FACULTY OF LAW, ECONOMICS AND FINANCE

CREA Discussion Paper 2015-16

Economics

Center for Research in Economics and Management
University of Luxembourg

On the Characterization of Steady-States in Three-Dimensional Discrete Dynamical Systems

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November, 2015

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On the Characterization of Steady-States in Three-Dimensional Discrete Dynamical Systems

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Abstract: This paper develops a new method to completely characterize the local stability properties of three-dimensional, first-order, nonlinear, autonomous discrete dynamical systems. This is acomplished analytically by means of the characteristic polynomial, i. e., by means of the trace, the determinant, and the sum of principal minors of order two of the Jacobi matrix. The intuition for this method relies on algebraic properties of the (cubic) characteristic polynomial.

Keywords: Non-Linear Discrete Dynamics, Local Stability Properties

JEL-Classification: C02, C62

This Version: December 1, 2015.

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I am grateful to Andreas Irmen, Christos Koulovatianos, Thomas Seegmuller and Benteng Zou for helpful comments and suggestions. Financial support from the University of Luxembourg under the program "Agecon C - Population Aging: An Exploration of its Effect on Economic Performance and Culture" is gratefully acknowledged.

1 Introduction

Many economic problems give rise to nonlinear dynamics for which there is rarely an explicit analytical solution. Instead, the economic theorist often turns to methods allowing to study the qualitative behavior of the dynamical economic system. Under certain conditions a lot may be learned about the dynamic behavior of a nonlinear system in the neighborhood of its steady-state equilibrium by approximating it with a linear system.

The present paper focuses on *three-dimensional, first-order, nonlinear, autonomous systems of difference equations* and proposes a simple method to fully characterize their local stability properties in the proximity of a steady-state equilibrium. The notion of a steady-state equilibrium is the mathematical counterpart of the notion of a long-run equilibrium in macroeconomic dynamical models: roughly speaking, a long-run equilibrium refers to a state that the economy tends to asymptotically and in which the variables either do not change or change only at a constant rate.

An important question is whether there are forces in the economy that tend to push the dynamical system back toward its steady-state equilibrium if the latter is perturbed, *i. e.*, whether the steady-state equilibrium is locally stable. To determine the local stability properties around the steady-state equilibrium of a nonlinear system one may use its linear approximation and study the roots of the associated characteristic equation. More precisely, the stability properties depend on the relative magnitude of the roots with respect to the unit circle. The equilibrium of any autonomous system of difference equations is locally stable (a sink) if all roots of the associated characteristic polynomial have modulus less than unity. Strictly speaking, if at least one root has modulus greater than unity, the system is unstable. It is called an unstable node (a source) if all roots have modulus greater than unity. The steady-state equilibrium is referred to as a saddle point if one or more but not all roots have modulus greater than unity.[2]

The above suggests that we do not need to solve for the roots of the characteristic equation explicitly. This is a great advantage since the coefficients of the characteristic polynomial often consist of unknown parameters of the underlying economic model. Rather, what we require is a way to determine the magnitude of the roots relative to the unit circle without having to solve explicitly for the roots of the characteristic polynomial.

The new method proposed in the present paper exploits algebraic properties of the characteristic equation, to fully characterize the local stability properties of a steady-state equilibrium. The paper builds on and extends previous work of Brooks (2004) who develops conditions for the system to be a sink. Similar to the well-known case of two-dimensional systems, we show how to fully characterize local stability of three-dimensional, first-order, nonlinear, autonomous systems by means of inequality conditions in terms of the trace, the sum of principal minors of order two and the determinant

[2]A precise classification of local stability properties is given in Section 2.

of the associated Jacobian matrix. These conditions allow to determine the position of the roots of the characteristic polynomial relative to the unit circle without having to solve the characteristic polynomial.[3]

In particular, the paper presents simple conditions to characterize saddle-point equilibria which arise frequently in economic dynamics.[4] Systems with the saddle-point property are in general dominated by the unstable root(s) and move away from the steady-state equilibrium. This is why they are unstable according to mathematical definition. However, there is some initial point from which those systems converge to the steady-state equilibrium. In many economic problems it is possible to place the system on precisely this initial point since not all initial values of the variables of interest are predetermined. While the initial value of state variables is predetermined by the history of the economy, the initial value of choice (jump) variable(s) may be discontinuously chosen. If there are as many jump variables as there are unstable roots, the values of the jump variables may be chosen so as to place the system on the unique stable path toward the steady state.[5] Therefore, contrary to mathematical definition, many economists have come to refer to economic models that feature saddle-point equilibria as being "saddle-path stable".[6] In light of this, having simple conditions which tell whether a unique convergent path, multiple paths, or no saddle-path exists is desirable.

The merits of the new method proposed here are threefold. First, it is simple as it requires no knowledge of advanced mathematics. Second, the underlying intuition is easy to grasp as it builds on the well-known characterization of two-dimensional systems. Third, as a consequence it has a relatively low calculatory demand.

The remainder of this paper is organized as follows. Section 2 outlines some important concepts. In particular, we define the notion of a steady-state equilibrium and provide

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[3]The method developed in the present paper is also complementary to Barinci and Drugeon (2005), who are interested in determinacy properties of three-dimensional discrete time systems. While these authors develop a geometric intuition for determinacy, which extends the well-known geometric representation of two-dimensional systems, the present method and its intuitive appeal are based on algebraic properties of the cubic characteristic equation.

[4]Examples of dynamic (macro-) economic models described by three difference equations include two-sector AK models of economic growth, growth models with habit formation à la catching-up with the Joneses (see e. g. Alonso-Carrera, Caballé, and Raurich (2005)), or the neoclassical growth model with capital- and labor-augmenting technical change (Irmen and Tabaković (2015)).

[5]If, however, there are fewer unstable roots than jump variables, it is not possible to determine a unique initial point in the state space from which the economy embarks on a convergent path. Roughly speaking, one (or more) jump variable remains "free" to take any value. Hence, in such a case there are infinitely many paths toward steady-state equilibrium, a feature which is not compatible with the rational expectations assumption in many economic models. If, on the other hand, there are more unstable roots than jump variables, the steady-state equilibrium is unstable and there is no point in the state space from which the system would converge.

[6]According to Azariadis (1993), p. 40, "Saddlepath solutions often make good economic sense... Actual economies, after all, rarely display "explosive" behavior."

a classification of its local stability properties. Section 3 contains the main result of the paper and develops the intuition for the proposed method. Section 4 concludes. All proofs are contained in Section 5, the Appendix.

2 Three-Dimensional Systems: Steady-State Equilibrium and Classification of Local Stability Properties

Consider the three-dimensional system of autonomous, nonlinear, first-order difference equations, where the evolution of the vector of state variables, x_t, is governed by the nonlinear system

$$x_{t+1} = \Phi(x_t), \quad t = 0, 1, 2, \cdots, \tag{2.1}$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^3$. More precisely,

$$\begin{aligned} x_{1,t+1} &= \phi^1(x_{1,t}, x_{2,t}, x_{3,t}) \\ x_{2,t+1} &= \phi^2(x_{1,t}, x_{2,t}, x_{3,t}) \\ x_{3,t+1} &= \phi^3(x_{1,t}, x_{2,t}, x_{3,t}), \end{aligned}$$

where $\phi^i : \mathbb{R}^3 \to \mathbb{R}, i = 1, 2, 3$, is a continuously differentiable, nonlinear and time-independent (autonomous) function, and the initial value of the vector of state variables, $x_0 = (x_{1,0}, x_{2,0}, x_{3,0})$, is given.

A solution to system (2.1) is a trajectory of the vector of state variables, $\{x_t\}_{t=0}^{\infty}$ such that the law of motion is satisfied at any time t. A value of the vector of state variables, x_t, that, if attained, is invariant, i. e., sustained forever, under the law of motion, is called a steady-state equilibrium (stationary equilibrium, rest point, fixed point, or period-1 solution) of the system.

