid analytic spaces in terms of formal schemes, roughly giving a dictionary between the two frameworks in the case that K is a complete discretely valued field. (See [Ray74].)

In the early 1990's, Serre suggested that results in Galois theory could also be obtained via rigid patching. Using this approach, results in [Hrb87] were then reproved by Q. Liu in [Liu95]. Later, this approach was used to prove many other results related to Galois theory. In [Ray94], M. Raynaud used rigid patching (and other methods) to prove Abhyankar's Conjecture in the case of the affine line;

this was one of the ingredients in the proof of the general case in [Hrb94]. F. Pop proved a number of results on the structure of absolute Galois groups of function fields in a series of papers including [Pop94] and [Pop95], the latter of which gave a rigid proof of the result proved at the same time in [Hrb95] using formal patching. See Section 4 of [Hrb03] for more about the use of rigid patching in Galois theory.

1.6 Algebraic patching

In the mid to late 1990's, another framework for patching was established, in work of D. Haran, M. Jarden and H. Völklein. This framework, called *algebraic patching*, avoids the machinery of Grothendieck and Tate, and is designed to isolate what is needed for applications to Galois theory. As the name indicates, the approach avoids mention of geometric objects, preferring to focus on the rings and fields involved. Like rigid patching, it relies on convergent power series with respect to a non-archimedean metric, but without mention of rigid analytic spaces. Key ingredients include versions of Cartan's Lemma and the Weierstrass Preparation Theorem.

Some of the results that had been shown by formal or rigid methods were reproven in this framework, as well as additional results concerning the Galois theory of function fields of curves. See [HV96] and [HJ98] for early papers in this direction. See also the lecture notes of Moshe Jarden at the Luxembourg Winter School, and also the volumes [Jar11] and [Völ96].

1.7 Patching over fields

Beginning in 2006, a framework of *patching over fields* was developed by Julia Hartmann and myself in [HH10], for the purpose of making patching more applicable to other algebraic contexts, and also to avoid heavy machinery. It uses and emphasizes the fraction fields of the rings that appear in formal and rigid approaches. In this way, the spaces that are used in those earlier approaches, and sheaves of functions on those spaces, are replaced by fields and vector spaces over them. As a consequence, this approach is more elementary in nature, but more general in application, since it permits uses in situations in which the objects are inherently defined over fields rather than rings or spaces.

Like algebraic patching, it relies on a form of Cartan's Lemma, and it uses a form of Weierstrass Preparation. But like formal patching, it relies on adic completions of rings (e.g. formal power series) and their fraction fields, rather than on convergent power series.

This approach is being used in work on quadratic forms, central simple algebras, and analogs of Galois theory (Galois theory of differential equations and Galois theory of division algebras). See in particular [HHK09] and [HHK11a], written jointly with Julia Hartmann and Daniel Krashen.

The remainder of this lecture series discusses this framework and some of its uses.

2 Patching algebraic structures

We begin by describing patching over fields in a basic situation: where the objects being patched are finite dimensional vector spaces. Afterwards we turn to other types of algebraic structures.

2.1 Patching vector spaces over fields

Motivated by the ideas in Section 1, we consider four fields F, F_1, F_2, F_0 that fit into a commutative diagram

(2.1)

$$
\begin{array}{ccc}
 & F_0 & \\
\nearrow & & \nwarrow \\
F_1 & & F_2 \\
\searrow & & \swarrow \\
 & F &
\end{array}
$$

where the lines represent inclusions. We assume here that F is the intersection of F_1 and F_2 inside the common overfield F_0.

As an example, we may consider the situation discussed before: a space S is covered by two subsets S_1, S_2; and S_0 is the intersection of $S_1 \cap S_2$. In this context we take F to be the field of (rational or meromorphic) functions on S and we let F_i be the field of functions on S_i, for $i = 0, 1, 2$. Given some structure over S_1 and over S_2, together with an isomorphism between their restrictions to S_0, we wish to "patch" them together in order to obtain a structure over the full space S that induces the given structures compatibly. From the algebraic point of view, given structures over F_1, F_2 with an isomorphism between the structures they induce over F_0, we want to show that there is a unique structure over F that induces them compatibly.

To make this more precise, we need to interpret the notion of "structure". As a first case, take the structures to be finite dimensional vector spaces. Given a field E, let $\mathrm{Vect}(E)$ be the category of finite dimensional vector spaces over E. If $E \subseteq E'$, there is a functor $\mathrm{Vect}(E) \to \mathrm{Vect}(E')$ given by $V \mapsto V \otimes_E E'$. For a diagram of four fields as in (2.1) above, there is a base change functor

(2.2)
$$\Phi : \mathrm{Vect}(F) \to \mathrm{Vect}(F_1) \times_{\mathrm{Vect}(F_0)} \mathrm{Vect}(F_2),$$

where the objects in the right hand category consist of triples (V_1, V_2, μ), with V_1 in $\mathrm{Vect}(F_i)$ and with μ an isomorphism of F_0-vector spaces $V_1 \otimes_{F_1} F_0 \to V_2 \otimes_{F_1} F_0$. (This is roughly a fiber product of categories. Actually, because of the choice of μ, it is not exactly a fiber product, but rather a "2-fiber product".) Objects in this category will be called *patching problems*. The desired patching assertion is that

the functor Φ is an equivalence of categories. In particular, this says that every patching problem has a *solution*; i.e. there is an object in $\mathrm{Vect}(F)$ that induces it, up to isomorphism. While we refer here to finite dimensional vector spaces, we will also later consider the corresponding situation for other interpretations of "structure"; e.g. for finite dimensional associative algebras, or Galois extensions with a given group, etc.

To be able to obtain an equivalence of categories as above, we need to choose the fields F_i appropriately. If we work classically (e.g. analytically, say with algebraic varieties over $\mathbb{R}$ or $\mathbb{C}$), we can take the sets S_i to be metric open sets, and take F_i to be the field of meromorphic or rational functions on S_i. But if we work with varieties over more general fields, there is the problem that the Zariski topology is too coarse: all open sets in an irreducible variety have the *same* function field. Our approach will be to consider varieties over fields K which (like $\mathbb{R}$ and $\mathbb{C}$) are complete with respect to a metric. Namely, we can take a base field K that is a complete discretely valued field such as $k((t))$ or $\mathbb{Q}_p$; i.e. the fraction field of a complete discrete valuation ring T (in these two examples, $k[[t]]$ and $\mathbb{Z}_p$).

2.2   Basic example: the line

As an example, let $T = k[[t]]$ and $K = k((t))$, and consider the T-curve $\widehat{X} = \mathbb{P}^1_T$, the projective x-line over T. There is then a structure morphism $\widehat{X} \to \mathrm{Spec}(T)$, where $\mathrm{Spec}(T)$ consists of two points: a closed point corresponding to the maximal ideal (t) of T, and the generic point corresponding to the zero ideal of T. The closed point is a copy of $\mathrm{Spec}(k)$, and the generic (open) point of $\mathrm{Spec}(T)$ is a copy of $\mathrm{Spec}(K)$. We may view $\mathrm{Spec}(T)$ as a "small neighborhood" of the origin in the t-line over k. The corresponding picture is:

$$\widehat{X} = \mathbb{P}^1_T \qquad\qquad \mathrm{Spec}(T)$$

Here the fiber X of $\widehat{X}$ over the closed point of $\mathrm{Spec}(T)$ is a copy of $\mathbb{P}^1_k$, and the fiber over the generic point of $\mathrm{Spec}(T)$ is a copy of $\mathbb{P}^1_K$. The affine T-line $\mathbb{A}^1_T = \mathrm{Spec}(k[[t]][x])$ is a Zariski open subset of $\widehat{X}$, and its fibers over the closed and generic points of $\mathrm{Spec}(T)$ are the affine lines over k and K, respectively.

$$\mathbb{A}^1_T \qquad\qquad \mathrm{Spec}(T)$$

The field of rational functions F on $\widehat{X}$ is the same as the field of rational functions on this open subset, viz. $k((t))(x)$ (this being the fraction field of $k[[t]][x]$).

We can instead consider the affine x-line over the base ring $T = \mathbb{Z}_p$ of p-adic integers, with uniformizer $t := p$. In this situation the residue field k is $\mathbb{F}_p$ and the fraction field K is $\mathbb{Q}_p$. The schematic pictures look the same as above, and the above discussion carries over to this case, with $k[[t]][x]$ replaced by $\mathbb{Z}_p[x]$ and $k((t))(x)$ replaced by $\mathbb{Q}_p(x)$.

2.3 Subsets and overfields

In general, consider a proper smooth curve $\widehat{X}$ over $\mathrm{Spec}(T)$, with T a complete discrete valuation ring having uniformizer t, and let F be the field of rational functions on $\widehat{X}$. Let $X \subset \widehat{X}$ be the closed fiber. For any subset $U \subset X$ that does not contain all the closed points of X, let R_U be the ring of rational functions on $\widehat{X}$ that restrict to rational functions on X and which are regular at the points of U. Let $\widehat{R}_U$ be the t-adic completion of R_U. Note that if U contains at least one closed point of X, then R_U and $\widehat{R}_U$ are two-dimensional domains; whereas in the case $U = \varnothing$, the rings $R_\varnothing$ and $\widehat{R}_\varnothing$ are discrete valuation rings. With U as above, the fraction field F_U of $\widehat{R}_U$ is an overfield of F. We also write $F_X = F$.

In the above example of the projective T-line, we can take the open set $U_1 = \mathbb{A}_k^1 = \mathrm{Spec}(k[x])$, the affine x-line over the residue field k of T. This is the complement of the point $x = \infty$ in $X = \mathbb{P}_k^1$. In the case that $T = k[[t]]$, the ring R_{U_1} is the subring of $F = k((t))(x)$ consisting of rational functions f/g where $f, g \in k[[t]][x]$ such that g does not vanish anywhere on U_1. That is, the reduction of g modulo t is a unit in $k[x]$, or equivalently a non-zero constant in k. The t-adic completion $\widehat{R}_{U_1}$ is given by $k[x][[t]]$. Observe that this ring strictly contains $k[[t]][x]$, the ring of regular functions on $\mathbb{A}_T^1$. For example, $\sum_0^\infty x^i t^i$ is contained in the former, but not the latter. This inclusion of rings corresponds to a morphism of their spectra in the other direction, viz. $\mathrm{Spec}(\widehat{R}_{U_1}) \to \mathbb{A}_T^1$. Note that $\sum_0^\infty x^i t^i = (1 - xt)^{-1} \in k[x][[t]]$, and so $(1 - xt)$ is the unit ideal in $k[x][[t]]$, and does not correspond to a point of $\mathrm{Spec}(\widehat{R}_{U_1})$. But $(1 - xt)$ does generate a prime ideal in $k[[t]][x]$, and does define a non-empty closed set in $\mathrm{Spec}(k[[t]][x]) = \mathbb{A}_T^1$. (Cf. the discussion in Section 1.3.)
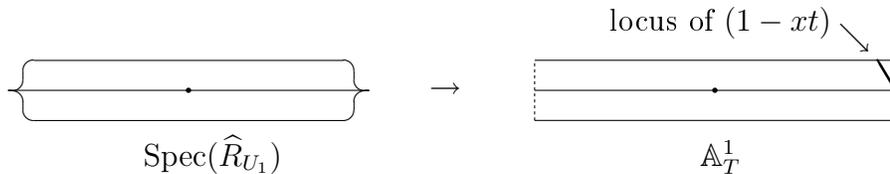
Similarly, the fraction field $F = k((t))(x)$ of $k[[t]][x]$ is strictly contained in the fraction field F_{U_1} of $\widehat{R}_{U_1} = k[x][[t]]$, which is in fact transcendental over F. Intuitively, we can regard $\mathrm{Spec}(\widehat{R}_{U_1})$ as an analytic open subset of the algebraic

(Zariski) open subset $\mathbb{A}_T^1 \subset \widehat{X}$; and regard $k[x][[t]]$ and its fraction field as consisting of the holomorphic and meromorphic functions on this set.

In the above example, we can also consider the open set $U_2 \subset X$ given by the complement of the point $x = 0$ on $X = \mathbb{P}_k^1$. This is another copy of the affine line over k, with ring of functions $k[x^{-1}]$. We then obtain $\widehat{R}_{U_2} = k[x^{-1}][[t]]$, with fraction field F_{U_2}. Let $U_0 = U_1 \cap U_2$; this is the complement of the two points $x = 0, \infty$ in X. We then have $\widehat{R}_0 = k[x, x^{-1}][[t]]$, with fraction field F_{U_0}.

$$\mathrm{Spec}(\widehat{R}_{U_2}) \qquad\qquad\qquad \mathrm{Spec}(\widehat{R}_{U_0})$$

If we take U to be the empty subset of X (or equivalently, the subset consisting just of the generic point of X), then $\widehat{R}_U = \widehat{R}_\varnothing = k(x)[[t]]$, which is a complete discrete valuation ring (unlike the rings above, which were two dimensional). Its quotient field is $F_\varnothing = k(x)((t))$.

As in Section 2.2, we can instead consider the analogous case of the x-line over $T = \mathbb{Z}_p$. The schematic pictures are again the same as in the power series case, though the rings are a bit more awkward to write down explicitly. With U_1 the affine line as above, the ring $\widehat{R}_{U_1}$ is the p-adic completion of $\mathbb{Z}_p[x]$, i.e. $\varprojlim \mathbb{Z}_p[x]/(p^n)$. For example $1 - px$ is a unit in this ring, with inverse $\sum_0^\infty p^i x^i$. On the other hand, the elements x, $x - p$, and p each define proper principal ideals, corresponding to curves in $\mathrm{Spec}(\widehat{R}_{U_1})$. Similarly, if we let U_2 be the complement of ∞ in $\mathbb{P}_T^1$ and let U_0 be the complement of the two points $0, \infty$ in $\mathbb{P}_T^1$, then $\widehat{R}_{U_2}$ and $\widehat{R}_{U_0}$ are the p-adic completions of $\mathbb{Z}_p[x^{-1}]$ and $\mathbb{Z}_p[x, x^{-1}]$. Note also that in this situation, $\widehat{R}_\varnothing$ is the p-adic completion of the discrete valuation ring $\mathbb{Z}_p[x]_{(p)}$. Its fraction field $F_\varnothing$ is the same as the completion of the field $\mathbb{Q}_p(x)$ with respect to the Gauss valuation; i.e. with respect to the metric induced on this field by the ideal $(p) \subset \mathbb{Z}_p[x]_{(p)}$.

2.4   Solutions to patching problems

As the above pictures suggest, we can regard $\widehat{X}$ as covered by the "analytic open sets" $S_1 := \mathrm{Spec}(\widehat{R}_{U_1})$ and $S_2 := \mathrm{Spec}(\widehat{R}_{U_2})$, with $S_0 := \mathrm{Spec}(\widehat{R}_{U_0})$ as the intersection of these sets. In fact, this can be made precise: the natural morphisms $S_0 \to S_i \to \widehat{X}$ $(i = 1, 2)$ define injections on the underlying sets of points, with the images of S_1 and S_2 covering $\widehat{X}$, and with the image of S_0 in $\widehat{X}$ being the intersection of those two images. Moreover, as we discuss below in Section 5, the fields F and $F_i := F_{U_i}$ $(i = 0, 1, 2)$ form a diagram of fields as in (2.1); and the corresponding functor Φ as in (2.2) turns out to be an equivalence of categories.