Definition 1 *(Steady-State Equilibrium) Consider the discrete time, nonlinear dynamical system (2.1). A steady-state equilibrium of (2.1) is a vector $\bar{x} \in \mathbb{R}^3$ such that*

$$\bar{x} = \Phi(\bar{x}).$$

If the system is in a steady state and upon a sufficiently small perturbation it converges asymptotically back to this steady-state equilibrium, then this equilibrium is locally stable. Suppose $\bar{x}$ is the unique steady-state equilibrium of system (2.1).[7] In order to characterize the local stability properties of the above system we need to linearly approximate it in the vicinity of the steady-state equilibrium, $\bar{x}$, which yields

[7] In general, there may be one steady-state equilibrium, many steady-state equilibria, or no steady-state equilibrium associated with a given nonlinear dynamical system.

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \end{bmatrix} = \begin{bmatrix} \frac{\partial \phi^1(\bar{x})}{\partial x_{1,t}} & \frac{\partial \phi^1(\bar{x})}{\partial x_{2,t}} & \frac{\partial \phi^1(\bar{x})}{\partial x_{3,t}} \\ \frac{\partial \phi^2(\bar{x})}{\partial x_{1,t}} & \frac{\partial \phi^2(\bar{x})}{\partial x_{2,t}} & \frac{\partial \phi^2(\bar{x})}{\partial x_{3,t}} \\ \frac{\partial \phi^3(\bar{x})}{\partial x_{1,t}} & \frac{\partial \phi^3(\bar{x})}{\partial x_{2,t}} & \frac{\partial \phi^3(\bar{x})}{\partial x_{3,t}} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} + \begin{bmatrix} \phi^1(\bar{x}) - \sum_{j=1}^{3} \frac{\partial \phi^1(\bar{x})}{\partial x_{j,t}} \bar{x}_j \\ \phi^2(\bar{x}) - \sum_{j=1}^{3} \frac{\partial \phi^2(\bar{x})}{\partial x_{j,t}} \bar{x}_j \\ \phi^3(\bar{x}) - \sum_{j=1}^{3} \frac{\partial \phi^3(\bar{x})}{\partial x_{j,t}} \bar{x}_j \end{bmatrix}.$$

Thus, the linear approximation to the nonlinear system in the vicinity of the steady-state equilibrium may be written as

$$x_{t+1} = Jx_t + Z, \tag{2.2}$$

where $J = D\Phi(\bar{x})$ is the Jacobian matrix of $\Phi(x_t)$ evaluated at steady-state equilibrium, $\bar{x}$, and Z is the constant column vector.

Let the three eigenvalues of the Jacobian matrix be given by λ_i, where $i = 1, 2, 3$. To find the eigenvalues we must find the solution to the associated characteristic polynomial given by

$$c(\lambda) \equiv \lambda^3 - a_1\lambda^2 + a_2\lambda - a_3 = 0, \tag{2.3}$$

where the coefficients a_1, a_2, a_3 are real numbers. Moreover, one can show that

$$\begin{aligned} a_1 &= tr(J), \\ a_2 &= \sum M_2(J), \\ a_3 &= det(J), \end{aligned}$$

where $tr(J), \sum M_2(J)$, and $det(J)$ denote, respectively, the trace, the sum of principal minors of order two, and the determinant of the Jacobian matrix, J. Thus, the eigenvalues of the Jacobian matrix are the roots of the characteristic polynomial $c(\lambda)$, and we will, henceforth, use the two interchangeably.

To rule out a Jacobi matrix that has eigenvalues with modulus one we invoke the concept of a hyperbolic steady-state equilibrium.

Definition 2 *(Hyperbolic Steady-State Equilibrium) The steady-state equilibrium, $\bar{x}$, is hyperbolic if none of the eigenvalues of $J = D\Phi(\bar{x})$ lies on the unit circle, i.e., if $|\lambda_i| \neq 1$, for $i = 1, 2, 3$.*

Given a hyperbolic steady-state equilibrium, the Hartman-Grobman Theorem states that we may study the local behavior of the linearly approximated system (2.2) to assess the local behavior of the nonlinear system (2.1) in the proximity of its steady-state equilibrium.[8]

[8] For a proof see Hartman (1964).

Hence, to characterize the local behavior of the nonlinear system in the proximity of the steady-state equilibrium we need to determine the magnitude of the eigenvalues of the Jacobian matrix relative to the unit circle. Before we move on to that task it is useful to classify the local stability properties of the nonlinear system (2.1) in terms of the magnitude of its eigenvalues.

Definition 3 *(Local Stability when all Eigenvalues are Real) Consider the discrete, nonlinear dynamical system in (2.1) with a steady-state equilibrium, $\bar{x}$. The linearized system is given by (2.2). The associated Jacobian matrix has three real eigenvalues λ_i ($i = 1, 2, 3$).*

(i) *The steady-state equilibrium $\bar{x}$ is called a stable node if $|\lambda_i| < 1$ for all $i = 1, 2, 3$.*

(ii) *The steady-state equilibrium $\bar{x}$ is called a two-dimensional saddle if one $|\lambda_i| > 1$.*

(iii) *The steady-state equilibrium $\bar{x}$ is called a one-dimensional saddle if one $|\lambda_i| < 1$.*

(iv) *The steady-state equilibrium $\bar{x}$ is called an unstable node if $|\lambda_i| > 1$ for all $i = 1, 2, 3$.*

Definition 4 *(Local Stability with Complex Eigenvalues) Consider the discrete, nonlinear dynamical system in (2.1) with a steady-state equilibrium, $\bar{x}$. The linearized system is given by (2.2). The associated Jacobian matrix has a pair of complex eigenvalues $\lambda_{1/2} = \rho \pm \omega i$ and one real eigenvalue λ_3.*

(i) *The steady-state equilibrium $\bar{x}$ is called a sink if $|\lambda_i| < 1$ for all $i = 1, 2, 3$.*

(ii) *The steady-state equilibrium $\bar{x}$ is called a two-dimensional saddle if one $|\lambda_3| > 1$.*

(iii) *The steady-state equilibrium $\bar{x}$ is called a one-dimensional saddle if one $|\lambda_3| < 1$.*

(iv) *The steady-state equilibrium $\bar{x}$ is called a source if $|\lambda_i| > 1$ for all $i = 1, 2, 3$.*

3 Characterization of Local Stability in 3D Discrete Systems

This section develops necessary and sufficient conditions for the characterization of local stability properties of three-dimensional systems as defined in Definitions 3 and 4 in terms of $tr(J)$, $\sum M_2(J)$ and $det(J)$. The intuition comes from the well-known case of two-dimensional systems. In the latter, if the eigenvalues of the Jacobian matrix are real and distinct, analytical conditions for the various types of stability of the steady-state equilibrium are obtained by evaluating the associated characteristic polynomial at $\lambda = \pm 1$. If, on the other hand, the system possesses complex eigenvalues, which occur if the discriminant of the characteristic polynomial is smaller than zero, the stability properties of the steady-state equilibrium may be inferred from the value of the determinant of the Jacobian matrix. Since the product of the eigenvalues is equal to the determinant of the

Jacobian matrix, $det(\tilde{J}) \lessgtr 1$ tells whether the system is a sink or a source, respectively.[9] We extend this approach to a system of three difference equations. To do that we will make use of Descartes' rule of signs and we will introduce an auxiliary characteristic equation, to be specified shortly.

3.1 An Intuitive Approach to Three-Dimensional Systems

For the purpose of gaining intuition it is useful to recall some basic properties of cubic equations. To that end consider the characteristic polynomial (2.3) of the linearized three-dimensional system which we restate for convenience

$$c(\lambda) \equiv \lambda^3 - tr(J)\lambda^2 + \sum M_2(J)\lambda - det(J). \tag{2.3}$$

Notice first that all cubic equations have either one real root, or three real roots. With a positive coefficient of λ^3 the function $c(\lambda)$ goes to negative infinity as $\lambda \to -\infty$ and to positive infinity as $\lambda \to \infty$. Therefore, by continuity, the graph of a cubic equation must cross the horizontal axis at least once giving at least one real root.

Evaluation of (2.3) at $\lambda = \pm 1$ will, in general, not determine the magnitude of all roots. Roughly speaking, this is because evaluating a cubic equation like (2.3) at $\lambda = \pm 1$ delivers two reference points while the equation has three roots. Therefore, there is typically not sufficient information to infer the position of all three eigenvalues relative to the interval $(-1, 1)$. To illustrate this consider the following example.

Example 1: (Characteristic Polynomial of a Three-Dimensional, First-Order Discrete Dynamical System)

Consider a linearly approximated nonlinear three-dimensional, first-order discrete dynamical system with associated characteristic polynomial,

$$c(\lambda) = \lambda^3 - tr(J)\lambda^2 + \sum M_2(J)\lambda - det(J) = 0,$$

and suppose the coefficients are such that the characteristic polynomial possesses two positive and one negative root. The crucial point to observe is that the two positive roots may come in three combinations. Both might be smaller than 1*, or both might be larger than* 1*, or one might be smaller than* 1 *and the other larger. Because there is only one negative root, clearly, evaluating* $c(\lambda)$ *at* -1 *will tell us whether the negative root has magnitude greater or smaller than* -1*. On the other hand, because there are two positive roots evaluating* $c(\lambda)$ *at* 1 *does not necessarily provide clear-cut information. To highlight the issue consider the three Panels in Figure 3.1 which depict the three possible cases with respect to the combination of the positive roots. In all three cases the negative root is chosen, without loss of generality, to have magnitude*

[9] For a detailed characterization of the qualitative local stability properties of planar systems see, e. g., Azariadis (1993) or Galor (2007).

greater than -1*. Thus, what we expect to see is that* $c(-1) < 0$ *which proves true upon a glance at each of the figures. Turning to the positive roots, and evaluating the characteristic polynomial at* 1 *we notice that it delivers a definite result if and only if one of the positive roots has magnitude smaller than* 1 *and the other larger than* 1*. Panel* 2 *of Figure 3.1 illuminates this case. If the characteristic polynomial possesses one negative and two positive real roots, then* $c(1) < 0$ *implies unambiguously that one of the positive roots is smaller than* 1 *and the other is larger. In contrast, if both positive roots are smaller or larger than* 1*, then* $c(1) > 0$ *and we cannot infer whether the roots are smaller or larger than* 1*. This case is illustrated in Panels* 1 *and* 3 *of Figure 3.1. In the same vein, one may gain intuition for why the information contained*

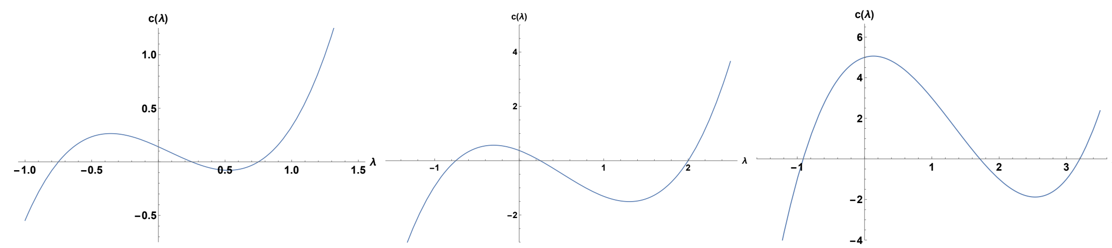

Figure 3.1: Characteristic equation of a linear three-dimensional, first-order discrete dynamical system. Coefficient values are chosen such that it possesses two positive and one negative root.

in the characteristic polynomial alone does not suffice for all other possible sign combinations of the real roots.

Moreover, when faced with a cubic equation there is no easy way to tell whether it possesses only real roots or not. One may evaluate the associated discriminant, which in case of equation (2.3) is given by

$$\begin{aligned}\Delta =& 18tr(J)\sum M_2(J)det(J) - 4(tr(J))^3det(J) + (tr(J))^2\left(\sum M_2(J)\right)^2 \\ & - 4\left(\sum M_2(J)\right)^3 - 27\left(det(J)\right)^2.\end{aligned}$$

If $\Delta > 0$, the cubic equation has three distinct real roots, if $\Delta = 0$ it has a multiple root and all roots are real, and if $\Delta < 0$, the equation has two complex roots and one real root. Knowing that $tr(J)$, $\sum M_2(J)$, and $det(J)$ usually all consist of a combination of unknown elements of the Jacobian matrix it will, in general, not be possible to sign Δ.[10]

In order to pin down the qualitative properties of the linearized system (2.2) we require additional sources of information. More precisely, we will use two additional sources of information: Descartes' rule of signs and an *auxiliary characteristic polynomial*, which we present in turn.

Descartes' rule of signs allows us to pin down the signs of the real root(s). It turns out to be useful to know, in case all roots are real, whether the characteristic equation

[10]Compare this to the discriminant $\Delta = (tr(J))^2 - 4det(J)$ associated with the characteristic polynomial of a two-dimensional system which is disproportionately easier to sign.

admits only roots of the same sign or of opposite signs. Descartes' rule of signs says that if the terms of a polynomial with real coefficients are ordered by descending variable exponent, then the number of positive roots of the polynomial is at most equal to the number of sign changes between consecutive nonzero coefficients, or is less than it by an even number. Moreover, the number of negative roots is at most equal to the number of continuations in the signs of the coefficients.[11]

Applying this rule to the characteristic polynomial (2.3) delivers the following possibilities:

1. either all roots are real and positive, or there is one real positive root and a pair of complex roots if and only if

$$tr(J) > 0, \quad \sum M_2(J) > 0, \quad \text{and} \quad det(J) > 0,$$

2. either all roots are real and negative, or there is one real negative root and a pair of complex roots if and only if

$$tr(J) < 0, \quad \sum M_2(J) > 0, \text{and} \quad det(J) < 0,$$

3. one real positive and the remaining two either real negative or a pair of complex roots occurs if either

$$tr(J) < 0, \quad \sum M_2(J) > 0, \quad \text{and} \quad det(J) > 0$$

or,

$$tr(J) > 0, \quad \sum M_2(J) < 0, \quad \text{and} \quad det(J) > 0,$$

or,

$$tr(J) < 0, \quad \sum M_2(J) < 0, \quad \text{and} \quad det(J) > 0,$$

4. one real negative root and the remaining two either real positive or a pair of complex roots occurs if either

$$tr(J) < 0, \quad \sum M_2(J) < 0, \quad \text{and} \quad det(J) < 0,$$

or,

$$tr(J) > 0, \quad \sum M_2(J) > 0, \quad \text{and} \quad det(J) < 0,$$

or,

$$tr(J) > 0, \quad \sum M_2(J) < 0, \quad \text{and} \quad det(J) < 0.$$

[11] See, e. g., Gandolfo (2009).

The second, crucial, source of information is an *auxiliary characteristic polynomial*. To obtain the *auxiliary characteristic polynomial* we make use of the connection between the eigenvalues of the Jacobian matrix and the coefficients in the associated characteristic polynomial

$$tr(J) = \lambda_1 + \lambda_2 + \lambda_3, \tag{3.1}$$

$$\sum M_2(J) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \tag{3.2}$$

$$det(J) = \lambda_1\lambda_2\lambda_3. \tag{3.3}$$

Notice that equations (3.1)-(3.3) form a system of three equations which may be solved for any product pair of eigenvalues $\lambda_i\lambda_j$, with $i = 1,2,3$, $j = 1,2,3$, and $i \neq j$, to result in the following Lemma:

Lemma 1 *(Auxiliary Characteristic Polynomial)*

The solution of equations (3.1)-(3.3) for any pair of eigenvalues $\lambda_i\lambda_j$, with $i = 1,2,3$, $j = 1,2,3$, and $i \neq j$, defines the auxiliary characteristic polynomial:

$$\mathcal{C}(\mu) \equiv \mu^3 - \sum M_2(J)\mu^2 + det(J)tr(J)\mu - [det(J)]^2 = 0, \tag{3.4}$$

where $\mu \equiv \lambda_i\lambda_j$, with $i = 1,2,3$, $j = 1,2,3$, and $i \neq j$.