More generally, let $\widehat{X}$ be any smooth projective T-curve with function field F and closed fiber X, and let U_1, U_2 be subsets of X with $U_1 \cup U_2 = X$. Suppose that neither U_i contains all the closed points of X. Write $U_0 = U_1 \cap U_2$ and $F_i := F_{U_i}$. Then the base change functor $\Phi : \text{Vect}(F) \to \text{Vect}(F_1) \times_{\text{Vect}(F_0)} \text{Vect}(F_2)$ is an equivalence of categories. Thus given finite dimensional vector spaces V_i over F_i ($i = 1, 2$) and an F_0-isomorphism between the F_0-vector spaces $V_i \otimes_{F_i} F_0$ that they induce, there is a unique finite dimensional F-vector space V inducing them *compatibly*. Moreover, if we identify V_i with its isomorphic image in $V_i \otimes_{F_i} F_0$, and if we identify $V_1 \otimes_{F_1} F_0$ with $V_0 := V_2 \otimes_{F_2} F_0$ via the isomorphism μ, then we get that V is equal to $V_1 \cap V_2$ inside V_0. In particular, if we let $n = \dim_{F_1}(V_1)$ (which is equal to $\dim_{F_2}(V_2)$), then $\dim_F(V_1 \cap V_2)$ is necessarily equal to n. See Theorem 5.3 below.

2.5 Patching for other objects

While the above specifically concerns finite dimensional vector spaces, many algebraic objects over a field consist of a finite dimensional vector space together with additional structure that is given by maps such that certain diagrams commute.

For example, consider finite dimensional associative algebras A over F. To give A is to give a finite dimensional F-vector space A together with an F-vector space homomorphism $A \otimes_F A \to A$ such that a certain diagram commutes (corresponding to the associative law). The above equivalence of categories for vector spaces (which preserves tensor products), together with the assertion that $V = V_1 \cap V_2$, yields the corresponding equivalence of categories for finite dimensional associative algebras, again with inverse given by intersection.

As explained in [HH10, Section 7], some other examples of algebraic objects for which equivalences of categories follow in this manner are these: finite dimensional associative F-algebras with identity; finite dimensional separable F-algebras; central simple F-algebras (i.e. the center is F and there are no non-trivial two-sided ideals); differential F-modules (i.e. finite dimensional F-vector spaces together with a derivation over F); separable F-algebras (i.e. products of finitely many finite separable field extensions of F); and G-Galois (commutative) F-algebras (for some finite group G).

An object of this last sort is by definition a separable F-algebra A of dimension equal to $|G|$ whose fixed field under G is F. These are of the form $\prod E$, a product of copies of a Galois field extension E/F of Galois group $H \subseteq G$, indexed by the cosets of H in G. If $H = G$, this is the same as a G-Galois field extension. At the other extreme, if $H = 1$, then A is a product of copies of F indexed by and permuted by G; this is called a *trivial* G-Galois F-algebra. If E/F is any G-Galois F-algebra, then $E_{\widetilde{F}} := E \otimes_F \widetilde{F}$ is a G-Galois $\widetilde{F}$-algebra for any field extension $\widetilde{F}/F$; and if $\widetilde{F}/F$ is sufficiently large (e.g. the separable closure of F), then the G-Galois $\widetilde{F}$-algebra $E_{\widetilde{F}}$ is trivial.

3   Applications of patching

Using patching for algebraic structures as discussed above, one can obtain applications of various sorts. For example, using the category of G-Galois F-algebras, one can obtain applications of patching to Galois theory. Using the category of differential modules, one can obtain applications to differential Galois theory (the Galois theory of linear differential equations). And using central simple algebras, one can obtain an application to a division algebra analog of Galois theory. We discuss these in turn below.

3.1   Applications to Galois theory

Suppose that we have a finite group G that we wish to realize as a Galois group over F, the function field of a curve over a complete discretely valued field. We can proceed inductively, generating G by two strictly smaller subgroups $H_1, H_2 \subset G$. Suppose that each H_i is the Galois group of a Galois field extension E_i/F_i, where the fields F_i are overfields of F as in (2.1). Suppose in addition that such extensions can be found for which $E_i \otimes_{F_i} F_0$ is a trivial H_i-Galois F_0-algebra. Taking a product of finitely many copies of E_i, indexed by the cosets of H_i in G, we obtain G-Galois F_i-algebras A_i for $i = 1, 2$. But note that A_i is *not* a field, because the index $(G : H_i)$ is greater than one. Nevertheless, we can use these algebras to obtain the desired field extension over F, using the composition μ of isomorphisms $A_1 \otimes_{F_1} F_0 \xrightarrow{\sim} \prod_G F_0 \xrightarrow{\sim} A_2 \otimes_{F_2} F_0$ arising from the triviality of A_i over F_0. Namely, the algebras A_i together with this isomorphism define a patching problem for G-Galois algebras, i.e. an element of the category $GAlg(F_1) \times_{GAlg(F_0)} GAlg(F_2)$, where $GAlg$ denotes G-Galois algebras. By the equivalence of categories Φ, this is induced by an object A in $GAlg(F)$, i.e. by a G-Galois F-algebra. Using the fact that H_1, H_2 generate G, together with the choice of patching isomorphism μ as above, one can show ([HH10, Section 7]) that A is actually a G-Galois *field* extension of F!

   This strategy explains how to realize individual finite groups as Galois groups over the field F, provided that the subgroups can themselves be realized. Since every finite group is generated by cyclic subgroups (in fact, even by cyclic subgroups of prime power order), it is sufficient to realize those, subject to the condition of being trivial over F_0. This is easy to do by Kummer theory if F has characteristic zero and contains all roots of unity. More generally this strategy works provided that G contains primitive n-th roots of unity ζ_n, where n ranges over the orders of the cyclic generating subgroups. Without this condition, if n is not divisible by the characteristic of F, one can construct a suitable n-cyclic Galois extension of $F_i(\zeta_n)$ by Kummer theory such that it is induced by some n-cyclic Galois extension of F_i. Finally, if $\text{char}(F) = p$, then Artin-Schreier theory and Witt vectors can be used to construct appropriate cyclic extensions of p-power order. As a result, it

follows that every finite group is a Galois group over F. See [Hrb87], where the details of the construction of these cyclic building blocks is given, though in the context of formal patching.

One can go further, by studying the absolute Galois group of F, viz. the Galois group $\text{Gal}(F) := \text{Gal}(F^{\text{sep}}/F)$ over F of the separable closure F^{sep} of F. This profinite group is the inverse limit of the Galois groups of the finite Galois extensions of F. So to understand the group $\text{Gal}(F)$, it suffices to find all the finite Galois groups over F and how they fit together in an inverse system (corresponding to the direct system of finite Galois extensions of F). For this, one wants to know when one can solve *embedding problems.* From the perspective of fields, the question is this: Given a finite group G and a quotient group H, and given an H-Galois field extension E/F, can E be embedded in a G-Galois field extension of F? Reinterpreting this in terms of groups, it asks: Given a surjection $\pi : G \to H$ of finite groups, and a surjection $f : \text{Gal}(F) \to H$, is there a surjection $\widetilde{f} : \text{Gal}(F) \to G$ such that $\pi\widetilde{f} = f$? If the answer is always yes, and if $\text{Gal}(F)$ is countably generated as a profinite group, then $\text{Gal}(F)$ must be a free profinite group, by a theorem of Iwasawa ([Iwa53, p.567]). A generalization of this theorem, due to Melnikov and Chatzidakis, handles the uncountable case [Jar95, Lemma 2.1].

Extending the patching strategy to this context has made it possible to show that many embedding problems can be solved; and using those results, it has been shown that the absolute Galois group of $k(x)$ is free if k is an algebraically closed field of arbitrary characteristic. (See [Hrb95], [Pop95], [HJ00], which respectively describe this proof in terms of formal, rigid and algebraic patching. This theorem had previously been shown just in characteristic zero, by relating k to $\mathbb{C}$; see [Dou64].) Note that $k(x)$ is *not* a field of the form F considered above, though $k((t))(x)$ is of that form. But it is possible to use results about embedding problems over $k((t))(x)$ to obtain such results over the field $k(x)$, by passing to a subfield of $k((t))(x)$ of finite type and then using that a variety over an algebraically closed field k has k-points. For more about this direction, see Section 5 of [Hrb03] and also the Luxembourg notes of Moshe Jarden.

3.2 Applications to differential algebra

Here we consider patching in the context of a *differential field,* i.e. a field F together with a derivation ∂. We assume that $\text{char}(F) = 0$, to avoid the unpleasant situation in which $\partial(x^p) = 0$. The *field of constants* in F is the subfield C of elements $c \in F$ such that $\partial(c) = 0$. (For example, in the differential field $F = K(x)$ with derivation $\partial/\partial x$, the constant field is K, using $\text{char}(F) = 0$.) A *differential module M* over F is a finite dimensional F-vector space together with an F-derivation ∂_M on M. That is, $\partial : M \to M$ is an additive map such that $\partial_M(fm) = \partial(f)m + f\partial_M(m)$ for $f \in F$ and $m \in M$.

We consider in particular a field F as before, i.e. the field of rational functions on a smooth projective curve $\widehat{X}$ over a complete discretely valued field T. Let U_1, U_2 be subsets of the closed fiber X, such that $X = U_1 \cup U_2$ and neither U_i contains all the closed points of X, and let $U_0 = U_1 \cap U_2$. Write $F_i = F_{U_i}$ for each i. Once we give F the structure of a differential field whose constant field is the fraction field K of T (e.g. taking $\partial = \partial/\partial x$ if $\widehat{X} = \mathbb{P}^1_T$), the overfields F_i obtain compatible structures as differential fields.

Write $\mathrm{DiffMod}(E)$ for the category of differential modules over a field E. As noted in Section 2.5, in the above context the functor

$$\Phi : \mathrm{DiffMod}(F) \to \mathrm{DiffMod}(F_1) \times_{\mathrm{DiffMod}(F_0)} \mathrm{DiffMod}(F_2),$$

is an equivalence of categories, and in particular every patching problem for differential modules has a solution.

This can be used to prove the analog of the inverse Galois problem in the context of differential algebra. Whereas ordinary Galois theory originated in the study of polynomial equations, differential Galois theory over a differential field F originated in the study of linear differential equations, which give rise to differential modules. Given such an equation (or the associated differential module), there is an analog of the splitting field of a polynomial, called the associated *Picard-Vessiot extension*. This is a differential field extension E/F that is generated by the solutions to the differential equation, and such that the field of constants of E is the same as the field of constants of F. It is known that a Picard-Vessiot extension E exists and is unique if the constant field C is algebraically closed (unlike for the fields F we have been considering above); for more general fields there are additional subtleties. The associated *differential Galois group* is the automorphism group of E/F as an extension of differential fields. This is a *linear algebraic group*, i.e. a (smooth) Zariski closed subgroup of $\mathrm{GL}_{n,C}$. See [MP03, Section 2] for more details.

The inverse differential Galois problem asks whether every linear algebraic group over C is a differential Galois group over F. The case that $C = \mathbb{C}$ was proven in [TT79], building on classical work of Plemelj on Hilbert's 21st problem concerning the realization of groups as monodromy groups of linear differential equations. In [Hrt05], it was shown that if $F = C(x)$ with C any algebraically closed of characteristic zero, then every linear algebraic group over C is the differential Galois group of some differential module. Using patching of differential modules, it can be shown that the same is true if F is instead a field as considered in the general discussion above, viz. the function field of a curve over a complete discretely valued field K of characteristic zero. (See [Hrt07].) Using this, it is possible to obtain another proof of the result for function fields over an algebraically closed field of characteristic zero.

In connection with the above historical comments, it is worth mentioning that the classical work of Plemelj, Birkhoff and others in this area relied on analytic

patching methods (e.g. see [Bir17]). A key idea was the use of matrix factorization. While the term "Riemann-Hilbert problem" was initially used to refer to versions of Hilbert's 21st problem, it has also come to mean problems related to matrix factorization in an analytic context. In other contexts involving patching, matrix factorization also plays a key role for related reasons, often under the name "Cartan's Lemma". See [Hrb03, Section 2.2] for a discussion of its use in Serre's GAGA; [Hrb84, Section 2] for its use in formal patching; the Luxembourg lectures of Moshe Jarden for its use in algebraic patching; and see Section 4 below for its use in patching over fields.

3.3 Application to Galois theory of division algebras

Given a field F, a (central) *division algebra* A over F is a finite dimensional associative F-algebra with identity such that A is a division ring and the center of A is equal to F. Two examples are the Hamilton quaternion algebra $\mathbb{H}$ over the field $\mathbb{R}$, and the $n \times n$ matrix algebra $\mathrm{Mat}_n(F)$ over any field F. The F-dimension of A is necessarily a square d^2, and d is called the *degree* of A as an F-algebra. Every subfield E of A that contains F satisfies $[E : F] \leq d$, and moreover there exists such a subfield E with $[E : F] = d$. Such a field E is called *maximal*. If such a maximal subfield $E \subseteq A$ is Galois over F, say with Galois group G, then the algebra structure of A can be described rather explicitly, and in these terms A is called a *crossed-product F-algebra* for G. (For example, see [Pie82].)

In [Sch68], Schacher defined a finite group G to be *admissible* over a field F if there is an F-division algebra A that contains a maximal subfield E that is Galois over F with group G. Equivalently, the condition is that there is a crossed-product F-algebra A for G. One can then ask the following analog of the inverse Galois problem: Given a field F, which finite groups are admissible over F?

Unlike the case of the usual inverse Galois problem, it is known that *not* every finite group is admissible over $\mathbb{Q}$. In [Sch68], Schacher showed that a necessary condition for a group G to be admissible over $\mathbb{Q}$ is that every Sylow subgroup of G is metacyclic (i.e. the extension of a cyclic group by a cyclic group). He conjectured that the converse should be true; but this remains open, although it is known in the case that the group is solvable ([Son83]). Work on this problem has also been done more generally for global fields, i.e. fields F that are finite extensions of either $\mathbb{Q}$ or $\mathbb{F}_p(x)$ for some prime p (e.g. see [Sch68, Corollary 10.3] in the function field case).

Using patching, the admissibility problem can be studied in the case that the field F is of the type that we have been considering in this manuscript, i.e. a finitely generated field of transcendence degree one over a complete discretely valued field K. In particular, in the case that $K = \mathbb{C}((t))$, it was shown in [HHK11] via patching over fields that a finite group G is admissible over F if and only if every Sylow subgroup of G is abelian metacyclic (or equivalently, a direct product of

two cyclic groups). Here, the abelian condition is related to the fact that this choice of K contains all the roots of unity.

4   Criterion for patching

We return to a more general situation, as at the beginning of Section 2, with four fields forming a diagram of inclusions:

$$
\begin{array}{ccc}
 & F_0 & \\
\diagup & & \diagdown \\
F_1 & & F_2 \\
\diagdown & & \diagup \\
 & F & \\
\end{array}
$$

We will give a criterion for patching to hold for vector spaces over these fields, for use in obtaining the type of results discussed above. That is, we give a necessary and sufficient condition for the functor

$$
\Phi : \mathrm{Vect}(F) \to \mathrm{Vect}(F_1) \times_{\mathrm{Vect}(F_0)} \mathrm{Vect}(F_2),
$$

to be an equivalence of categories.