Thus, by construction, the three roots of (3.4) are the three product pairs of the three roots of the characteristic polynomial (2.3), say $\mu_1 = \lambda_1\lambda_2$, $\mu_2 = \lambda_1\lambda_3$, and $\mu_3 = \lambda_2\lambda_3$. Notice that this is indispensable for determining the magnitude of all three roots, irrespective of them being all real or not.[12]

Descartes' rule of signs, the characteristic polynomial (2.3) and the *auxiliary characteristic polynomial* (3.4) are all we need to determine the qualitative properties of the steady-state equilibrium of the three-dimensional system (2.1) irrespective of whether the roots of the characteristic polynomial are real or complex. To illustrate how the *auxiliary characteristic polynomial* may be used to analyze the stability properties of a steady-state equilibrium we apply it to the previous example.

Example 2 (ctd.) *Recall that we consider a system that possesses only real roots, two of which are positive. Moreover, consider the situation in which the positive roots are either both larger or both smaller than 1. Let λ_2 and λ_3 be the positive roots of $c(\lambda)$. Since the three roots of $\mathcal{C}(\mu)$ are the product pairs of the roots of $c(\lambda)$ we know that only one root of $\mathcal{C}(\mu)$ is positive, say μ_3. Moreover, using the fact that λ_2 and λ_3 are both either greater or smaller than one, we know that $\mu_3 < 1$ if and only if $\lambda_{2,3} < 1$, and, similarly, $\mu_3 > 1$ if and only if $\lambda_{2,3} > 1$. Hence, $\mathcal{C}(1) \gtrless 0$ if and only if $\mu_3 \lessgtr 1$, and, consequently, the auxiliary characteristic polynomial provides the additional information we need to characterize the stability of a steady-state equilibrium.*

[12]See also Brooks (2004).

3.2 Local Stability in 3D Discrete Systems

We are now ready to state the main results of the paper as Propositions 1-4.

Proposition 1 *(Real and Distinct Eigenvalues of Same Sign) Consider the linearized discrete three-dimensional dynamical system (2.2). If the associated eigenvalues are real, distinct and have the same sign, i.e., if either*

$$tr(J) > 0, \sum M_2(J) > 0, det(J) > 0$$

or

$$tr(J) < 0, \sum M_2(J) > 0, det(J) < 0,$$

then a steady-state equilibrium is a:

(i) Stable Node (i.e. $\mid \lambda_i \mid < 1$, $i = 1, 2, 3$)

if and only if

$$c(1) > 0, \quad c(-1) < 0, \quad \mathcal{C}(1) > 0, \quad and \quad det(J) < 1$$

(ii) Two-Dimensional Saddle (i.e. one $\mid \lambda_i \mid > 1$, $i = 1, 2, 3$)

if

$$c(1) < 0, \quad c(-1) > 0, \quad and \quad \mathcal{C}(1) > 0,$$

or

$$c(1) < 0, \quad c(-1) > 0, \quad \mathcal{C}(1) < 0, \quad and \quad det(J) < 1$$

(iii) One-Dimensional Saddle (i.e. one $\mid \lambda_i \mid < 1$, $i = 1, 2, 3$)

if and only if

$$c(1) > 0, \quad c(-1) < 0, \quad and \quad \mathcal{C}(1) < 0,$$

or

$$c(1) > 0, \quad c(-1) < 0, \quad \mathcal{C}(1) > 0, \quad and \quad det(J) > 1$$

(iv) Unstable Node (i.e. $\mid \lambda_1 \mid > 1$, $i = 1, 2, 3$)

if and only if

$$c(1) < 0, \quad c(-1) > 0, \quad \mathcal{C}(1) < 0, \quad and \quad det(J) > 1$$

Proposition 2 *(One Real and a Pair of Complex Eigenvalues) Consider the linearized discrete three-dimensional dynamical system (2.2). If the associated eigenvalues consist of one real, and a pair of complex eigenvalues, which is possible in any configuration of $tr(J), \sum M_2(J), det(J)$, then a steady-state equilibrium is a:*

(i) Stable Node (i. e. $\mid \lambda_i \mid < 1$, $i = 1, 2, 3$)

$$\textit{if and only if}$$
$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) > 0$$

(ii) Two-Dimensional Saddle (i. e. one $\mid \lambda_i \mid > 1$, $i = 1, 2, 3$)

$$\textit{if}$$
$$c(1) < 0, \quad c(-1) > 0, \quad \textit{and} \quad \mathcal{C}(1) > 0,$$

(iii) One-Dimensional Saddle (i. e. one $\mid \lambda_i \mid < 1$, $i = 1, 2, 3$)

$$\textit{if and only if}$$
$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) < 0,$$

(iv) Unstable Node (i. e. $\mid \lambda_1 \mid > 1$, $i = 1, 2, 3$)

$$\textit{if and only if}$$
$$c(1) < 0, \quad c(-1) > 0, \quad \mathcal{C}(1) < 0, \quad \textit{and} \quad det(J) > 1$$

Proposition 3 *(Real and Distinct Eigenvalues of Different Sign - Two Positive, One Negative) Consider the linearized discrete three-dimensional dynamical system (2.2). If the associated eigenvalues are real and distinct of which two are positive and one is negative, i. e., if either*

$$tr(J) > 0, \sum M_2(J) > 0, det(J) < 0,$$
$$\textit{or,}$$
$$tr(J) > 0, \sum M_2(J) < 0, det(J) < 0,$$
$$\textit{or,}$$
$$tr(J) < 0, \sum M_2(J) < 0, det(J) < 0,$$

then a steady-state equilibrium is a:

(i) Stable Node (i. e. $\mid \lambda_i \mid < 1$, $i = 1, 2, 3$)

$$\textit{if and only if}$$
$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) > 0$$

(ii) *Two-Dimensional Saddle (i. e. one* $|\lambda_i| > 1$, $i = 1, 2, 3$*)*

if and only if

$$c(1) > 0, \quad c(-1) > 0, \quad \textit{and} \quad \mathcal{C}(1) > 0,$$

or

$$c(1) < 0, \quad \textit{and} \quad c(-1) < 0$$

(iii) *One-Dimensional Saddle (i. e. one* $|\lambda_i| < 1$, $i = 1, 2, 3$*)*

if and only if

$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) < 0,$$

or

$$c(1) < 0, \quad \textit{and} \quad c(-1) > 0$$

(iv) *Unstable Node (i. e.* $|\lambda_1| > 1$, $i = 1, 2, 3$*)*

if and only if

$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) > 0$$

Proposition 4 *(Real and Distinct Eigenvalues of Different Sign - Two Negative, One Positive) Consider the linearized discrete three-dimensional dynamical system (2.2). If the associated eigenvalues are real and distinct of which two are negative and one is positive, i. e., if either*

$$tr(J) > 0, \sum M_2(J) < 0, det(J) > 0,$$

or

$$tr(J) < 0, \sum M_2(J) > 0, det(J) > 0,$$

or,

$$tr(J) < 0, \sum M_2(J) < 0, det(J) > 0,$$

then a steady-state equilibrium is a:

(i) *Stable Node (i. e.* $|\lambda_i| < 1$, $i = 1, 2, 3$*)*

if and only if

$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) > 0$$

(ii) Two-Dimensional Saddle (i.e. one $|\lambda_i| > 1$, $i = 1, 2, 3$*)*

$$\textit{if and only if}$$
$$c(1) < 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) > 0,$$
$$\textit{or}$$
$$c(1) > 0, \quad \textit{and} \quad c(-1) > 0$$

(iii) One-Dimensional Saddle (i.e. one $|\lambda_i| < 1$, $i = 1, 2, 3$*)*

$$\textit{if and only if}$$
$$c(1) > 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) < 0,$$
$$\textit{or}$$
$$c(1) < 0, \quad \textit{and} \quad c(-1) > 0$$

(iv) Unstable Node (i.e. $|\lambda_1| > 1$, $i = 1, 2, 3$*)*

$$\textit{if and only if}$$
$$c(1) < 0, \quad c(-1) < 0, \quad \textit{and} \quad \mathcal{C}(1) < 0$$

4 Concluding Remarks

The qualitative analysis of nonlinear dynamical systems for which there is no explicit solution is an important source of information about their dynamical behavior. One may study the local stability properties of such systems based on their linear approximation in the neighborhood of a steady-state equilibrium. The stability properties depend on how many roots of the associated characteristic polynomial lie within the unit circle.

The present paper develops a characterization of local stability of three-dimensional systems in terms of $tr(J)$, $\sum M_2(J)$ and $det(J)$. The underlying idea is intuitive as it draws upon general properties of cubic functions and builds on the existing method of characterizing the qualitative properties of two-dimensional (planar) systems in terms of $tr(J)$ and $det(J)$.

Several other methods have been developed to determine the stability of discrete dynamical systems relying on inequalities that involve the coefficients of the characteristic polynomial, such as those by Schur (1917) and Cohn (1986) (the Schur-Cohn conditions), as well as those introduced by Samuelson (1941).[13] By contrast, Bistritz (1984) introduces

[13]For the statement of these conditions for three-dimensional (and, more generally, n-dimensional) systems consult Gandolfo (2009). Notice that for three-dimensional systems the Schur-Cohn and the Samuelson conditions are equivalent, as has been shown by Okuguchi and Irie (1990).

a method based on a sequence of symmetric polynomials of descending degrees for the characteristic polynomial. The sign variation pattern of the symmetric polynomials in the sequence then determines the relative magnitude of the roots with respect to the unit circle.

Nevertheless, for economic problems that give rise to three-dimensional systems it may, at times, be exceedingly complicated to determine stability properties using the thus far known methods. The reason is that coefficients of the characteristic polynomial tend to be composed of several unknown parameters and reduced forms of the underlying economic model, which makes transformations of any kind calculatory very demanding. Our paper provides for an alternative, less complicated solution for three-dimensional systems. In this respect it adds an additional tool to the toolbox of the economic theorist studying discrete dynamical systems.

5 Appendix: Proofs

5.1 Proof of Lemma 1

Consider equations 3.1-3.3 and notice that

$$(\lambda_1\lambda_2\lambda_3 - det(J))\,(\lambda_1+\lambda_2+\lambda_3 - tr(J))\,\left(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 - \sum M_2(J)\right) = 0.$$

Our goal is to obtain a cubic equation in a product pair of the eigenvalues λ_i, $i=1,2,3$. W.l.o.g. let us consider the product $\lambda_2\lambda_3$.

Calculating the product of the terms in the first two parentheses and multiplying with the third delivers

$$\begin{aligned}
0 &= (\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3) - (\lambda_2\lambda_3)\,\lambda_1\lambda_2\lambda_3 tr(J) - (\lambda_2\lambda_3)\,det(J)\,(\lambda_1+\lambda_2+\lambda_3) + (\lambda_2\lambda_3)\,det(J)tr(J)\\
&+ (\lambda_1\lambda_2)\,\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3) - (\lambda_1\lambda_2)\,\lambda_1\lambda_2\lambda_3 tr(J) - (\lambda_1\lambda_2)\,det(J)\,(\lambda_1+\lambda_2+\lambda_3) + (\lambda_1\lambda_2)\,det(J)tr(J)\\
&+ (\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3) - (\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3 tr(J) - (\lambda_1\lambda_3)\,det(J)\,(\lambda_1+\lambda_2+\lambda_3) + (\lambda_1\lambda_3)\,det(J)tr(J)\\
&-\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3)\sum M_2(J) + \lambda_1\lambda_2\lambda_3 tr(J)\sum M_2(J) + det(J)\,(\lambda_1+\lambda_2+\lambda_3)\sum M_2(J)\\
&-det(J)tr(J)\sum M_2(J)
\end{aligned}$$

The above equation may be simplified as follows. Notice that the first row may be simplified to obtain

$$(\lambda_2\lambda_3)^2\,[\lambda_1\,(\lambda_1+\lambda_2+\lambda_3)] - \lambda_2\lambda_3\,(\lambda_1\lambda_2\lambda_3)\,(\lambda_1+\lambda_2+\lambda_3)\,.$$

Moreover, the 2nd and the 3rd row may be reduced to

$$\begin{aligned}
&(\lambda_1\lambda_2+\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3) - (\lambda_1\lambda_2+\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3 tr(J)\\
&- (\lambda_1\lambda_2+\lambda_1\lambda_3)\,det(J)\,(\lambda_1+\lambda_2+\lambda_3) + (\lambda_1\lambda_2+\lambda_1\lambda_3)\,det(J)tr(J).
\end{aligned}$$

Next, the last row may be written as

$$det(J)tr(J)\sum M_2(J) = (det(J))^2 + det(J)\,(\lambda_1+\lambda_2+\lambda_3)\,(\lambda_1\lambda_2\lambda_3) + det(J)\,(\lambda_2+\lambda_3)\,\lambda_2\lambda_3.$$

Combining the above delivers

$$\begin{aligned}
0 &= (\lambda_2\lambda_3)^2\,[\lambda_1\,(\lambda_1+\lambda_2+\lambda_3)] - \lambda_2\lambda_3\,(\lambda_1\lambda_2\lambda_3)\,(\lambda_1+\lambda_2+\lambda_3)\\
&+ (\lambda_1\lambda_2+\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3) - (\lambda_1\lambda_2+\lambda_1\lambda_3)\,\lambda_1\lambda_2\lambda_3 tr(J)\\
&- (\lambda_1\lambda_2+\lambda_1\lambda_3)\,det(J)\,(\lambda_1+\lambda_2+\lambda_3) + (\lambda_1\lambda_2+\lambda_1\lambda_3)\,det(J)tr(J)\\
&-\lambda_1\lambda_2\lambda_3\,(\lambda_1+\lambda_2+\lambda_3)\sum M_2(J) + \lambda_1\lambda_2\lambda_3 tr(J)\sum M_2(J) + det(J)\,(\lambda_1+\lambda_2+\lambda_3)\sum M_2(J)\\
&- (det(J))^2 - det(J)\,(\lambda_1+\lambda_2+\lambda_3)\,(\lambda_1\lambda_2+\lambda_1\lambda_3) - det(J)\,(\lambda_2+\lambda_3)\,\lambda_2\lambda_3.
\end{aligned}$$

Cancelling some terms yields

$$
\begin{aligned}
0 = & (\lambda_2\lambda_3)^2 \left[\lambda_1 \left(\lambda_1 + \lambda_2 + \lambda_3\right)\right] \\
& -\lambda_2\lambda_3 \left(\lambda_1\lambda_2\lambda_3\right) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& -\left(\lambda_1\lambda_2 + \lambda_1\lambda_3\right) det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& +det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \sum M_2(J) \\
& -\left(det(J)\right)^2 - det(J) \left(\lambda_2 + \lambda_3\right) \lambda_2\lambda_3.
\end{aligned}
$$

Next, we add to and subtract from the above equation $(\lambda_2\lambda_3)^3$ and rearrange terms in the first row to obtain

$$
\begin{aligned}
0 = & -(\lambda_2\lambda_3)^3 + (\lambda_2\lambda_3)^2 \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) + \left(\lambda_1\lambda_2\lambda_3\right)^2 \\
& -\lambda_2\lambda_3 \left(\lambda_1\lambda_2\lambda_3\right) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& -\left(\lambda_1\lambda_2 + \lambda_1\lambda_3\right) det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& +det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \sum M_2(J) \\
& -\left(det(J)\right)^2 - det(J) \left(\lambda_2 + \lambda_3\right) \lambda_2\lambda_3.
\end{aligned}
$$

Multiplication by (-1) delivers

$$
\begin{aligned}
0 = & (\lambda_2\lambda_3)^3 - (\lambda_2\lambda_3)^2 \left(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\right) - \left(\lambda_1\lambda_2\lambda_3\right)^2 \\
& +\lambda_2\lambda_3 \left(\lambda_1\lambda_2\lambda_3\right) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& +\left(\lambda_1\lambda_2 + \lambda_1\lambda_3\right) det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \\
& -det(J) \left(\lambda_1 + \lambda_2 + \lambda_3\right) \sum M_2(J) \\
& +det(J)^2 + det(J) \left(\lambda_2 + \lambda_3\right) \lambda_2\lambda_3.
\end{aligned}
$$

It is straightforward to show that the last three rows of above equation sum up to zero and the first two rows are immediately recognized as the auxiliary characteristic polynomial (3.4). ■

5.2 Proof of Proposition 1

To prove Proposition 1 notice first that we need to distinguish two cases:

1.) all roots are positive

2.) all roots are negative

1.) Consider first the case of all positive roots.