To state this criterion, we consider two conditions:

Condition 4.1. *Factorization property ("Cartan's Lemma"): For every $n \geq 1$, every matrix $A_0 \in \mathrm{GL}_n(F_0)$ can be factored as $A_1 A_2$ with $A_i \in \mathrm{GL}_n(F_i)$.*

Condition 4.2. *Intersection property: $F = F_1 \cap F_2 \subseteq F_0$.*

As at the beginning of Section 2, Condition 4.2 is intuitively plausible if we think of F as the field of rational functions on a space S; F_1, F_2 as the fields of functions on two subspaces S_1, S_2 with $S_1 \cup S_2 = S$; and F_0 as the field of functions on $S_0 = S_1 \cap S_2$. Note also that one may write the factorization with one of the factors A_i replaced by A_i^{-1}; sometimes this is a more natural way to write it.

Theorem 4.3. Φ *is an equivalence of categories if and only if Conditions 4.1 and 4.2 hold.*

Proof. For the reverse direction, the key assertion to show is that Conditions 4.1 and 4.2 imply that every patching problem has a solution. To do this, suppose that we are given a vector space patching problem $\mathcal{V}$ for fields satisfying these conditions. This patching problem consists of finite dimensional F_i-vector spaces V_i for $i = 1, 2$, of a common dimension n, together with an isomorphism

$$
\mu : V_1 \otimes_{F_1} F_0 \to V_2 \otimes_{F_1} F_0 =: V_0
$$

of F_0-vector spaces. We may then identify $V_1 \otimes_{F_1} F_0$ with V_0 via μ, and we can thus view V_1, V_2 as subsets of V_0. With respect to these inclusions, we will show

that the intersection $V := V_1 \cap V_2$, viewed as a vector space over F, is a solution to the given patching problem.

To do this, we will find a common basis B for V_1, V_2 over the fields F_1, F_2 respectively, and will show that V is the F-span of B. First, let B_i be any F_i-basis for V_i, for $i = 1, 2$. Thus B_1, B_2 are both F_0-bases of V_0, and there is a transition matrix $A_0 \in \mathrm{GL}_n(F_0)$ between them, satisfying $B_1 = A_0(B_2)$.
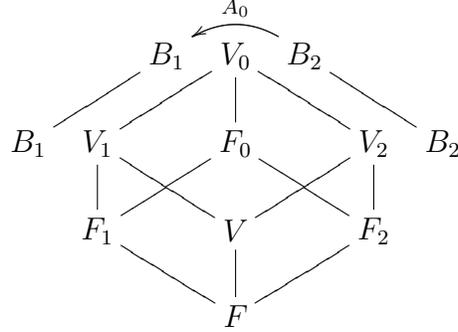
Next, by Condition 4.1, we may write $A_0 = A_1 A_2$ with $A_i \in \mathrm{GL}_n(F_i)$, $i = 1, 2$. Let $B = A_2(B_2) = A_1^{-1}(B_1)$. Then B is also an F_i-basis of V_i, for $i = 0, 1, 2$. The intersection $V = V_1 \cap V_2$ is thus the F-span of the common basis B, since $F = F_1 \cap F_2$ by Condition 4.2.

The F-vector space V therefore induces V_1 and V_2 with respect to the above inclusions of V_1, V_2 into V_0. This shows that V, together with the corresponding isomorphisms, provides a solution to the patching problem. Equivalently, the functor Φ is essentially surjective, i.e. surjective on isomorphism classes.

To complete the proof in this direction, note that Condition 4.2 yields the short exact sequence

(4.1) $$0 \to F \xrightarrow{\Delta} F_1 \times F_2 \xrightarrow{-} F_0 \to 0$$

of F-vector spaces, where Δ is the diagonal inclusion and $-$ is the subtraction map. Tensoring over F with $\mathrm{Hom}(V, W)$ yields the exact sequence

$$0 \to \mathrm{Hom}_F(V, W) \xrightarrow{\Delta} \mathrm{Hom}_{F_1}(V_1, W_1) \times \mathrm{Hom}_{F_2}(V_2, W_2) \xrightarrow{-} \mathrm{Hom}_{F_0}(V_0, W_0) \to 0,$$

using that $F_i \otimes_F \mathrm{Hom}_F(V, W) = \mathrm{Hom}_{F_i}(V_i, W_i)$. This says that that the natural map $\mathrm{Hom}(V, W) \to \mathrm{Hom}(\Phi(V), \Phi(W))$ is a bijection, i.e. that the functor Φ is fully faithful. Thus it is an equivalence of categories, completing the proof of the reverse direction.

For the forward direction, given a quadruple of fields as above, note that each $A_0 \in \mathrm{GL}_n(F_0)$ defines a patching problem with A_0 the transition matrix between given bases of the vector spaces over F_1, F_2. A basis for a solution to this patching problem yields a factorization of A_0. So Condition 4.1 holds. Now let $F' := F_1 \cap F_2$; thus $F \subseteq F'$. The reverse direction of the theorem (proven above) implies that the base change functor $\mathrm{Vect}(F) \to \mathrm{Vect}(F')$ is an equivalence of categories; and this implies that $F = F' = F_1 \cap F_2$, giving Condition 4.2. $\qquad\square$

5   Satisfying the patching criterion

We would like to use Theorem 4.3 to show that patching holds for quadruples of fields as in Section 2.4. That is, F is the function field of a smooth projective T-curve $\widehat{X}$ having closed fiber X, where T is a complete discrete valuation ring with fraction field K. We take subsets $U_1, U_2 \subset X$, neither containing all the closed points of X; and we write $U = U_1 \cup U_2$ and $U_0 = U_1 \cap U_2$. Consider the fraction fields $F_i := F_{U_i}$ of $\widehat{R}_i := \widehat{R}_{U_i}$. By Theorem 4.3, proving that patching holds for the fields $F_U \subset F_1, F_2 \subset F_0$ is equivalent to proving that these fields satisfy Conditions 4.1 and 4.2. (Recall that $F_U = F$ by definition, if $U = X$.) We will describe the proof of patching in explicit examples, for simplicity of exposition. For full proofs, see [HH10, Section 4].

   We begin with the factorization property, and illustrate it for the example of the quadruple considered in Sections 2.3 and 2.4 above. That is, we take $T = k[[t]]$ and $K = k((t))$; we let $\widehat{X}$ be the projective line $\mathbb{P}^1_T$, with function field $F = k((t))(x)$; and we take U_1, U_2, U_0 to be the complements in $\mathbb{P}^1_k$ of the sets $\{\infty\}, \{0\}, \{0, \infty\}$, respectively. Thus $\widehat{R}_1 = k[x][[t]]$, $\widehat{R}_2 = k[x^{-1}][[t]]$, and $\widehat{R}_0 = k[x, x^{-1}][[t]]$.

   We first explain why Condition 4.1 holds in the key case in which

(5.1) $$A_0 \in \mathrm{GL}_n(\widehat{R}_0) \text{ and } A_0 \equiv I \ \mathrm{mod}\ t.$$

In this situation we will obtain that in fact $A_0 = A_1 A_2$ where $A_i \in \mathrm{GL}_n(\widehat{R}_i)$ for $i = 1, 2$. We will do this by constructing A_1, A_2 modulo t^j inductively on j, thereby finding the successive coefficients of the powers of t in the entries of the matrices A_i.

   To start the inductive process, let A_1, A_2 be congruent to I modulo t. To do the inductive step, we use that every element in $k[x, x^{-1}]$ is the sum of elements in $k[x]$ and $k[x^{-1}]$. As an example, take $n = 1$, and consider the 1×1 matrix

$$A_0 = (1 + (x + 1 + x^{-1})t) \in \mathrm{GL}_1(\widehat{R}_0) = \widehat{R}_0^{\times}.$$

Modulo t, the factorization is just $1 \cdot 1$. For the factorization modulo t^2, we write $x + 1 + x^{-1}$ as the sum of $x + 1 \in k[x]$ and $x^{-1} \in k[x^{-1}]$, and obtain

$$A_0 \equiv (1 + (x + 1)t)(1 + x^{-1}t) \ \ \mathrm{mod}\ t^2.$$

The discrepancy between the left and right hand sides is $(1 + x^{-1})t^2$, and so at the next step we write

$$A_0 \equiv (1 + (x + 1)t - t^2))(1 + x^{-1}t - x^{-1}t^2) \ \ \mathrm{mod}\ t^3.$$

We continue in this way, and in the limit we get the desired matrices A_i. The same strategy handles the $n \times n$ case, again provided that A_0 is in the above key case.

This approach can be also used if we take two more general sets U_1, U_2 whose union is $\mathbb{P}^1_k$, neither of which contains all the closed points of $\mathbb{P}^1_k$. The inductive procedure as above still works in the key case (5.1), since again we can write every regular function on $U_0 := U_1 \cap U_2$ as a sum of regular functions on U_1 and U_2, because of the partial fractions decomposition. For example, if U_1 is the complement of the point $x = 1$ and U_2 is the complement of $x = -1$, and if $A_0 = (1 + \frac{1}{x^2-1}t)$, then we can write $\frac{1}{(x-1)(x+1)} = \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$, and proceed as before in order to factor A_0. (This works even if U_1, U_2 are not necessarily Zariski open; e.g. if U_1 is a closed point of $\mathbb{P}^1_k$ and U_2 is its complement.) In this way, we see that Condition 4.1 holds in the key case (5.1). The approach also works for a more general choice of complete discrete valuation ring T.

Before turning to the general case, we observe an important consequence of Condition 4.1 under (5.1), taking $n = 1$:

Theorem 5.1 (Weierstrass Preparation Theorem). *If U is an open subset of the closed fiber X, then every $f \in F_U$ can be written in the form $f = au$ with $a \in F$ and $u \in \widehat{R}_U^\times$.*

Proof. For simplicity of exposition, we explain the proof in the special case that $T = k[[t]]$, with $\widehat{X} = \mathbb{P}^1_T$ and $U = \mathbb{A}^1_k$, the affine x-line over k. Since F_U is the fraction field of $\widehat{R}_U$, we may assume that f lies in $\widehat{R}_U = k[x][[t]]$. Let $f_0 \in k[x]$ be the constant term of f, viewing f as a power series in t. Thus $f/f_0 \in k(x)((t))$, and $f/f_0 \equiv 1 \mod t$. Writing $U_1 = \{\infty\}$ and $U_2 = U$, Condition 4.1 under (5.1) with $n = 1$ asserts that $f/f_0 = f_1 f_2$ for some $f_i \in \mathrm{GL}_1(R_i) = R_i^\times$ for $i = 1, 2$. But $f_0 f_1 = f f_2^{-1}$, where the left hand side lies in $\widehat{R}_1[x] = k[x^{-1}]_{(x^{-1})}[[t]][x]$ and the right hand side lies in $\widehat{R}_2 = k[x][[t]]$. Since the left hand side has bounded degree in x, and the right hand side has no x^{-1} terms, this common element lies in $k[[t]][x] \subset F$. So we may take $a = f_0 f_1$ and $u = f_2$. $\qquad\square$

In the case that $\widehat{X} = \mathbb{P}^1_T$, this is equivalent to the classical Weierstrass Preparation Theorem (see [Bou72], Proposition VII.3.8.6).

Corollary 5.2. *Let U_1, U_2 be open subsets of the closed fiber X, let $U_0 = U_1 \cap U_2$, and write $U = U_1 \cup U_2$ and $F_i = F_{U_i}$. Then $F_U = F_1 \cap F_2 \subseteq F_0$.*

Proof. For simplicity, we take the special case of $T = k[[t]]$; $\widehat{X} = \mathbb{P}^1_T$; U_1 is the complement of the point 0; and U_2 is the complement of the point ∞. Here $U_i = \mathrm{Spec}(\widehat{R}_i)$ with $\widehat{R}_1 = k[x^{-1}][[t]]$ and $\widehat{R}_2 = k[x][[t]]$. Write $R' = \widehat{R}_1[x] \cap \widehat{R}_2$. Thus $R' = k[[t]][x]$, with fraction field equal to F.

Say $f \in F_1 \cap F_2$. By Theorem 5.1, we may write $f = f_1 u_1 = f_2 u_2$ with $f_i \in F$ and $u_i \in \widehat{R}_i^\times$. Since F is the fraction field of R', we can write $f_i = a_i/b_i$ with $a_i, b_i \in R'$. So $f = a_1 u_1 / b_1 = a_2 u_2 / b_2$. The common element $a_1 b_2 u_1 = a_2 b_1 u_2$ lies in R', since the left hand side lies in $R'\widehat{R}_1^\times \subseteq \widehat{R}_1[x]$ and the right hand side lies in $R'\widehat{R}_2 = \widehat{R}_2$. Therefore $f = a_1 b_2 u_1 / b_1 b_2$ lies in F, being the ratio of two elements of R'. $\qquad\square$

Using this, we obtain that the criterion of Section 4 is satisfied for our fields, and hence patching holds.

Theorem 5.3. *Let U_1, U_2 be subsets of the closed fiber X of $\widehat{X}$, and write $U = U_1 \cup U_2$ and $U_0 = U_1 \cap U_2$. Write $F_i = F_{U_i}$. Then the four fields $F_U \subseteq F_1, F_2 \subseteq F_0$ satisfy Conditions 4.1 and 4.2, and hence the base change functor Φ of (2.2) is an equivalence of categories. That is, patching holds for finite dimensional vector spaces over these fields.*

Proof. Condition 4.2 follows from Corollary 5.2, and the last part of the assertion follows from the first part together with Theorem 4.3. So it remains to show that Condition 4.1 holds.

For this, one first reduces to the case that U_1 and U_2 are disjoint. This is done by applying the disjoint case to an invertible matrix over $F_0 \subseteq F_\varnothing$, with respect to the fields F_1 and F_2', where $F_2' = F_{U_2'}$ with U_2' the complement of U_0 in U_2. One checks that the second factor is an invertible matrix over F_2, by applying Corollary 5.2 to get $F_0 \cap F_2' = F_2$.

In the disjoint case, we are given $A_0 \in \mathrm{GL}_n(F_\varnothing)$; after multiplying by a power of t, we may assume that A_0 is a matrix over the complete discrete valuation ring $\widehat{R}_\varnothing$. By the t-adic density of $R_\varnothing \subseteq F \subseteq F_U$ in $\widehat{R}_\varnothing$, there is a matrix C over $R_\varnothing$ such that $A_0 C \equiv I \mod t$. We then conclude by using that Condition 4.1 holds in the Key Case (5.1). $\square$

As in Section 2.5, in our geometric situation we then have patching for many other algebraic objects, which consist of finite dimensional vector spaces with additional structure. And as in Section 3, various applications then follow.

6   Variants on the patching set-up

6.1   Using more than two open sets

Rather than covering a subset U of the closed fiber X with just two proper subsets, it is possible to use a larger number of subsets. This can be useful in constructions, e.g. in the situations described in Section 3. Namely, suppose that $U \subseteq X$ is the union of subsets U_i, for $i = 1, \ldots, n$. For simplicity we assume that all double intersections $U_i \cap U_j$ are equal to a common set U_0, for $i \neq j$. Suppose we are given finite dimensional vector spaces V_i over each $F_i := F_{U_i}$ together with isomorphisms $\nu_i : V_i \otimes_{F_i} F_0 \to V_0$ for $i = 1, \ldots, n$. Then the arguments in Section 5 can be generalized to show that there is a unique choice of a finite dimensional F_U-vector space V together with isomorphisms $\alpha_i : V \otimes_{F_U} F_i \to V_i$ for $i = 0, \ldots, n$ such that $\nu_i \circ (\alpha_i \otimes F_0) = \alpha_0$ for $i = 1, \ldots, n$. Moreover this defines an equivalence between the category $\mathrm{Vect}(F_U)$ and the category of patching problems that consist of data $(V_i, \nu_i)_i$ as above. This equivalence of categories can be proven from the

equivalence with two open sets via induction. See [HH10, Theorem 4.14] for details.