Since all three roots are positive we must have $c(-1) < 0$. Depending on whether $c(1) \lesseqgtr 0$ we obtain the four different local stability properties of the steady-state equilibrium, $(i) - (iv)$, as given in Proposition 1.

We start by considering $c(1) < 0$.

By the continuity of $c(\lambda)$ and the fact that all three roots are positive, $c(1) < 0$ implies either

a1) $\lambda_i > 1, i = 1, 2, 3$, or

a2) $\lambda_i < 1$, $i = 1, 2$, and $\lambda_3 > 1$.

Construct the auxiliary characteristic polynomial (3.4), $\mathcal{C}(\mu) \equiv \mu^3 - \sum M_2(J)\mu^2 + det(J)tr(J)\mu - [det(J)]^2 = 0$, which inherits the properties of $c(\lambda)$, in particular, $\lim_{\mu\to\pm\infty} \mathcal{C}(\mu) \to \pm\infty$. Notice that we need to consider the possibility that $\mathcal{C}(1) > 0$ as well as $\mathcal{C}(1) < 0$.

Suppose first that $\mathcal{C}(1) > 0$ holds. Then continuity of $\mathcal{C}(\mu)$ and the fact that all its roots μ_j, $j = 1, 2, 3$, must be positive implies

b1) $\mu_j < 1$, $j = 1, 2, 3$, or

b2) $\mu_j > 1$, $j = 1, 2$ and $\mu_3 < 1$.

By construction the three roots μ_j, $j = 1, 2, 3$, are equal to the three product pairs of the roots of $c(\lambda)$. Now consider the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. If $a1)$ were true, then all three roots of $\mathcal{C}(\mu)$ would be strictly greater than 1, which is not compatible with $\mathcal{C}(1) > 0$. Hence $a2)$ must be true, and we have that if $c(1) < 0$ and $\mathcal{C}(1) > 0$, then it must be that $\lambda_i < 1$, $i = 1, 2$ and $\lambda_3 > 1$, i. e., the steady-state equilibrium is a two-dimensional saddle.

Next, suppose that $\mathcal{C}(1) < 0$. Then

b11) $\mu_j > 1$ $j = 1, 2, 3$, or

b22) $\mu_j < 1$, $j = 1, 2$ and $\mu_3 > 1$.

Notice that both $b11)$ and $b22)$ are compatible with each of the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. Thus, we still do not have enough information to infer which one of the two alternatives, $a1)$ or $a2)$, holds. However, if in addition we know whether $det(J) \gtrless 0$, then we may proceed as follows.

Suppose $det(J) < 1$. Then since $det(J) = \lambda_1\lambda_2\lambda_3 < 1$, $c(1) < 0$ is only compatible with $a2)$. Thus, we may state that if $c(1) < 0$, $\mathcal{C}(1) < 0$ and $det(J) < 1$, then it must be that $\lambda_i < 1$, $i = 1, 2$, and $\lambda_3 > 1$, i. e., the steady-state equilibrium is a two-dimensional saddle.

If, on the other hand, $det(J) > 1$, both $a1)$ and $a2)$ are possible, in general. Suppose $a2)$ is true, i. e., two eigenvalues of the characteristic polynomial (2.3) are smaller than 1 and one eigenvalue is greater than 1, say, $\lambda_i < 1$, $i = 1, 2$, and $\lambda_3 > 1$. Then it must be that $\lambda_1\lambda_2 < \lambda_i < 1$ $i = 1, 2$. Together with $det(J) > 1$ it then must hold that

$$\begin{aligned} \lambda_i\lambda_3 &> \lambda_1\lambda_2\lambda_3, \quad \text{for} \quad i = 1, 2, \\ &= det(J) \\ &> 1, \end{aligned}$$

which, in turn, implies that two of the three roots of $\mathcal{C}(\mu)$ must be greater and one smaller than 1, say, $\mu_1 = \lambda_1\lambda_2 < 1$, $\mu_2 = \lambda_1\lambda_3 > 1$ and $\mu_3 = \lambda_2\lambda_3 > 1$. But this contradicts the fact that $\mathcal{C}(1) < 0$, which is only compatible with either $b11)$ (i. e., $\mu_j > 1$, $j = 1, 2, 3$) or with $b22)$ (i. e., $\mu_j < 1$, $j = 1, 2$ and $\mu_3 > 1$). Hence, we may state that if $c(1) < 0$, $\mathcal{C}(1) < 0$ and $det(J) > 1$, then it must be that $\lambda_i > 1$, $i = 1, 2, 3$, i. e., the steady-state equilibrium is an unstable node.

Now, consider the case $c(1) > 0$ which implies either a1) $\lambda_i < 1, i = 1, 2, 3$, or

a2) $\lambda_i > 1$, $i = 1, 2$, and $\lambda_3 < 1$.

To be able to distinguish the two cases reconsider the auxiliary characteristic polynomial (3.4),

$$\mathcal{C}(\mu) \equiv \mu^3 - \sum M_2(J)\mu^2 + det(J)tr(J)\mu - [det(J)]^2 = 0.$$

As before, we need to take into account the possibility of $\mathcal{C}(1) > 0$ as well as $\mathcal{C}(1) < 0$.

Suppose first that $\mathcal{C}(1) > 0$ holds. Then continuity of $\mathcal{C}(\mu)$ and the fact that all its roots μ_j, $j = 1, 2, 3$, must be positive implies

b1) $\mu_j < 1$, $j = 1, 2, 3$, or

b2) $\mu_j > 1$, $j = 1, 2$ and $\mu_3 < 1$.

Notice that both $b1)$ and $b2)$ are compatible with each of the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. Thus, we still do not have enough information to infer which one of the two alternatives, $a1)$ or $a2)$, holds.

However, if in addition we know whether $det(J) \gtrless 0$, then we may proceed as follows.

Suppose $det(J) > 1$. Then since $det(J) = \lambda_1\lambda_2\lambda_3 > 1$, $c(1) > 0$ is only compatible with $a2)$. Thus, we may state that if $c(1) > 0$, $\mathcal{C}(1) > 0$ and $det(J) > 1$, then it must be that $\lambda_i > 1$, $i = 1, 2$, and $\lambda_3 < 1$, i. e., the steady-state equilibrium is a one-dimensional saddle.

If, on the other hand, $det(J) < 1$, both $a1)$ and $a2)$ are possible, in general. Suppose $a2)$ is true, i. e., $\lambda_1 > 1$, $\lambda_2 > 1$ and $\lambda_3 < 1$. This implies that $\lambda_1\lambda_2 > \lambda_i > 1$ $i = 1, 2$. But then it must hold that

$$\begin{aligned} 1 > det(J) &= (\lambda_1\lambda_2)\lambda_3 \\ &> \lambda_i\lambda_3, \quad \text{for} \quad i = 1, 2, \end{aligned}$$

which, in turn, implies that two of the three roots of the auxiliary characteristic polynomial $\mathcal{C}(\mu)$ are smaller than one, say, $\mu_1 = \lambda_1\lambda_3 < 1$ and $\mu_2 = \lambda_2\lambda_3 < 1$. But this contradicts the fact that $\mathcal{C}(1) > 0$ since it is not compatible with neither $b1)$ nor $b2)$. Hence, we may state that if $c(1) > 0$, $\mathcal{C}(1) > 0$ and $det(J) < 1$, then it must be that $a1)$ holds, i. e., the steady-state equilibrium is a stable node.

Next, suppose that $\mathcal{C}(1) < 0$. Then

b11) $\mu_j > 1$ $j = 1, 2, 3$, or

b22) $\mu_j < 1$, $j = 1, 2$ and $\mu_3 > 1$.