Since the above functor is an equivalence of categories and preserves tensor products, it carries over from finite dimensional vector spaces to other categories of objects that consist of such vector spaces together with additional structure, as in the situation of Section 2.5. As a result, applications can be obtained in a single step, rather than building up object inductively as in Section 3.1. In that situation, a finite group G can be generated by cyclic subgroups G_i for $i = 1, \ldots, n$. By covering X by n open subsets U_i as above, and building a G_i-Galois extension of F_i for each i (e.g. by Kummer theory), one can then obtain a G-Galois extension of F. The agreement over F_0 can be achieved by choosing the cyclic extensions of the field F_i each to induce trivial extensions over the field F_0.

6.2   Using a point and a complement

Additional flexibility in constructions can be obtained by allowing patches that come from points, rather than from subsets of the closed fiber X.

For any point $P \in X$, let R_P be the local ring of $\widehat{X}$ at P, and let $\widehat{R}_P$ be the completion of this local ring at its maximal ideal $\mathfrak{m}_P$. This is a domain, and its fraction field will be denoted by F_P. (Note that although $R_P = R_{\{P\}}$, the completions $\widehat{R}_P$ and $\widehat{R}_{\{P\}}$ are different, the former being the $\mathfrak{m}_P$-adic completion and the latter being the t-adic completion. Thus F_P and $F_{\{P\}}$ are also different, with the former containing the latter.)

For example, if $T = k[[t]]$ and $\widehat{X} = \mathbb{P}^1_T$, we may consider the point P where $x = t = 0$. Then $\widehat{R}_P = k[[x, t]]$ and F_P is the fraction field $k((x, t))$ of $k[[x, t]]$. Taking U to be the complement of P in the closed fiber $X = \mathbb{P}^1_k$, so that $\widehat{R}_U = k[x^{-1}][[t]]$, we obtain the following picture:

$$\mathrm{Spec}(\widehat{R}_U) \qquad\qquad\qquad \mathrm{Spec}(\widehat{R}_P)$$
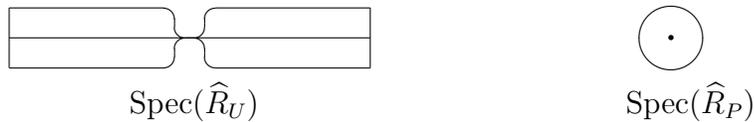
Patching can then be carried out using the fields F_P and F_U in place of the two fields F_{U_1} and F_{U_2} of Section 2. To do this, we also need to take an appropriate overfield of F_U and F_P, which will take the place of the field F_{U_0} in Section 2. In the above example, we will take the field $F_0 := k((x))((t))$, which is the fraction field of the domain $\widehat{R}_0 := k((x))[[t]]$. Note that $\widehat{R}_0$ is t-adically complete, and contains the rings $\widehat{R}_U$ and $\widehat{R}_P$; and similarly F_0 contains F_U and F_P. This choice of $\widehat{R}_0$ makes sense intuitively, because $\mathrm{Spec}(U)$ is the complement of the point $(x = 0)$ in X, while $\mathrm{Spec}\big(k[[x]]\big)$ can be viewed as a small neighborhood of this

point in X, so that $\mathrm{Spec}\big(k((x))\big)$ can be viewed as the corresponding punctured neighborhood (since $k((x)) = k[[x]][x^{-1}]$).

More generally, if T is a complete discrete valuation ring and $\widehat{X}$ is a smooth projective T-curve, then we may pick a closed point P on the closed fiber X of $\widehat{X}$, and consider the ring $\widehat{R}_P$ and its fraction field F_P. The complement U of P in X is a smooth affine curve, and we may also consider the ring $\widehat{R}_U$ and its fraction field F_U. Let $\pi \in R_P$ be an element whose reduction is a uniformizer of the complete local ring of X at P. We then take $\widehat{R}_0$ to be the t-adic completion of $R_P[\pi^{-1}]$, and F_0 to be its fraction field.

In this situation, the strategy of Section 4 can be carried over, to show that Conditions 4.1 and 4.2 hold. As a result, the corresponding base change functor

$$\Phi : \mathrm{Vect}(F) \to \mathrm{Vect}(F_U) \times_{\mathrm{Vect}(F_0)} \mathrm{Vect}(F_P),$$

is an equivalence of categories. (For details, see [HH10, Theorem 5.9]. There is also a generalization that allows more than one point P; see [HH10, Theorem 5.10].)

This approach is useful for certain applications, such as split embedding problems in Galois theory. In such a problem, one is given a surjection of finite groups $f : G \to H$ together with a section $s : H \to G$ of f, and also an H-Galois field extension E/F. The problem is then to embed E into a G-Galois field extension E'/F, such that the Galois correspondence associates the inclusion $E \hookrightarrow E'$ to the surjection f. (See the discussion in Section 3.1, where no splitting condition was assumed.) The difficulty with using just the fields F_U is that the extension E/F does not in general become trivial over any F_U. But E/F will become trivial over F_P if P is unramified and split in E/F. This approach has been used to obtain a variety of results about embedding problems in Galois theory, and those in turn have been used to obtain information about the structure of absolute Galois groups (e.g. as discussed in Sections 1.4–1.6 above).

6.3   Patching over singular curves

Until now, we have considered smooth projective T-curves $\widehat{X}$. But given a function field F of transcendence degree one over the fraction field K of a complete discrete valuation ring T, there need not be a smooth model $\widehat{X}$ of F over T. That is, there need not be a smooth projective T-curve $\widehat{X}$ whose function field is the K-algebra F. For greater generality, we will now permit projective T-curves $\widehat{X}$ that are merely assumed to be *normal* as schemes, i.e. the local rings are integrally closed domains. Given any function field F of transcendence degree one over K, one can easily obtain a normal projective model $\widehat{X}$, e.g. by writing F as a finite extension of $K(x)$, and then taking the normalization of $\mathbb{P}^1_T$ in F. (In fact, by [Abh69] and [Lip75], for each such F there even exist regular projective models $\widehat{X}$ over T, i.e. ones for which every local ring $\mathcal{O}_{\widehat{X},P}$ is a regular local ring. These can be obtained by starting with any projective model and then applying a suitable

combination of normalization and blowing up. Such regular models, however, still need not be smooth, the latter condition being equivalent to the closed fiber being smooth over the residue field k of T.)

Typically, the closed fiber X of a normal projective model $\widehat{X}$ of F over T will have several irreducible components, which will meet at several closed points. Let $\mathcal{P}$ be a non-empty finite set of closed points of X that contains all of these intersection points, and let $\mathcal{U}$ be the set of irreducible components of the complement of $\mathcal{P}$ in X. For each $P \in \mathcal{P}$ we can consider the complete local ring $\widehat{R}_P$ of $\widehat{X}$ at P, and its fraction field F_P. For each $U \in \mathcal{U}$ we can consider the t-adic completion $\widehat{R}_U$ of the ring R_U of rational functions that are regular on U; and take F_U to be the fraction field of this domain. We then have a set-up that combines the situations of the previous two subsections, using fields both of the types F_U and F_P, and typically using more than two fields in total.

For example, the following picture illustrates a choice of $\widehat{X}$ in which the closed fiber X has irreducible components X_1, X_2, X_3, where X_2 meets each of the other two components at a single point (P_1, P_2 respectively). The open subsets U_1, U_2, U_3 of X are the connected components of $X \smallsetminus \{P_1, P_2\}$, with U_i being a Zariski open dense subset of X_i. Thus U_1 (resp. U_2, resp. U_3) is the complement of P_1 (resp. P_1 and P_2, resp. P_2) in X_1 (resp. X_2, resp. X_3). The picture also illustrates the spectrum of $\widehat{R}_{P_i}$ for $i = 1, 2$, as a small neighborhood of P_i. (The spectrum of $\widehat{R}_{U_i}$, for $i = 1, 2, 3$, could similarly be illustrated as a neighborhood of U_i in $\widehat{X}$; but for visual clarity these spectra are not shown.)

As in Section 6.2, we need to define overfields for the fields F_P ($P \in \mathcal{P}$) and F_U ($U \in \mathcal{U}$). But unlike Section 6.1, there is no common overfield for all these fields. Instead, we define an overfield for F_U and F_P only if P and U are incident; i.e. if P is a point in the closure of U.

Thus in the above example, there will be four overfields, arising the pairs (U_1, P_1), (U_2, P_1), (U_2, P_2), (U_3, P_2). The overfield arising from a pair (U_i, P_j) will contain the fields F_{U_i} and F_{P_j}.

Moreover, this will be done in a way that if the closed fiber is smooth (and therefore irreducible) and there is just one point P chosen, the resulting overfield will be the same as the overfield F_0 that was considered in Section 6.1. Finally, and most crucially, the fields F_U, F_P, and these overfields will satisfy a generalization of Conditions 4.1 and 4.2; and as a result, the associated base change functor Φ (as in (2.2)) will be an equivalence of categories.

To define these overfields, we use the notion of *branches*. To illustrate, consider the affine curve C in the x, y-plane given by $y^2 = x^3 + x^2$. This is shaped like

the letter α, with the node at the origin O. The curve C is irreducible, and so the local ring $\mathcal{O}_{C,O}$ is a domain (with fraction field equal to the function field of C). But the completion $\widehat{\mathcal{O}}_{C,O}$ of the local ring is not a domain. Explicitly, the completion is isomorphic to the ring $k[[x,z]]/(z^2-x^2)$, via the isomorphism taking y to zf, where $f = (1+x)^{1/2} \in k[[x]]$. Geometrically, the spectrum of $\widehat{\mathcal{O}}_{C,O}$ has two irreducible components, corresponding to the two "branches" of C at O. They are respectively defined by the two minimal primes $\wp_1 = (z-x)$, $\wp_2 = (z+x)$ of $\widehat{\mathcal{O}}_{C,O}$. This observation can be used to motivate the formal definition: a *branch* of a variety V at a point P is a minimal prime of $\widehat{\mathcal{O}}_{V,P}$.

Returning to our situation, for each $P \in \mathcal{P}$ and for each branch of X at P, we wish to associate an overfield of F_P; this should also be an overfield of F_U, where $U \in \mathcal{U}$ is the unique element on whose closure $\bar{U}$ the branch lies (i.e. such that it is a branch of $\bar{U}$ at P). To illustrate, in the example with the reduced closed fiber pictured above, there will be two branches of X at P_1 (along the closures of U_1 and U_2) and two at P_2 (along the closures of U_2 and U_3); these four branches will correspond to the four desired overfields.

For $P \in \mathcal{P}$, the inclusion $X \to \widehat{X}$ induces a surjection $\widehat{R}_P = \widehat{\mathcal{O}}_{\widehat{X},P} \to \widehat{\mathcal{O}}_{X,P}$ whose kernel is the radical of (t). By taking inverse images under this surjection, the minimal primes of $\widehat{\mathcal{O}}_{X,P}$ can then be identified with the height one primes of $\widehat{R}_P$ that contain t. Thus we may regard each of these height one primes $\wp$ as a *branch* of X at P. Viewed as a point of $\mathrm{Spec}(\widehat{R}_P)$, the image of $\wp$ in $\widehat{X}$ lies on X, and in fact is the generic point of an irreducible component of X. We regard this as the component on which the branch lies; and the unique $U \in \mathcal{U}$ that is contained in this component is an open set that is incident to P. In this situation, we can now define an overfield $F_\wp$ of F_U and F_P, associated to this branch.

Namely, for $\wp$ a height one prime of $\widehat{R}_P$ that contains t, let $R_\wp$ be the local ring of $\widehat{R}_P$ at $\wp$. This is a discrete valuation ring, with t lying in its maximal ideal. Let $\widehat{R}_\wp$ be its completion; this is a complete discrete valuation ring, whose fraction field will be denoted by $F_\wp$. This is the desired overfield of F_U and F_P, where $\wp$ is a branch on the closure of U.

We then have a finite inverse system of fields $F_U, F_P, F_\wp$, indexed by the disjoint union $\mathcal{U} \sqcup \mathcal{P} \sqcup \mathcal{B}$, where $\mathcal{B}$ is the set of branches of X at the points of $\mathcal{P}$. Here we have inclusions of F_U and F_P into $F_\wp$ if $\wp$ is a branch of X at the point P lying on the closure of U. In the special case of Section 6.2, there is just one branch $\wp$ to consider, viz. the unique branch of X at P; this lies on U. The associated field $\wp$ is then the same as the field F_0 considered in that section. As another example, suppose that the closed fiber X is irreducible but singular, and is isomorphic to the nodal curve C discussed above. Taking $\mathcal{P}$ to consist just of the nodal point P, the set $\mathcal{U}$ will then consist just of the complement $U := X \smallsetminus P$. But there will be *two* branches $\wp_1, \wp_2$ of X at P, both lying on the closure of U. Thus in this case we have two overfields $F_{\wp_1}, F_{\wp_2}$ in the inverse system, and we will need to consider both.

In general in the above situation, it can be shown that the inverse limit of the fields $F_U, F_P, F_\wp$ is just the field F, which is a subfield of each of them. In the special case of Section 6.2, this is another way of asserting that F is the intersection of F_U and F_P inside $F_0 = F_\wp$. Thus this *inverse limit property* is a way of generalizing the intersection Condition 4.2 to the case of singular curves, where there may be multiple patching fields F_U, F_P and multiple overfields $F_\wp$. (See [HH10, Proposition 6.3] for details. There this was shown by reducing to the case of $\mathbb{P}^1_T$ with $\mathcal{P} = \{\infty\}$, by choosing a finite morphism $\widehat{X} \to \mathbb{P}^1_T$.)

The factorization Condition 4.1 can also be generalized to the current situation. The condition then becomes a *simultaneous factorization property*. That is, given elements $A_\wp \in \mathrm{GL}_n(F_\wp)$ for each $\wp \in \mathcal{B}$, there exist elements $A_U \in \mathrm{GL}_n(F_U)$ for each $U \in \mathcal{U}$ and elements $A_P \in \mathrm{GL}_n(F_P)$ for each $P \in \mathcal{P}$, satisfying the following condition: For each triple $U, P, \wp$ with $\wp$ a branch at P lying on the closure of U, the identity $A_\wp = A_U A_P$ holds in $\mathrm{GL}_n(F_\wp)$. Here we regard F_U, F_P as subfields of $F_\wp$. In the above situation, this property always holds. (Again, this can be shown by reducing to the case of $\mathbb{P}^1_T$ with $\mathcal{P} = \{\infty\}$, which was discussed in Section 6.2. A stronger result was proven this way in [HHK09, Theorem 3.6].)

Consider, for example, the model $\widehat{X}$ displayed in the picture above. There are four branches $\wp_1, \wp_2, \wp_3, \wp_4$, which are respectively associated to the four pairs $(U_1, P_1), (U_2, P_1), (U_2, P_2), (U_3, P_2)$. Suppose we are given elements $A_{\wp_i} \in \mathrm{GL}_n(F_{\wp_i})$ for $i = 1, 2, 3, 4$. The simultaneous factorization property asserts that there exist elements $A_{U_j} \in \mathrm{GL}_n(F_{U_j})$ for $j = 1, 2, 3$, and elements $A_{P_\ell} \in \mathrm{GL}_n(F_{P_\ell})$ for $\ell = 1, 2$, such that $A_{\wp_1} = A_{U_1} A_{P_1}$, $A_{\wp_2} = A_{U_2} A_{P_1}$, $A_{\wp_3} = A_{U_2} A_{P_2}$, and $A_{\wp_4} = A_{U_3} A_{P_2}$, in the respective groups $\mathrm{GL}_n(F_{\wp_i})$.

The analog of Theorem 4.3 also holds in the current situation; i.e. the base change functor is an equivalence of categories if and only if the simultaneous factorization and inverse limit properties hold. (This can be shown by replacing the fields F_1, F_2, F_0 in Theorem 4.3 by the F-algebras $\prod F_P, \prod F_U, \prod F_\wp$, and proceeding as before.) As a consequence, patching holds for finite dimensional vector spaces in this situation. More precisely, we have the following (see also [HH10, Theorem 6.4]):

Theorem 6.1. *Given a projective normal curve $\widehat{X}$ over T, with $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as above, the associated base change functor*

$$\Phi : \mathrm{Vect}(F) \to \varprojlim \mathrm{Vect}(F_\xi)$$

is an equivalence of categories, where ξ ranges over $\mathcal{P} \sqcup \mathcal{U} \sqcup \mathcal{B}$.

Thus if we are given finite dimensional vector spaces over the fields F_U and F_P, together with isomorphisms between the vector spaces that they induce over the overlap fields $F_\wp$, then there is a unique vector space over F that induces all of them compatibly; and this is an equivalence of categories. This assertion automatically carries over to other algebraic objects that consist of finite dimensional vector spaces with additional structure, as in Section 2.5.

7 Patching torsors

Until now, we have been patching structures that consist of finite dimensional vector spaces with additional structure (i.e. such that some diagrams commute). For example, G-Galois F-algebras are finite dimensional because the group G is finite. But what if we allow *infinite* dimensional vector spaces? For example, can we generalize G-Galois F-algebras to infinite groups G, such as matrix groups? To consider this, we introduce the notion of torsors.

7.1 Introduction to torsors

Say G is a linear algebraic group over a field F, i.e. a smooth Zariski closed subgroup of GL_n for some n. Here G need not be connected, and in particular all ordinary finite groups G are linear algebraic groups (as groups of permutation matrices). A *G-torsor* over F is a principal homogeneous G-space over F. That is, it is an F-variety H together with a (right) G-action $\alpha : H \times G \to H$ that is simply transitive. To be more precise, this simple transitivity property asserts that the morphism $(\mathrm{pr}_1, \alpha) : H \times G \to H \times H$ is an isomorphism of F-schemes, where pr_1 is the first projection map. So intuitively, given two points of H, there is a unique element of G taking one to the other.

A G-torsor H is *trivial* if it is F-isomorphic to G, with the G-action being given by right multiplication, with respect to this isomorphism. Note that a G-torsor H is trivial if and only if it has an F-point. Namely, in the forward direction, the identity element of G corresponds to an F-point of H; and in the reverse direction, by sending an F-point of H to the identity of G we obtain a unique isomorphism $H \to G$ that is compatible with the right G-actions. Thus if F is algebraically closed, then every G-torsor is trivial, since it automatically has an F-point.

Torsors are important in part because they classify various algebraic objects. For example, torsors for the orthogonal group $\mathrm{O}(n)$ classify quadratic forms in n variables over F, provided that $\mathrm{char}(F) \neq 2$ (see [KMRT98, (29.28)]).

In the case that G is a finite group in the classical sense, if we regard G as a torsor over F, then G-torsors are of the form $\mathrm{Spec}(A)$, where A is a G-Galois F-algebra. (Note that G acts on A on the left, since it acts on the torsor on the right.) So in this case, the theory of G-torsors is just the theory of G-Galois F-algebras. At one extreme, if A is a G-Galois *field* extension E/F, then $\mathrm{Spec}(A)$ consists of just one point, viz. $\mathrm{Spec}(E)$. At the other extreme, if A is a product of copies of F indexed by the elements of G, then $\mathrm{Spec}(A)$ consists of $|G|$ copies of $\mathrm{Spec}(F)$, again indexed by G. This is a *trivial G-torsor*.

7.2 Torsors and cohomology

It is convenient to study torsors in terms of Galois cohomology. Say G is a linear algebraic group over a field F. Consider maps $\chi : \mathrm{Gal}(F) \to G(F^{\mathrm{sep}})$ such that

$$\chi(\sigma\tau) = \chi(\sigma)\sigma(\chi(\tau)) \ \text{ for all } \sigma, \tau \in \mathrm{Gal}(F).$$

Such maps χ are called 1-*cocycles* for $\mathrm{Gal}(F)$ in $G(F^{\mathrm{sep}})$. Two 1-cocycles χ, χ' are called *cohomologous* if there is a $g \in G(F^{\mathrm{sep}})$ such that

$$\chi(\sigma) = g^{-1}\chi'(\sigma)\sigma(g) \ \text{ for all } \sigma \in \mathrm{Gal}(F).$$

The set of cohomology classes of 1-cocycles is denoted by $H^1(F, G)$. This is the *first Galois cohomology set* of F with coefficients in G. It is a group if G is commutative; otherwise it is just a pointed set (whose distinguished element corresponds to the trivial cocycle χ).

The key point is that there is a natural bijection

$$(7.1) \qquad \{\text{isomorphism classes of } G-\text{torsors over } F\} \ \leftrightarrow \ H^1(F, G).$$

Namely, given a G-torsor H, pick a point $P \in H(F^{\mathrm{sep}})$. For each $\sigma \in \mathrm{Gal}(F)$, we have the two points $P, \sigma(P) \in H(F^{\mathrm{sep}})$. By the torsor property, there is a unique element $g \in G(F^{\mathrm{sep}})$ such that $\sigma(P) = Pg$. Write $\chi(\sigma) = g$. Then the map $\chi : \mathrm{Gal}(F) \to G(F^{\mathrm{sep}})$ is a 1-cocycle. It depends on the choice of P; but it is straightforward to check that changing P does not change the equivalence class of χ. Thus we have a well-defined element $[\chi]$ of $H^1(F, G)$ that is associated to the G-torsor H. One can then check that the association $H \mapsto [\chi]$ defines a bijection as in (7.1). (Note also that this bijection parallels the fact in topology that the principal G-bundles over a space X are classified by $H^1(X, G)$.)

One can also define $H^n(F, G)$ for other values of n, though for $n > 1$ one requires G to be commutative. In particular, $H^0(F, G)$ is just $G(F)$. Moreover $H^0(F, E/J)$ can also be defined for any subgroup $J \subseteq E$ of a group E; this is the set of $\mathrm{Gal}(F)$-invariant cosets $eJ_{F^{\mathrm{sep}}}$ of $J_{F^{\mathrm{sep}}}$, with $e \in E(F^{\mathrm{sep}})$.

See [Ser00, I, Section 5] for more about non-abelian Galois cohomology.

7.3 Torsors and patching

We now return to the question of patching. As in Section 3.1, it is possible in the context of our fields to patch G-Galois F-algebras, or equivalently G-torsors, where G is a finite group. More generally, let G be a linear algebraic group over F. If G is infinite, then any G-torsor is of the form $H = \mathrm{Spec}(A)$, where A is an F-algebra that is *infinite* dimensional over F. It turns out that it is still possible to patch G-torsors!

To see this, we first relate torsors to matrices. View $G \subseteq \mathrm{GL}_n$ over F. For $h \in \mathrm{GL}_n(F^{\mathrm{sep}})$, consider the translate (or equivalently, coset) $hG \subseteq \mathrm{GL}_n$. If

$h \in \mathrm{GL}_n(F)$, then via left multiplication by h (which is an F-isomorphism), hG is isomorphic to the trivial G-torsor $hG = G$. If $h \notin \mathrm{GL}_n(F)$, then the translate hG need not be defined over F; but if it is, then it defines a G-torsor that will in general be non-trivial. To say that hG is defined over F is equivalent to the condition that for every $\sigma \in \mathrm{Gal}(F)$, $hG = (hG)^\sigma = h^\sigma G$; i.e. that $h^\sigma \in hG$.

As an example, let $F = \mathbb{R}$, let G be the orthogonal group $\mathrm{O}(2)$, and let h be the matrix $\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$. Then hG defines a non-trivial G-torsor over $\mathbb{R}$.

The above construction in fact gives rise to *all* torsors; i.e. every G-torsor over F arises from a translate hG as above. This follows from the exact sequence

(7.2)
$$1 \to H^0(F, G) \to H^0(F, \mathrm{GL}_n) \to H^0(F, \mathrm{GL}_n/G) \to H^1(F, G) \to H^1(F, \mathrm{GL}_n)$$

(see [Ser00, I.5.4, Proposition 36]) and the fact that $H^1(F, \mathrm{GL}_n)$ is trivial by Hilbert's Theorem 90 ([KMRT98, Theorem 9.2]). In fact this shows that there is a bijection of pointed sets $\mathrm{GL}_n(F) \backslash H^0(F, \mathrm{GL}_n/G) \to H^1(F, G)$, which classifies G-torsors over F.

So we can study torsors by studying matrices. Using this approach, one can show in an abstract context:

Theorem 7.1. *Given a finite inverse system of fields, if patching holds for finite dimensional vector spaces, then patching also holds for G-torsors, for all linear algebraic groups G.*

That is, if F is the inverse limit of a finite inverse system of fields $(F_\xi)_{\xi \in I}$, and if the base change functor

$$\Phi : \mathrm{Vect}(F) \to \varprojlim \mathrm{Vect}(F_\xi)$$

on vector spaces is an equivalence of categories, then so is the base change functor

$$\Phi_G : G\mathrm{Tors}(F) \to \varprojlim G\mathrm{Tors}(F_\xi)$$

for any linear algebraic group G over F, where $G\mathrm{Tors}$ denotes the category of G-torsors over the given field.

We explain the proof in the basic context of four fields as in (2.1) and (2.2). Suppose that these fields $F \subseteq F_1, F_2 \subseteq F_0$ satisfy patching for finite dimensional vector spaces. By Theorem 4.3, the factorization Condition 4.1 and the intersection Condition 4.2 hold for these fields. Suppose we are given G-torsors H_i over F_i for $i = 1, 2$, together with an isomorphism over F_0. By the above discussion, we may write $H_i = h_i G$, with $h_i \in \mathrm{GL}_n(F_i^{\mathrm{sep}})$; and we are given an F_0-isomorphism $h_2 G \to h_1 G$. This map is defined by left multiplication by some $g_0 \in \mathrm{GL}_n(F_0)$. Thus $h_1 G = g_0 h_2 G$. Applying Condition 4.1 to g_0, we obtain elements $g_i \in \mathrm{GL}_n(F_i)$ for $i = 1, 2$ such that $g_1^{-1} g_2 = g_0$. Let $h_i' = g_i h_i \in \mathrm{GL}_n(F_i^{\mathrm{sep}})$ for $i = 1, 2$. Thus $h_1' G = h_2' G$ over F_0. That is, the translates $h_i' G$, for $i = 1, 2$,

define the *same* F_0-point on the quotient GL_n/G, which is an F-variety (e.g. see [Bor91, Theorem II.6.8]). This point of GL_n/G is then defined over both F_1 and F_2, and hence over F by Condition 4.2. The point thus defines an element of $H^0(F, \mathrm{GL}_n/G)$, and hence a G-torsor defined over F, by the above exact sequence (7.2).

Essentially the same argument holds for a more complicated finite inverse system of fields. We thus obtain that patching for finite dimensional vector spaces implies patching for torsors.

In the context of one-variable function fields F over a complete discretely valued field K, with a normal projective model $\widehat{X}$ and sets $\mathcal{P}, \mathcal{U}, \mathcal{B}$ as before, patching for finite dimensional vector spaces holds by Theorem 6.1. Hence Theorem 7.1 yields:

Corollary 7.2. *In the situation of Section 6.3, patching for G-torsors holds with respect to the index set $\mathcal{P} \sqcup \mathcal{U} \sqcup \mathcal{B}$, for any linear algebraic group G over F.*

8   Local-global principles

Patching for torsors can be used to study local-global principles for algebraic objects. For example, using torsors under the orthogonal group $\mathrm{O}(n)$, local-global principles can be obtained for quadratic forms.

The most classical local-global principle, in fact, concerns quadratic forms, though over a different type of field. The Hasse-Minkowski theorem states that a quadratic form over $\mathbb{Q}$ is isotropic (i.e. has a non-trivial zero) if and only if it is isotropic over each field $\mathbb{Q}_p$ and also over $\mathbb{R}$. More generally, the analogous assertion holds for any global field K, with respect to its completions K_v with respect to the absolute values v on K. In the case of equal characteristic global fields (i.e. function fields of curves over a finite field), this is equivalent to taking the completions at the closed points of the associated smooth projective curve.

A related classical local-global principle is the theorem of Albert, Brauer, Hasse, and Noether. That theorem concerns central simple algebras over global fields. It says that such an algebra is split (i.e. is isomorphic to a matrix algebra over the given field) if and only it is split over each completion.

Local-global principles can typically be rephrased in terms of the existence of rational points on varieties. For example, in the context of the Hasse-Minkowski theorem, let q be a quadratic form over a global field K. If q is a form in n variables, then it defines a quadric hypersurface Q in $\mathbb{P}_K^{n-1}$; and q is isotropic over K if and only if Q has a K-point. From this point of view, local-global principles assert that a variety has a point over K if and only if it has a point over each completion of K.

To go beyond the context of global fields, we broaden the notion of a local-global principle: it is an assertion that a given property (such as the presence of

points on varieties) holds over a given field F if and only if it holds over each of a given set of overfields F_ξ of F, where ξ ranges over some index set I.

In the situation of torsors, we can consider the following local-global principle: A G-torsor H over F is trivial (or equivalently, has an F-point) if and only if the induced torsor $H_\xi := H \times_F F_\xi$ is trivial over F_ξ for each $\xi \in I$. This assertion can rephrased in terms of Galois cohomology, since the G-torsors over F are classified by $H^1(F, G)$. Namely, there is a natural *local-global map* $H^1(F, G) \to \prod_\xi H^1(F_\xi, G)$ of pointed sets (or of groups, if G is commutative). The local-global principle then states that the kernel of this map is trivial; i.e. that only the trivial element of $H^1(F, G)$ maps to the trivial element in the product. This need not always hold, however; and a related problem is then to determine the kernel of this map, and in particular to determine whether it is finite (even if not necessarily trivial).

An important classical example considers a global field K and its completions K_v, along with an abelian variety A over K (e.g. an elliptic curve over K). In this case the local-global map is a group homomorphism, and its kernel is a group, called the *Tate-Shafarevich group* $\Sha(K, A)$. A major open question is whether its order is necessarily finite. In fact, its order has been conjectured in the case of elliptic curves, in terms of special values of L-functions; this is the conjecture of Birch and Swinnerton-Dyer.

In the case that G is a linear algebraic group over a number field K, it was shown by Borel and Serre ([BS64]) that $\Sha(K, G)$ is finite. (Here we again take the local-global map with respect to the set of absolute values on K.) For the analogous problem in the function field case of global fields, the finiteness of $\Sha(K, G)$ was shown by Brian Conrad ([Con12]).

9 Local-global principles in the patching context

In our context, we will take F to be the function field of a projective normal curve $\widehat{X}$ over a complete discrete valuation ring T, and we will let F_ξ range over a finite set of overfields corresponding to patches F_U and F_P as in Section 6.3. Note that this involves using just a *finite* collection of overfields, unlike the classical situation in which infinitely many overfields are considered. The local-global principle for torsors then says that a G-torsor H over F is trivial if and only if H_ξ is trivial over F_ξ for each $\xi \in \mathcal{P} \sqcup \mathcal{U}$. Equivalently, it says that the kernel of the local-global map

$$H^1(F, G) \to \prod_{P \in \mathcal{P}} H^1(F_P, G) \times \prod_{U \in \mathcal{U}} H^1(F_U, G)$$

is trivial. The kernel of this map will be denoted by $\Sha_{\widehat{X}, \mathcal{P}}(F, G)$, where $\mathcal{P}$ determines $\mathcal{U}$. If the model $\widehat{X}$ of F is understood, we will simply write $\Sha_{\mathcal{P}}(F, G)$.