By construction the three roots μ_j, $j = 1, 2, 3$, are equal to the three product pairs of the roots of $c(\lambda)$. Now consider the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. If $a1)$ were true, then all three roots of $\mathcal{C}(\mu)$ would be strictly smaller than 1, which is not compatible with $\mathcal{C}(1) < 0$. Hence $a2)$ must be true, and we have that if $c(1) > 0$ and $\mathcal{C}(1) < 0$, then it must be that $\lambda_i > 1$, $i = 1, 2$ and $\lambda_3 < 1$, i. e., the steady-state equilibrium is a one-dimensional saddle.

2.) Consider now the case when all roots are negative.

Since all three roots are negative we must have $c(1) > 0$. Depending on whether $c(-1) \lesseqgtr 0$ we obtain the four different local stability properties of the steady-state equilibrium, $(i) - (iv)$, as given in Proposition 1.

We start by considering $c(-1) < 0$.

By the continuity of $c(\lambda)$ and the fact that all three roots are negative, $c(-1) < 0$ implies either a1) $\lambda_i > -1, i = 1, 2, 3$, or

a2) $\lambda_i < -1$, $i = 1, 2$, and $\lambda_3 > -1$.

To be able to pin down which of the two cases holds consider the auxiliary characteristic polynomial (3.4), $\mathcal{C}(\mu)$. Notice that since $\lambda_i < 0$, $i = 1, 2, 3$, all roots of $\mathcal{C}(\mu)$ are positive. Again, we need to consider the possibility that $\mathcal{C}(1) > 0$ as well as $\mathcal{C}(1) < 0$.

Suppose first that $\mathcal{C}(1) > 0$ holds. Then continuity of $\mathcal{C}(\mu)$ and the fact that all its roots μ_j, $j = 1, 2, 3$, must be positive implies either

b1) $\mu_j < 1$, $j = 1, 2, 3$, or

b2) $\mu_j > 1$, $j = 1, 2$ and $\mu_3 < 1$.

Notice that both $b1)$ and $b2)$ are compatible with each of the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. Thus, we do not have enough information to infer which one of the two alternatives, $a1)$ or $a2)$, holds.

However, if in addition we know whether $det(J) \gtrless 0$, then we may proceed as follows.

Suppose $det(J) > 1$. Then since $det(J) = \lambda_1\lambda_2\lambda_3 > 1$, $c(-1) < 0$ is only compatible with $a2)$. Thus, we may state that if $c(-1) < 0$, $\mathcal{C}(1) > 0$ and $det(J) > 1$, then it must be that $\lambda_i < -1$, $i = 1, 2$, and $\lambda_3 > -1$, i. e., the steady-state equilibrium is a one-dimensional saddle.

If, on the other hand, $det(J) < 1$, both $a1)$ and $a2)$ are possible, in general. Suppose $a2)$ is true, i. e., $\lambda_1 < -1$, $\lambda_2 < -1$ and $\lambda_3 > -1$. Thus, $det(J) = \lambda_1\lambda_2 > \lambda_i < 1$ $i = 1, 2$. But then it must hold that

$$\begin{aligned} 1 > det(J) &= (\lambda_1\lambda_2)\lambda_3 \\ &> \lambda_i\lambda_3, \quad \text{for} \quad i = 1, 2, \end{aligned}$$

which, in turn, implies that two of the three roots of the auxiliary characteristic polynomial $\mathcal{C}(\mu)$ are $\in (0, 1)$, say, $\mu_1 = \lambda_1\lambda_3 \in (0, 1)$ and $\mu_2 = \lambda_2\lambda_3 \in (0, 1)$. But this contradicts the fact that $\mathcal{C}(1) > 0$ since it is not compatible with neither $b1)$ nor $b2)$. Hence, we may state that if $c(1) > 0$, $\mathcal{C}(1) > 0$ and $det(J) < 1$, then it must be that $a1)$ holds, i. e., the steady-state equilibrium is a stable node.

Next, suppose that $\mathcal{C}(1) < 0$. Then

b11) $\mu_j > 1$ $j = 1, 2, 3$, or

b22) $\mu_j < 1$, $j = 1, 2$ and $\mu_3 > 1$.

By construction the three roots μ_j, $j = 1, 2, 3$, are equal to the three product pairs of the roots of $c(\lambda)$. Now consider the two alternative combinations $a1)$ and $a2)$ that describe the eigenvalues of the characteristic polynomial $c(\lambda)$. If $a1)$ were true, then all three roots of $\mathcal{C}(\mu)$ would be strictly smaller than 1, which is not compatible with $\mathcal{C}(1) < 0$. Hence $a2)$ must be true, and we have that if $c(-1) < 0$ and $\mathcal{C}(1) < 0$, then it must be that $\lambda_i < -1$, $i = 1, 2$ and $\lambda_3 > -1$, i. e., the steady-state equilibrium is a two-dimensional saddle.

Now, consider the case $c(-1) > 0$ which implies either

a1) $\lambda_i \in (-\infty, -1), i = 1, 2, 3$, or

a2) $\lambda_i \in (-1, 0)$, $i = 1, 2$, and $\lambda_3 \in (-\infty, -1)$.

To pin down which of the two cases holds consider the auxiliary characteristic polynomial (3.4), $\mathcal{C}(\mu)$. Notice that since $\lambda_i < 0$, $i = 1, 2, 3$, all roots of $\mathcal{C}(\mu)$ are positive. Again, we need to consider the possibility that $\mathcal{C}(1) > 0$ as well as $\mathcal{C}(1) < 0$.

Suppose first that $\mathcal{C}(1) > 0$ holds. Then continuity of $\mathcal{C}(\mu)$ and the fact that all its roots μ_j, $j = 1, 2, 3$, must be positive implies either

b1) $\mu_j < 1$, $j = 1, 2, 3$, or

b2) $\mu_j > 1$, $j = 1, 2$ and $\mu_3 < 1$.

Notice that based on $\mathcal{C}(1) > 0$ alone we know that $a1)$ cannot be true. $\mathcal{C}(1) > 0$ requires either all three roots or just one root to be $\in (0, 1)$. None of the two are possible with $a1)$. Therefore, we may state that if $c(-1) > 0$ and $\mathcal{C}(1) > 0$, then $\lambda_i \in (-1, 0)$, $i = 1, 2$, and $\lambda_3 \in (-\infty, -1)$ must hold, i. e., the steady-state equilibrium is a two-dimensional saddle.

Next, suppose that $\mathcal{C}(1) < 0$. Then

b11) $\mu_j > 1$ $j = 1, 2, 3$, or

b22) $\mu_j < 1$, $j = 1, 2$ and $\mu_3 > 1$.

Since $a2)$ potentially satisfies both $b11)$ and $b22)$ and $a1)$ would satisfy $b2)$ we need more information to pin down the two alternatives $a1)$ and $a2)$.

If in addition we know whether $det(J) \gtrless 0$, then we may proceed as follows.

Suppose $det(J) < 1$. This is only compatible with $a2)$. Thus, we may state that if $c(-1) > 0$, $\mathcal{C}(1) < 0$ and $det(J) < 1$, then it must be that $\lambda_i \in (-1, 0)$, $i = 1, 2$, and $\lambda_3 \in (-\infty, -1)$, i. e., the steady-state equilibrium is a two-dimensional saddle.