For example, suppose that $F = K(x)$, where as before K is the fraction field of T, and take $\widehat{X} = \mathbb{P}_T^1$. Let $\mathcal{P}$ consist just of the point $P = \infty$ on the closed fiber $X = \mathbb{P}_k^1$, so that $\mathcal{U}$ consists just of the single open set $U = \mathbb{A}_k^1$. In this case $\text{Ш}_{\mathcal{P}}(F, G)$ is the kernel of $H^1(F, G) \to H^1(F_P, G) \times H^1(F_U, G)$. Let $\wp$ be the unique branch of X at P. Using the fact that patching holds for finite dimensional vector spaces in this context, one then obtains the following Mayer-Vietoris type exact sequence (see [HHK11a, Theorem 3.5]):

Theorem 9.1. *There is an exact sequence of pointed sets*

$$1 \longrightarrow H^0(F, G) \longrightarrow H^0(F_P, G) \times H^0(F_U, G) \xrightarrow{\quad \delta \quad} H^0(F_\wp, G)$$

$$H^1(F, G) \longrightarrow H^1(F_P, G) \times H^1(F_U, G) \rightrightarrows H^1(F_\wp, G).$$

Proof. Define the maps on $H^i(F, G)$ to be the diagonal inclusions. Define the last arrow on the first line as the quotient $i_U^{-1} i_P$ of the inclusion maps from the factors in the middle term to the group $H^0(F_\wp, G)$. (This quotient map is not in general a group homomorphism.) The two arrows at the end of second line are the maps arising from the inclusions of F_P, F_U into $F_\wp$. Exactness at the middle term of that line means that the equalizer of these two maps is equal to the image of $H^1(F, G)$. (If G is commutative then $H^1(F_\wp, G)$ is a group and we could take the corresponding quotient instead of using a double arrow.)

The coboundary map δ is defined as follows: Take trivial G-torsors H_P, H_U over F_P, F_U with rational points x_P, x_U. Given $g_\wp \in H^0(F_\wp, G)$, we have an isomorphism

$$H_P \times_{F_P} F_\wp \to H_U \times_{F_U} F_\wp$$

of trivial G-torsors over $F_\wp$, taking x_P to $x_U g_\wp$. This defines a patching problem for G-torsors. By Corollary 7.2, there is a solution H to this patching problem, viz. a G-torsor H over F corresponding to an element of $H^1(F, G)$. This element is then defined to be $\delta(g_\wp)$.

With the above maps, one can then check that the sequence is exact. $\qquad\square$

In the example of the line, exactness implies that $\text{Ш}_{\mathcal{P}}(F, G)$ is the cokernel of the last map of H^0 terms, i.e. $G(F_P) \times G(F_U) \to G(F_\wp)$, given by $(g_P, g_U) \mapsto g_U^{-1} g_P$, where we regard $G(F_P), G(F_U)$ as contained in $G(F_\wp)$. But the surjectivity of this map is equivalent to the factorization property for the group G. (Note that until now we have considered factorization only for the groups GL_n, not for other linear algebraic groups G.) Hence the local-global principle for G-torsors is equivalent to the factorization property for the group G.

While the above is for the example of the line and with $\mathcal{P} = \{\infty\}$, Theorem 9.1 carries over to arbitrary normal T-curves $\widehat{X}$ together with $\mathcal{P} \subset X$. (See [HHK11a, Theorem 3.5].) There the middle and last terms on each line of the Mayer-Vietoris

exact sequence are replaced by products, where $P, U, \wp$ respectively range over $\mathcal{P}$, $\mathcal{U}$, $\mathcal{B}$. Just as the simultaneous factorization property for GL_n was considered in Section 6.3, one can also consider this property for an arbitrary linear algebraic group G. We then obtain:

Corollary 9.2. *In the situation of Section 6.3, if G is a linear algebraic group over F, then the local-global principle for G-torsors is equivalent to the simultaneous factorization property for the group G.*

10   Obstructions to local-global principles

In the previous section, local-global principles for torsors in the context of patches were reformulated in terms of factorization. Using this, we can now examine when such principles hold, and whether the obstruction $\mathrm{III}_\mathcal{P}(F, G)$ is finite even if not necessarily trivial.

10.1   Case of rational connected groups

As we discuss below, simultaneous factorization holds for G, and hence $\mathrm{III}_\mathcal{P}(F, G)$ vanishes, provided that the group G is connected and rational (i.e. rational as an F-variety, meaning that it is birationally isomorphic to $\mathbb{A}_F^n$ for some n). For example, the special orthogonal group $\mathrm{SO}(n)$ is a connected rational group, by the Cayley parametrization. In fact many connected linear algebraic groups are known to be rational; and over an *algebraically closed field* it is known that every linear algebraic group is rational.

We describe the idea of the proof, and to simplify the discussion we again return to the example of the projective line and $\mathcal{P} = \{\infty\}$. As in the case of proving factorization for the group GL_n, we will construct the factors inductively, modulo successive powers of the uniformizer t of T. But unlike in the GL_n case, we have to be careful to ensure that the limit of these inductive sequences of mod t^i factors will itself lie in the group G. (This was automatic for GL_n, since any matrix that is congruent to the identity modulo t will lie in GL_n.) To do this, we will use the rationality of the group G.

Specifically, since G is a connected rational variety, say of dimension m, there are Zariski open neighborhoods $N \subseteq G$ and $N' \subseteq \mathbb{A}^m$ of $1 \in G$ and $0 \in \mathbb{A}^m$, respectively, together with an isomorphism of F-varieties $\eta : N \to N'$. The multiplication map $\mu : G \times G \to G$ on G, restricted to an open neighborhood of $(1, 1)$ in $N \times N$, corresponds under η to a map $f : \tilde{N}' \to N'$, where $\tilde{N}'$ is an open dense subset of $N' \times N'$ that contains the origin. By an inductive construction that generalizes the factorization construction for GL_n, we can show that for some open neighborhood M' of the origin in N', every $B_0 \in M'(F_\wp)$ is of the form $f(B_U, B_P)$, for some $B_U \in N'(F_U)$ and $B_P \in N'(F_P)$ with $(B_U, B_P) \in \tilde{N}'$. Now

take any $A_0 \in G(F_\wp)$ that we wish to factor. After translation, we may assume that $A_0 \in N(F_\wp)$ and that $B_0 := \eta(A_0) \in M'(F_\wp)$. Taking B_U, B_P as above, the elements $A_U := \eta^{-1}(B_U) \in G(F_U)$ and $A_P := \eta^{-1}(B_P) \in G(F_P)$ then satisfy $A_U A_P = A_0$, as desired. (See [HHK09], Theorems 3.2 and 3.4, for more details.)

In the more general situation, where $\mathcal{P}, \mathcal{U}$, and $\mathcal{B}$ can each have more than one element, one can generalize this and prove *simultaneous* factorization for collections of elements of $G(F_\wp)$, $\wp \in \mathcal{B}$. (See [HHK09], Theorem 3.6.) By Corollary 9.2, it follows that the local-global principle holds for G-torsors, or equivalently $\Sha_\mathcal{P}(F, G)$, where G is any rational connected linear algebraic group over F.

10.2 Case of finite groups

Above, the groups considered were connected. This leaves open the question of what happens if the group is disconnected. For example, suppose that G is an ordinary finite group, viewed as a finite linear algebraic group. As discussed in Section 7, G-torsors over F are then just the spectra of G-Galois F-algebras. For such an F-algebra A, we can take the normalization of $\widehat{X}$ in A, and obtain a G-Galois branched cover $\widehat{Y} \to \widehat{X}$ whose corresponding extension of F is A. (Here, the *normalization* is obtained by taking the integral closure S of R in A for every Zariski affine open subset $\mathrm{Spec}(R) \subset \widehat{X}$; and then taking $\widehat{Y}$ to be the union of the affine schemes $\mathrm{Spec}(S)$. More formally, $\widehat{Y} = \underline{\mathrm{Spec}}(\mathcal{S})$, where $\mathcal{S}$ is the integral closure of the structure sheaf $\mathcal{O}_{\widehat{X}}$ in A.) Here $\widehat{Y}$ is connected (in fact irreducible) if and only if A is a field.

Thus $H^1(F, G)$ is in bijection with the set of isomorphism classes of G-Galois branched covers of $\widehat{X}$. The subset $\Sha_\mathcal{P}(F, G) \subseteq H^1(F, G)$ corresponds to those covers that are trivial over each F_P and each F_U. Covers of that type, which we call *split covers*, are necessarily unramified over $\widehat{X}$, since they are unramified (in fact, trivial) locally.

As we will discuss, the local-global principle for G can fail in this situation. But the obstruction $\Sha_\mathcal{P}(F, G)$ is finite, and can be computed in terms of a graph Γ that is associated to the closed fiber X of $\widehat{X}$. Namely, Γ can be regarded as a topological space, and $\Sha_\mathcal{P}(F, G)$ can be described in terms of its fundamental group.

This graph, called the *reduction graph*, is easiest to describe in the special case that there are exactly two branches of X at each point of $\mathcal{P}$, and that these two branches lie on distinct irreducible components of X. In this case, we can use the definition given in [DM69, p. 86]: the vertices of the graph correspond to the irreducible components of X, the edges correspond to the (singular) points that lie on two distinct components, and the vertices of a given edge correspond to the two components on which the corresponding point lies.

For example, the graph associated to the configuration pictured in Section 6.3 is as follows:

$$\underset{U_1}{\bullet} \qquad P_1 \qquad \underset{U_2}{\bullet} \qquad P_2 \qquad \underset{U_3}{\bullet}$$
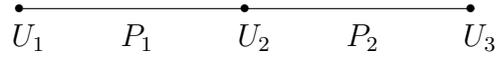
In the case that more than two components of X can meet at a point, the above definition does not apply. But one can instead consider a modified version, which in the above special case gives the barycentric subdivision of the graph as described above. This graph will then be homotopic to the graph above, and that will suffice for our purposes (for which only the homotopy class of the graph will be relevant).

Namely, we construct a bipartite graph, i.e. a graph whose vertex set $\mathcal{V}$ is partitioned into two subsets $\mathcal{V}_1, \mathcal{V}_2$ such that every edge connects a vertex in $\mathcal{V}_1$ to one in $\mathcal{V}_2$. Choose a non-empty finite subset $\mathcal{P}$ of X that contains all the points where the closed fiber X has more than one branch. (This contains in particular all the points where distinct irreducible components of X meet.) Let $\mathcal{U}$ consist of the connected components of $X \smallsetminus \mathcal{P}$; its elements U are in bijection with the irreducible components of X, by taking Zariski closures. We also obtain a set $\mathcal{B}$, consisting of the branches of X at the points of $\mathcal{P}$; each branch lies on the closure of a unique $U \in \mathcal{U}$. For the bipartite graph, let $\mathcal{V}_1 = \mathcal{P}$ and $\mathcal{V}_2 = \mathcal{U}$. The edges of the graph correspond to the branches of X at the points of $\mathcal{P}$, with the vertices of a branch corresponding to the associated point $P \in \mathcal{P}$ and element $U \in \mathcal{U}$.

As an example, consider the situation of a closed fiber X consisting of three components X_1, X_2, X_3 that all meet at two points P_1, P_2. Thus $U_i = X_i \smallsetminus \{P_1, P_2\}$. There are six branches $\wp_1, \ldots, \wp_6$ on X at the points of $\mathcal{P}$, one for each pair (P_i, U_j). This configuration can be pictured as follows:
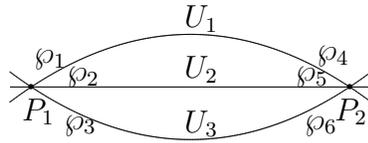
The associated bipartite graph is then as follows, where the five vertices correspond to $\mathcal{P} \sqcup \mathcal{U}$ and the six edges correspond to the six branches $\wp_i$.
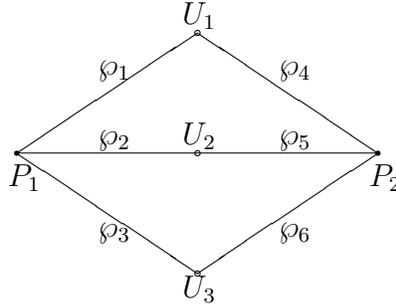
In general, if we enlarge $\mathcal{P}$ by adding an additional closed point (and shrinking the corresponding set $U \in \mathcal{U}$ by deleting that point), then the homotopy class of the graph is unchanged. The same is true if we blow up $\widehat{X}$ at a regular point $P \in \mathcal{P}$.

Coming back to local-global principles, consider a finite group G, which we can also consider as a finite torsor over F. An element of $\mathrm{III}_{\mathcal{P}}(F, G)$ induces a G-Galois covering space of the associated reduction graph Γ, as follows. This covering space is locally trivial over a neighborhood of each vertex, including the adjacent open edges. The local covers are glued over the overlaps, according to the patching data of the given element of $\mathrm{III}_{\mathcal{P}}(F, G) \subseteq H^1(F, G)$. This data consists of the given $F_\wp$-isomorphisms of the trivial G-torsors over F_U and F_P that are obtained via base change from the given torsor over F. (These base changes are trivial G-torsors because the given torsor lies in $\mathrm{III}_{\mathcal{P}}(F, G)$.) Conversely, a covering space of Γ induces an element of $\mathrm{III}_{\mathcal{P}}(F, G)$, by patching.

Thus the elements of $\mathrm{III}_{\mathcal{P}}(F, G)$ are classified by $\mathrm{Hom}(\pi_1(\Gamma), G)/\sim$, where $\sim$ is the equivalence relation given by conjugating maps by elements of G. This is because this set classifies the (possibly disconnected) G-Galois covering spaces of the graph Γ. (If we instead classified *pointed* G-Galois covering spaces of Γ, we would not need to mod out by this equivalence relation.)

Given $\widehat{X}$ and $\mathcal{P}$, we can find Γ explicitly, and then compute

$$\mathrm{III}_{\mathcal{P}}(F, G) = \mathrm{Hom}(\pi_1(\Gamma), G)/\sim,$$

which is a finite set. Namely, $\pi_1(\Gamma)$ is a free group of some finite rank r, where r is the "number of loops in Γ". We can then identify $\mathrm{Hom}(\pi_1(\Gamma), G)$ with G^r. Thus $\mathrm{III}_{\mathcal{P}}(F, G)$ can be identified with G^r/G, where the action of G on G^r is by uniform conjugation.

Note that this description of $\mathrm{III}_{\mathcal{P}}(F, G)$ is independent of the choice of $\mathcal{P}$ and of $\widehat{X}$ (which can be varied by blowing up), since the homotopy class of Γ is independent of these choices. Note also, for $G \neq 1$, that $\mathrm{III}_{\mathcal{P}}(F, G)$ vanishes (or equivalently, the local-global principle holds) if and only if Γ is a tree.

10.3 Disconnected rational groups

We have considered the obstructions $\text{Ш}_{\mathcal{P}}(F, G)$ to local-global principles in the special cases that G is a rational connected group, or if G arises from a finite group. Here we turn to linear algebraic groups such as $\text{O}(n)$ that are at neither of those two extremes.

We say that a (not necessarily connected) linear algebraic group G is *rational* if each connected component of the F-variety G is a rational F-variety. An equivalent characterization is that the identity component G^0 of G is a rational connected group and that each connected component of G has an F-point. Another equivalent characterization is that G^0 is a rational connected group and the quotient $\bar{G} := G/G^0$ is a finite constant group scheme (i.e. arises from an ordinary finite group) such that $G(F) \to \bar{G}(F)$ is surjective.

For such a group G, we have a short exact sequence

$$1 \to G^0 \to G \to \bar{G} \to 1$$

of groups, with G^0 connected and $\bar{G}$ a finite constant group. Associated to this is a long exact cohomology sequence, which involves just H^0 and H^1 if G is not commutative (see [Ser00, I.5.5, Proposition 38]). Combining that with the Mayer-Vietoris sequence in Theorem 9.1, one obtains a short exact sequence of pointed sets (and of groups, if G is commutative):

$$1 \to \text{Ш}_{\mathcal{P}}(F, G^0) \to \text{Ш}_{\mathcal{P}}(F, G) \to \text{Ш}_{\mathcal{P}}(F, \bar{G}) \to 1.$$

(See Corollary 2.6 of [HHK11a].) Here the first term vanishes, as discussed in Section 10.1; and the third term is given by $\text{Hom}(\pi_1(\Gamma), \bar{G})/\sim)$, as in Section 10.2. We then obtain the following description of the obstruction $\text{Ш}_{\mathcal{P}}(F, G)$ to the local-global principle for G-torsors (where some extra work is needed if G is not commutative):

Theorem 10.1. *Let $\widehat{X}$ be a normal projective model of a field F of transcendence degree one over a complete discretely valued field. Let $\mathcal{P}$ be a non-empty finite subset of the closed fiber X that contains all the points where X has more than one branch. Let G be a rational linear algebraic group over F, and write $\bar{G} = G/G^0$. Then*

$$\text{Ш}_{\mathcal{P}}(F, G) = \text{Hom}(\pi_1(\Gamma), \bar{G})/\sim .$$

As a consequence, $\text{Ш}_{\mathcal{P}}(F, G)$ is finite, and its order can be explicitly computed, for G rational. Moreover we obtain the precise condition for the local-global principle to hold:

Corollary 10.2. *With F as above and G a rational linear algebraic group over F, the local-global principle for G-torsors holds if and only if either G is connected or the reduction graph is a tree.*

Namely, these are precisely the conditions under which $\mathrm{Hom}(\pi_1(\Gamma), \bar{G})$, and hence also $\mathrm{Hom}(\pi_1(\Gamma), \bar{G})/ \sim$, is trivial.

The above discussion answers the question of whether there is a local-global principle, and more generally what the obstruction is to such a principle, in the case of *torsors* for rational groups. But the question remains about local-global principles for other homogeneous spaces for such groups.

To be more precise, consider an F-variety H together with a right G-action $\alpha : H \times G \to H$. We say that G *acts transitively on* H if for every field extension E/F, the action of $G(E)$ on $H(E)$ is transitive. Every torsor has this property; but not conversely, since transitive actions can have stabilizers.

We can then show:

Proposition 10.3. *Let G be a linear algebraic group over a field F as above. If the local-global principle holds for G-torsors, then it also holds for **all** F-varieties H on which G acts transitively. Equivalently, the simultaneous factorization condition for G implies that local-global principles hold for all transitive G-varieties.*

Proof. The first part follows from the second part together with Corollary 9.2. Concerning the second part, for simplicity, we sketch the proof in the simple case where we have just one U and one P. More abstractly, we have four fields $F \subseteq F_1, F_2 \subseteq F_0$ satisfying Conditions 4.1 and 4.2 (the latter also being satisfied here, as discussed in Section 6).

Say that we have points $P_1 \in H(F_1)$ and $P_2 \in H(F_2)$. We wish to find a point $P \in H(F)$. Viewing $P_i \in H(F_0)$, the transitivity property implies that there exists some $A_0 \in G(F_0)$ such that $P_1 A_0 = P_2$. By the factorization condition, there exist $A_1 \in G(F_1)$ and $A_2 \in G(F_2)$ such that $A_0 = A_1 A_2$. Let $P_1' = P_1 A_1 \in H(F_1)$ and let $P_2' = P_2 A_2^{-1} \in H(F_2)$. Then P_1', P_2' define the same point in $H(F_0)$. Since $F_1 \cap F_2 = F$, it follows that this common point P is defined over F, i.e. lies in $H(F)$, as desired. $\square$

11 Applications of local-global principles

11.1 Applications to quadratic forms

As before, let F be the function field of a curve over a complete discretely valued field K, and let $\widehat{X}$ be a normal projective model of F over the valuation ring T of K. Suppose now that $\mathrm{char}(F) \neq 2$. If q is a quadratic form over F, then after a change of variables it can be diagonalized, i.e. written as $q = \sum_{i=1}^n a_i z_i^2$. We assume that q is regular, i.e. each $a_i \neq 0$. As in Section 8, q defines a quadric hypersurface Q in $\mathbb{P}_K^{n-1}$; and q is isotropic over F if and only if Q has an F-point.

In this situation, we may choose a non-empty finite subset $\mathcal{P} \subset X$ of the closed fiber with properties as before, and let $\mathcal{U}$ be the set of connected components of the complement. The local-global principle for q over F with respect to the overfields

F_ξ, for $\xi \in \mathcal{P} \sqcup \mathcal{U}$, then asserts that q is isotropic over F if and only if it is isotropic over each F_ξ; or equivalently,

$$Q(F_\xi) \neq \varnothing \text{ for all } \xi \in \mathcal{P} \sqcup \mathcal{U} \ \Leftrightarrow \ Q(F) \neq \varnothing.$$

The question is whether this principle holds.

Let $\mathrm{O}(q)$ be the orthogonal group associated to the quadratic form q; i.e. the subgroup of GL_n that preserves the form q. The projective hypersurface Q is a homogeneous space for the group $\mathrm{O}(q)$, though not a torsor for this group. This group is rational, by the Cayley parametrization ([KMRT98], p. 201, Exercise 9), but not connected. In fact, it has two connected components, with the identity component being $\mathrm{SO}(q)$.

Note that $\dim(Q) = n-2$. In particular, Q is of dimension zero if q is a binary quadratic form. In that case, Q can consist of two points, and be disconnected (e.g. if $q = x^2 - y^2$). But if $n > 2$, then Q is connected, and hence the connected subgroup $\mathrm{SO}(q)$ also acts transitively on Q. Since the group $\mathrm{SO}(q)$ is both rational and connected, the local-global principle holds for Q, using Corollary 10.2 and Proposition 10.3. Rephasing this, we have:

Theorem 11.1. *The local-global principle for isotropy of quadratic forms over F, with respect to patches F_ξ, holds for forms of dimension $n > 2$.*

This result is analogous to the classical Hasse-Minkowski theorem for quadratic forms over global fields (see Section 8). But unlike that situation, here there can be an exception, in the case of binary forms. And in fact, there really do exist examples of binary forms in which the principle does not hold. By Corollary 10.2, any such example must involve a field F for which the reduction graph associated to a model is not a tree. The simplest case of this is a Tate curve, where the general fiber is a genus one curve over K, and the closed fiber consists of one or more projective lines whose crossings form a "loop". One such possibility consists of two projective lines that cross each other at two points. The closed fiber then looks like
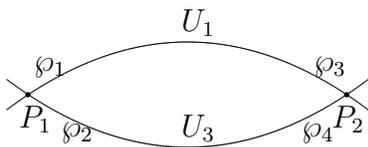
and the reduction graph looks like
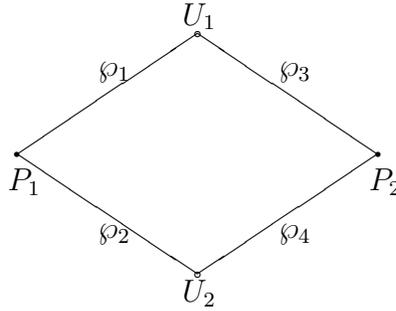
which is not a tree. An explicit example of this situation is the double cover $\widehat{X}$ of $\mathbb{P}^1_T$ given in affine coordinates by $y^2 = x(x-t)(1-xt)$, with P_1, P_2 being the points $x = 0, \infty$ on the closed fiber ($t = 0$). The form $q = x(x-t)z_1^2 - z_2^2$ is locally isotropic but not isotropic over the function field of $\widehat{X}$. (This phenomenon was observed by J.-L. Colliot-Thélène, based on [Sai83, Example 2.7]. See [CPS12, Remark 4.4] and [HHK09, Example 4.4] for Tate curve examples with an irreducible closed fiber.)

There is another way to understand the fact that the local global principle will hold for *all* quadratic forms over F (without any restriction on dimension) if and only if the associated reduction graph Γ is a tree. This concerns the *Witt group* of the field F. This group $W(F)$ is defined to be the set of equivalence classes of quadratic forms, with two forms being considered equivalent if they differ by a hyperbolic form $\sum_{i=1}^m (x_i^2 - y_i^2)$, up to a change of variables. These equivalence classes form a group under orthogonal direct sum, i.e. adding representative forms in disjoint sets of variables. The local-global principle for the Witt group would assert that $W(F)$ is trivial if and only if each group $W(F_\xi)$ is trivial. As a result of the two-dimensional exception in Theorem 11.1, this will not always hold, due to the presence of two-dimensional forms that become isotropic over each F_ξ but are not isotropic over F. (A two-dimensional form is hyperbolic if and only if it is isotropic.) Instead, the obstruction to this principle can be found explicitly, using the results described above:

Proposition 11.2. *The kernel of the local-global map*

$$W(F) \to \prod_{\xi \in \mathcal{U} \sqcup \mathcal{P}} W(F_\xi)$$

on Witt groups is $\mathrm{Hom}(\pi_1(\Gamma), \mathbb{Z}/2)$, *where Γ is the reduction graph of any regular projective model of F. Hence the local-global principle for Witt groups holds if and only if Γ is a tree.*

The key ingredients in the proof are Theorem 11.1 and Theorem 10.1 with G an orthogonal group. For more details, see [HHK11a, Theorem 9.6] (where the assertion was phrased somewhat differently).

A numerical application of the above ideas is motivated by classical quadratic form theory over global fields. Namely, the Hasse-Minkowski theorem implies the following theorem of Meyer: If q is a quadratic form over $\mathbb{Q}$ of dimension greater than four, and if q is not positive or negative definite, then q is isotropic over $\mathbb{Q}$. (See [Ser73, Section IV.3.2].) More generally, this assertion holds over any global field. Note that in the function field case, no forms are positive or negative definite, and the assertion simplifies, to say that every quadratic form in more than four variables is isotropic.

In the situation of the fields F that we have been considering, i.e. one-variable function fields over a complete discretely valued field K, Theorem 11.1 can similarly be used to obtain an analogous result concerning isotropy of forms in "too many variables." (Since we are in the function field case, there is no issue of being positive or negative definite.) In particular, if $K = \mathbb{Q}_p$ with $p \neq 2$, and if q is a quadratic form over F of dimension greater than 8, then q is isotropic over F. And to give another example, if $K = \mathbb{Q}_p((t))$ with $p \neq 2$, and if q is a quadratic form over F of dimension greater than 16, then q is isotropic over F. These and other related results (e.g. for fields of the form F_ξ with $K = \mathbb{Q}_p$) were shown in [HHK09, Section 4] using the above methods.

Until the 1990's, it was not known if there were results of this sort, even in the case of $K = \mathbb{Q}_p$. The above result in that case was first shown by a different approach in [PS10]. Another proof, in [Lee12], later showed that that result holds for $K = \mathbb{Q}_2$ as well. Building on [HHK09, Section 4] and [HHK11a], it has been recently shown more generally in [PS13] that if K is a complete discretely valued field of characteristic zero whose residue field is a perfect field of characteristic two, then any quadratic form of dimension greater than 8 must be isotropic over F. These results, however, were obtained by proofs that did not apply to such cases as $K = \mathbb{Q}_p((t))$.

11.2   Applications to central simple algebras

By a similar approach, local-global principles can be obtained for central simple algebras over our field F. We begin by reviewing some background; see also [Pie82].

Recall that Wedderburn's Theorem states that every (finite dimensional) central simple algebra A over a field K is of the form $\mathrm{Mat}_n(D)$, where D is a (central) division algebra over K. Moreover the integer n is uniquely determined by A, and D is unique up to isomorphism. The *index* of A is the degree of the division algebra D (see Section 3.3); equivalently, it is the minimal value of $[E : F]$ where E/F is a field extension such that $A_E := A \otimes_F E$ is split over E (i.e. a matrix algebra).

One says that two central simple F-algebras A, A' are *Brauer equivalent* if the associated division algebras are isomorphic. The set of Brauer equivalence classes form a group under tensor product, called the *Brauer group* $\mathrm{Br}(K)$ of

K. By the above, its elements are in bijection with isomorphism classes of K-division algebras. One says that A is *split* if its Brauer class is the trivial class; i.e. $A = \mathrm{Mat}_n(K)$ for some n. This is equivalent to the condition $\mathrm{ind}(A) = 1$. As mentioned in Section 8, the classical theorem of Albert, Brauer, Hasse, and Noether says that if K is a *global* field and if A is a central simple K-algebra, then A is split over K if and only if $A_v := A \otimes_K K_v$ is split over K_v for every completion K_v of K.

Analogously in our situation, with F, $\widehat{X}$, $\mathcal{P}$, and $\mathcal{U}$ as before, we can show the following local-global principle for central simple algebras ([HHK09], Theorem 5.1):

Theorem 11.3. *A central simple F-algebra A is split over F if and only if $A_\xi := A \otimes_F F_\xi$ is split over F_ξ for every $\xi \in \mathcal{P} \sqcup \mathcal{U}$. In fact even more is true:* $\mathrm{ind}(A) = \mathrm{lcm}_{\xi \in \mathcal{P} \sqcup \mathcal{U}} \mathrm{ind}(A_\xi)$.

The proof parallels that of Theorem 11.1. But instead of using the rational connected group $\mathrm{SO}(q)$, we use the group $\mathrm{GL}_1(A)$. If the degree of A is d (i.e. $\dim_F(A) = d^2$), then $\mathrm{GL}_1(A)$ is a Zariski open subset of $\mathbb{A}_F^{d^2}$, and hence it is a rational and connected F-group. There are canonically defined varieties $\mathrm{SB}_i(A)$, known as *generalized Severi-Brauer varieties* of A, on which $\mathrm{GL}_1(A)$ acts transitively. (See [VdB88], p. 334, and [See99], Theorem 3.6.) Using these in place of the hypersurface Q, the argument in the quadratic form case carries over to provide the desired local-global principle for central simple F-algebras. Note that here, unlike in the quadratic forms situation, there is no exception to the principle. That is because here we consider varieties on which a rational connected group acts transitively, whereas in the quadratic forms case the connectivity property can fail for $n = 2$.

As in the case of quadratic forms, the local-global principle for central simple algebras can be used to obtain the values of numerical invariants associated to F that concern the behavior of central simple algebras. If A is a central simple F-algebra, then the Brauer class of A has finite order in $\mathrm{Br}(F)$; this is called the *period* of A. Regardless of the ground field, the period always divides the index, and moreover those two integers are divisible by precisely the same set of primes. Thus for every A there is an integer e such that $\mathrm{ind}(A)$ divides $\mathrm{per}(A)^e$. An important question is whether there is a value of e that works for *all A* (or at least, all A whose period is not divisible by the relevant characteristic). Paralleling the argument in the case of quadratic forms, such uniform values of e can be found in many cases. For example, if $K = \mathbb{Q}_p((t))$ then for algebras of period not divisible by p, we have $e = 2$ for A over K, and $e = 3$ for A over F. As in the situation of quadratic forms, there are also similar results over the fields F_P and F_U. See [HHK09, Section 5], where the above is carried out. See [Lie11] for a different proof in the case of the field F. See also [PS13] for recent results in the case that K is of mixed characteristic $(0, p)$ and the period of the algebra A is a power of p.

12 Complements

12.1 Other local-global set-ups

Above, we have considered local-global principles for one-variable function fields F over a complete discretely valued field K. These have been phrased in terms of a finite set of overfields F_ξ of F, corresponding to a choice of patches on a normal projective model $\widehat{X}$ of F over the valuation ring T of K. While inspired by classical local-global principles for global fields, this set-up is not quite analogous, since in the classical case one takes infinitely many overfields, corresponding to all the completions of the global field. Also, the classical local-global principles do not depend on making a choice, unlike our situation above, where we choose a non-empty finite subset $\mathcal{P}$ of the closed fiber X of $\widehat{X}$, in order to define our set of overfields.

The above framework can be modified, however, to be closer in spirit to the classical situation. We describe two ways to do this.

The first of these begins with a normal model $\widehat{X}$ of F, and considers the set of *all* the fields F_P for $P \in X$. This includes not only the infinitely many closed points of X, but also the finitely many non-closed points of X. These latter points are the generic points η of the (finitely many) irreducible components U of X. (Here F_η is the fraction field of the completion $\widehat{R}_\eta$ of the local ring $R_\eta = \mathcal{O}_{\widehat{X},\eta}$ of $\widehat{X}$ at η.)

In this situation, unlike our prior set-up above, we do not have overfields that would play the role of the fields $F_\wp$. So we cannot ask for patching to hold. But we can still ask for local-global principles. In the case of G-torsors, for G a linear algebraic group over F, this says that the local-global map

$$H^1(F, G) \to \prod_{P \in X} H^1(F_P, G)$$

is injective. Let $\text{Ш}_X(F, G)$ denote the kernel of this map (with the model $\widehat{X}$ being understood). It turns out that if G is rational over F, then $\text{Ш}_X(F, G)$ is naturally isomorphic to $\text{Ш}_\mathcal{P}(F, G)$, for any choice of a finite non-empty subset $\mathcal{P}$ of X that contains all the points where X has more than one branch ([HHK11a, Corollary 5.6]). Thus the obstruction $\text{Ш}_\mathcal{P}(F, G)$ is more canonical than it had appeared to be, as it depends only on the model $\widehat{X}$. Moreover $\text{Ш}_X(F, G)$ can therefore be identified with $\text{Hom}(\pi_1(\Gamma), \bar{G})/\sim$, where the reduction graph Γ is taken with respect to any choice of $\mathcal{P}$ as above (and where $\bar{G} = G/G^0$ as before).

The second modification instead considers the set Ω_F of (equivalence classes of) discrete valuations v on the field F, just as in the classical case of global function fields one considers the set of discrete valuations. For G a linear algebraic group

over F, let $\text{Ш}(F, G)$ be the kernel of the local-global map

$$H^1(F, G) \rightarrow \prod_{v \in \Omega_F} H^1(F_v, G).$$

Then $\text{Ш}(F, G)$ is naturally contained in $\text{Ш}_X(F, G)$ ([HHK11a, Proposition 8.2]), and the question is whether they are equal. In general this is unknown, but it is known in several cases (see [HHK11a, Theorem 8.10]), e.g. if G is rational and the residue field k of T is algebraically closed of characteristic zero. Moreover the local-global principle, in this sense, is known to hold for quadratic forms of dimension greater than two provided that the residue field k is not of characteristic two (see [CPS12], Theorem 3.1), thereby carrying over Theorem 11.1 to this situation. Moreover the local-global principle for central simple algebras, given in the first part of Theorem 11.3, also carries over; see [CPS12, Theorem 4.3(ii)] and [HHK11a, Theorem 9.13]. The second part of Theorem 11.3 also has an analog for discrete valuations, at least in the presence of sufficiently many roots of unity; see [RS13, Theorem 2], whose proof relies on Theorem 11.3.

12.2 Non-rational groups

Although our patching results for torsors do not require that the linear algebraic group is rational, the proofs of the results above concerning local-global principles do require that. There is then the question of whether such results hold more generally. In particular, there is the question of whether local-global principles hold for connected linear algebraic groups that are not rational.

In certain cases where a connected linear algebraic group G over F is not known to be rational, local-global principles have been shown. In particular, this was done for groups of type G_2 in [HHK11a, Example 9.4], by using local-global principles for quadratic forms. This has also been done for various other types of groups by combining cohomological invariants with local-global principles for higher Galois cohomology; see [CPS12, Section 5], [Hu12], and [HHK12, Section 4]. Also, in [Kra10], it was shown that local-global principles hold for connected linear algebraic groups that are *retract rational*, a condition that is strictly weaker than being rational.

These results suggest the possibility that local-global principles might hold for all connected linear algebraic groups over our fields F. But in fact, this is not the case. In [CPS13], examples were obtained of a connected linear algebraic group G over a field F as above, such that the local-global principle for G-torsors fails. In fact, it fails in each of the three settings discussed above: with respect to patches, point on the closed fiber, and discrete valuations.

In light of this, it would be very interesting to find a necessary and sufficient condition on connected linear algebraic groups G over F for local-global principles to hold.

References

[Abh69] Shreeram S. Abhankar. *Resolution of singularities of algebraic surfaces.* In *1969 Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968)*, pp. 1–11, Oxford Univ. Press, London.

[Bir17] George David Birkhoff. *A theorem on matrices of analytic functions.* Math. Ann. 74 (1917), 240–251.

[Bor91] Armand Borel. *Linear Algebraic Groups*, second edition. Graduate Texts in Mathematics, volume 126. Springer-Verlag, New York, Berlin and Heidelberg, 1991.

[BS64] Armand Borel and Jean-Pierre Serre. *Théorèmes de finitude en cohomologie galoisienne.* Comment. Math. Helv. **39** (1964), 111–164.

[Bou72] Nicolas Bourbaki. *Commutative Algebra.* Addison-Wesley Publishing Co., 1972.

[CPS12] Jean-Louis Colliot-Thélène, R. Parimala, and V. Suresh. *Patching and local-global principles for homogeneous spaces over function fields of p-adic curves.* Comment. Math. Helv. **87** (2012), 1011–1033.

[CPS13] Jean-Louis Colliot-Thélène, R. Parimala, and V. Suresh. *Lois de réciprocité supérieures et points rationnels.* 2013 manuscript. Available at arXiv:math/1302.2377.

[Con12] Brian Conrad. *Finiteness theorems for algebraic groups over function fields.* Compos. Math. **148** (2012), 555–639.

[DM69] Pierre Deligne and David Mumford. *The irreducibility of the space of curves of given genus.* Publ. Math. IHES, vol. 36, 75–109, 1969.

[Dou64] Adrien Douady. *Détermination d'un groupe de Galois.* C.R. Acad. Sci. Paris **258** (1964), 5305–5308.

[Fr95] Michael Fried, ed. *Recent Developments in the Inverse Galois Problem*, AMS Contemporary Math. Series, vol. 186, 1995.

[GH78] Phillip Griffiths, Joseph Harris. *Principles of algebraic geometry.* Pure and Applied Mathematics. Wiley-Interscience, New York, 1978.

[Gro59] Alexander Grothendieck. *Géométrie formelle et géométrie algebrique.* Sem. Bourbaki **182** (1959), 1–28.

[Gro61] Alexander Grothendieck. *Élements de géométrie algebrique, III.* Publ. Math. IHES, vol. 11, 1961.

[HJ98] Dan Haran and Moshe Jarden. *Regular split embedding problems over function fields of one variable over ample fields.* J. Algebra **208** (1998), 147–164.

[HJ00] Dan Haran and Moshe Jarden. *The absolute Galois group of $C(x)$.* Pacific J. Math. **196** (2000), 445–459.

[HV96] Dan Haran and Helmut Völklein. *Galois groups over complete valued fields.* Israel J. Math. **93** (1996), 9–27.

[Hrb84] David Harbater. *Convergent arithmetic power series.* Amer. J. Math. **106** (1984), 801–846.

[Hrb87] David Harbater. *Galois covers of the affine line.* In: *Number Theory: New York, 1984-85.* Springer LNM, vol. 1240 (1987), pp. 165–195.

[Hrb94] David Harbater. *Abhyankar's conjecture on Galois groups over curves.* Inventiones Math., **117** (1994), 1–25.

[Hrb95] David Harbater. *Fundamental groups and embedding problems in characteristic p.* In [Fr95], pp. 353–369.

[Hrb03] David Harbater. *Patching and Galois theory.* In [Sch03], pp.313–424.

[HH10] David Harbater and Julia Hartmann. *Patching over fields.* Israel J. Math. **176** (2010), 61–107.

[HHK09] David Harbater, Julia Hartmann, and Daniel Krashen. *Applications of Patching to Quadratic Forms and Central Simple Algebras.* Inventiones Math. **178** (2009), 231–263.

[HHK11] David Harbater, Julia Hartmann, and Daniel Krashen. *Patching subfields of division algebras.* Transactions of the AMS, **363** (2011), 3335–3349.

[HHK11a] David Harbater, Julia Hartmann, and Daniel Krashen. *Local–global principles for torsors over arithmetic curves.* 2011 manuscript. Available at arXiv:math/1108.3323.

[HHK12] David Harbater, Julia Hartmann, and Daniel Krashen. *Local–global principles for Galois cohomology.* 2012 manuscript. To appear in Commentarii Mathematici Helvetici. Available at arXiv:math/1208.6359.

[Hrt05] Julia Hartmann. *On the inverse problem in differential Galois theory.* J. Reine Angew. Math. **586** (2005), 21–44.

[Hrt07] Julia Hartmann. *Patching and differential Galois groups.* In: Arithmetic and Differential Galois Groups, Oberwolfach reports vol. 4, no.2, EMS 2007.

[Hts77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York - Heidelberg, 1977.

[Hu12] Yong Hu. *Hasse principle for simply connected groups over function fields of surfaces*. 2012 manuscript. Available at arXiv:math/1203.1075.

[Iwa53] Kenkichi Iwasawa. *On solvable extensions of algebraic number fields*. Annals of Math. **58** (1953), 548–572.

[Jar95] Moshe Jarden. *On free profinite groups of uncountable rank*. In [Fr95], pp. 371–383.

[Jar11] Moshe Jarden. *Algebraic patching*. Springer Monographs in Mathematics. Springer, Heidelberg, 2011.

[KMRT98] Max-Albert Knus, Alexander S. Merkurjev, Markus Rost, and Jean-Pierre Tignol. *The Book of Involutions*. American Mathematical Society, Providence, RI, 1998.

[Kra10] Daniel Krashen. *Field patching, factorization and local-global principles* (S. Garibaldi, R. Sujatha, and V. Suresh, eds.). In: *Quadratic forms, linear algebraic groups, and cohomology*. Developments in Mathematics, Vol. 18, pp. 57–82, Springer, 2010.

[Lee12] David Leep. *The u-invariant of p-adic function fields*. J. Reine Angew. Math., DOI 10.1515/crelle.2012.029, 2012.

[Lie11] Max Lieblich. *Period and index in the Brauer group of an arithmetic surface*, with an appendix by Daniel Krashen. J. Reine Angew. Math. **659** (2011), 1–41.

[Lip75] Joseph Lipman. *Introduction to resolution of singularities*. In *Algebraic geometry (Proc. Sympos. Pure Math., Vol. 29, Humboldt State Univ., Arcata, Calif., 1974)*, pages 187–230. Amer. Math. Soc., Providence, R.I., 1975.

[Liu95] Qing Liu. *Tout groupe finit est un groupe de Galois sur $\mathbb{Q}_p(T)$*, d'aprés Harbater. In [Fr95], pp. 261–265.

[MP03] B. Heinrich Matzat and Marius van der Put. *Constructive differential Galois theory*. In [Sch03], 425–467.

[PS10] R. Parimala and V. Suresh. *The u-invariant of the function fields of p-adic curves*. Annals of Math. **172** (2010), 1391–1405.

[PS13] R. Parimala and V. Suresh. *Period-index and u-invariant questions for function fields over complete discretely valued fields*. 2013 manuscript. Available at arXiv:math/1304.2214.

[Pie82] Richard S. Pierce. *Associative algebras*. Graduate Texts in Mathematics, No. 88. Springer-Verlag, New York - Heidelberg - Berlin, 1980.

[Pop94] Florian Pop. *Half Riemann Existence Theorem with Galois action.* In: *Algebra and number theory*, G. Frey and J. Ritter, eds., de Gruyter Proceedings in Math. (1994), 1–26.

[Pop95] Florian Pop. *Étale Galois covers of affine smooth curves*. Inventiones Math. **120** (1995), 555–578.

[Ray74] Michel Raynaud. *Géométrie analytique rigide d'aprés Tate, Kiehl,....* Bull. Soc. Math. France, Mémoire 39–40 (1974), 319–327.

[Ray94] Michel Raynaud. *Revêtements de la droite affine en caractéristique $p > 0$ et conjecture d'Abhyankar*. Inventiones Math. **116** (1994), 425–462.

[RS13] B. Surendranath Reddy, V. Suresh. *Admissibility of groups over function fields of p-adic curves*. Adv. Math. **237** (2013), 316–330.

[Sai83] Shuji Saito, *Class field theory for curves over local fields*. J. Number Theory **21** (1985), 44–80.

[Sch68] Murray M. Schacher. *Subfields of division rings. I*. J. Algebra **9** (1968), 451–477.

[Sch03] Leila Schneps, ed. *Galois groups and fundamental groups*. MSRI Publications series, vol. 41, Cambridge University Press, 2003.

[See99] George F. Seelinger. *Brauer-Severi schemes of finitely generated algebras*. Israel J. of Math. **111** (1999), 321–337.

[Ser56] Jean-Pierre Serre. *Géométrie algébrique et géométrie analytique*. Ann. Inst. Fourier, Grenoble **6** (1956), 1–42.

[Ser73] Jean-Pierre Serre. *A Course in Arithmetic*. Graduate Texts in Mathematics, No. 7. Springer-Verlag, New York - Heidelberg - Berlin, 1973.

[Ser00] Jean-Pierre Serre. *Galois Cohomology*. Springer Monographs in Mathematics, Berlin, Heidelberg and New York, 2000.

[Son83] Jack Sonn. $\mathbb{Q}$-*admissibility of solvable groups*. J. Algebra **84** (1983), 411–419.

[Tat71] John Tate. *Rigid analytic spaces*. Inventiones Math. **12** (1971), 257–289.

[TT79] Carol Tretkoff and Marvin Tretkoff. *Solution of the inverse problem of differential Galois theory in the classical case*. Amer. J. Math. **101** (1979), 1327–1332.

[VdB88] Michel Van den Bergh. *The Brauer-Severi scheme of the trace ring of generic matrices.* In: *Perspectives in ring theory (Antwerp, 1987)*, pp. 333–338, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 233, Kluwer Acad. Publ., Dordrecht, 1988.

[Völ96] Helmut Völklein. *Groups as Galois groups. An introduction.* Cambridge Studies in Advanced Mathematics, vol. 53. Cambridge University Press, Cambridge, 1996.

David Harbater

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA

email: harbater@math.upenn.edu