If, on the other hand, $det(J) > 1$, both $a1)$ and $a2)$ are possible, in general. Suppose $a2)$ is true, i. e., $\lambda_i \in (-1, 0)$, $i = 1, 2$, and $\lambda_3 \in (-\infty, -1)$. But then it must hold that

$$\begin{aligned} \lambda_i \lambda_3 &> (\lambda_1 \lambda_2) \lambda_3, \quad \text{for} \quad i = 1, 2, \\ &= det(J) \\ &> 1 \end{aligned}$$

which, in turn, implies that two of the three roots of the auxiliary characteristic polynomial $\mathcal{C}(\mu)$ are greater than one, say, $\mu_1 = \lambda_1 \lambda_3 > 1$ and $\mu_2 = \lambda_2 \lambda_3 > 1$. But this contradicts the fact that $\mathcal{C}(1) < 0$ since it is not compatible with neither $b1)$ nor $b2)$. Hence, we may state that if $c(-1) > 0$, $\mathcal{C}(1) < 0$ and $det(J) > 1$, then it must be that $a1)$ holds, i. e., the steady-state equilibrium is an unstable node. ■

5.3 Proof of Proposition 2

Consider the characteristic polynomial (2.3) and suppose it has one real root and a pair of complex conjugate roots. Let λ_1 denote the real root and let $\lambda_i = \rho \pm \omega i$, $i = 2, 3$, denote the complex roots. The real root may be either positive or negative. The complex roots satisfy either of the following

a) $\mid \lambda_2 \lambda_3 \mid = \mid \rho \pm \omega i \mid < 1$

b) $\mid \lambda_2\lambda_3 \mid = \mid \rho \pm \omega i \mid > 1$

CASE 1: $\lambda_1 > 0$.

Since λ_1 is the only positive root $c(1) > 0$ implies $\lambda_1 < 1$ and $c(1) < 0$ implies $\lambda_1 > 1$.

Suppose $c(1) > 0$, i.e., $\lambda_1 < 1$.

To determine whether $\mid \lambda_2\lambda_3 \mid \gtreqless 1$ consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one real and two complex roots, (3.4) must have one real and two complex roots.

But then $\mathcal{C}(1) > 0$ implies that the only real root of (3.4) must be smaller than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with a). Thus, we may state that if $c(1) > 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 < 1$ and $\mid \lambda_2\lambda_3 \mid < 1$, i.e., the steady-state equilibrium is a stable node.

On the other hand, $\mathcal{C}(1) < 0$ implies that the onlyreal root of (3.4) must be greater than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with b). Thus, we may state that if $c(1) > 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 < 1$ and $\mid \lambda_2\lambda_3 \mid > 1$, i.e., the steady-state equilibrium is a one-dimensional saddle.

Suppose $c(1) < 0$, i.e., $\lambda_1 > 1$.

To determine whether $\mid \lambda_2\lambda_3 \mid \gtreqless 1$ consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one real and two complex roots, (3.4) must have one real and two complex roots.

But then $\mathcal{C}(1) > 0$ implies that the only real root of (3.4) must be smaller than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with a). Thus, we may state that if $c(1) < 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 > 1$ and $\mid \lambda_2\lambda_3 \mid < 1$, i.e., the steady-state equilibrium is a two-dimensional saddle.

On the other hand, $\mathcal{C}(1) < 0$ implies that the onlyreal root of (3.4) must be greater than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with b). Thus, we may state that if $c(1) < 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 > 1$ and $\mid \lambda_2\lambda_3 \mid > 1$, i.e., the steady-state equilibrium is an unstable node.

CASE 2: $\lambda_1 < 0$.

Since λ_1 is the only negative root $c(-1) > 0$ implies $\lambda_1 \in (-\infty, -1)$ and $c(-1) < 0$ implies $\lambda_1 \in (-1, 0)$.

Suppose $c(-1) < 0$, i.e., $\lambda_1 \in (-1, 0)$.

To determine whether $\mid \lambda_2\lambda_3 \mid \gtreqless 1$ consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one real and two complex roots, (3.4) must have one real and two complex roots.

But then $\mathcal{C}(1) > 0$ implies that the only real root of (3.4) must be smaller than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with a). Thus, we may state that if $c(-1) < 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 \in (-1, 0)$ and $\mid \lambda_2\lambda_3 \mid < 1$, i.e., the steady-state equilibrium is a stable node.

On the other hand, $\mathcal{C}(1) < 0$ implies that the only real root of (3.4) must be greater than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with b). Thus, we may state that if $c(-1) < 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 \in (-1, 0)$ and $\mid \lambda_2\lambda_3 \mid > 1$, i.e., the steady-state equilibrium is a one-dimensional saddle.

Suppose $c(-1) > 0$, i. e., $\lambda_1 \in (-\infty, -1)$.

To determine whether $\mid \lambda_2\lambda_3 \mid \gtrless 1$ consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one real and two complex roots, (3.4) must have one real and two complex roots.

But then $\mathcal{C}(1) > 0$ implies that the only real root of (3.4) must be smaller than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with a). Thus, we may state that if $c(-1) > 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 \in (-\infty, -1)$ and $\mid \lambda_2\lambda_3 \mid < 1$, i. e., the steady-state equilibrium is a two-dimensional saddle.

On the other hand, $\mathcal{C}(1) < 0$ implies that the onlyreal root of (3.4) must be greater than one. Since this root is the product of the two complex roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with b). Thus, we may state that if $c(-1) > 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 \in (-\infty, -1)$ and $\mid \lambda_2\lambda_3 \mid > 1$, i. e., the steady-state equilibrium is an unstable node. ■

5.4 Proof of Proposition 3

Let λ_1 be the only negative root. Notice that λ_1 may either be greater or smaller than minus one. We will consider these two cases in turn.

1.) CASE 1: $\lambda_1 \in (-1, 0)$. Consider the characteristic polynomial (2.3). Suppose two of its roots are positive and one is negative. Let λ_1 denote the negative root and let λ_2 and λ_3 denote the positive roots. Then, irrespective of the magnitude of the positive roots, for $\lambda_1 \in (-1, 0)$ to hold it must be that $c(-1) < 0$.

Next, notice that the positive roots can arise in three different combinations:

a) $\lambda_2 \in (0, 1)$, and $\lambda_3 > 1$

b) $\lambda_i \in (0, 1)$, $i = 2, 3$

c) $\lambda_i > 1$, $i = 2, 3$

We would like to pin down the conditions under which either of the three above alternatives occur. To do that we start by evaluating the characteristic polynomial at $\lambda = 1$. Notice that it may either be that $c(1) > 0$ or that $c(1) < 0$.

Suppose $c(1) < 0$.

Given that $c(-1) < 0$ the properties of a cubic equation imply that $\lambda_2 \in (0, 1)$, and $\lambda_3 > 1$ if and only if $c(1) < 0$. Hence, no additional information is required, and the method familiar from planar systems is sufficient.Thus, we may state that if $c(-1) < 0$ and $c(1) < 0$, then the steady-state equilibrium is a two-dimensional saddle.

Suppose $c(1) > 0$.

Given that $c(1) > 0$ the properties of a cubic equation imply that, in general, $c(1) > 0$ is compatible with b) and c). To distinguish between these two alternatives consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one negative and two positive roots, (3.4) must have one positive and two negative roots.

But then $\mathcal{C}(1) > 0$ implies that the only positive root of (3.4) must be smaller than one. Since this root is the product of the two positive roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with b). Thus, we may state that if $c(-1) < 0$, $c(1) > 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 \in (-1, 0)$ and $\lambda_i \in (0, 1)$, $i = 2, 3$, i. e., the steady-state equilibrium is a stable node.

On the other hand, $\mathcal{C}(1) < 0$ implies that the only positive root of (3.4) must be greater than one. Since this root is the product of the two positive roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with c). Thus, we may state that if $c(-1) < 0$, $c(1) > 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 \in (-1, 0)$ and $\lambda_i > 1$, $i = 2, 3$, i. e., the steady-state equilibrium is a one-dimensional saddle.

2.) CASE 2: $\lambda_1 \in (-\infty, 1)$. Consider the characteristic polynomial (2.3). Suppose two of its roots are positive and one is negative. Let λ_1 denote the negative root and let λ_2 and λ_3 denote the positive roots. Then, irrespective of the magnitude of the positive roots, for $\lambda_1 \in (-\infty, 1)$ to hold it must be that $c(-1) > 0$.

As above the positive roots can arise in the three different combinations, repeated here for convenience:

a) $\lambda_2 \in (0, 1)$, and $\lambda_3 > 1$
b) $\lambda_i \in (0, 1)$, $i = 2, 3$
c) $\lambda_i > 1$, $i = 2, 3$

We would like to pin down the conditions under which either of the three above alternatives occur. To do that we start by evaluating the characteristic polynomial at $\lambda = 1$. Notice that it may either be that $c(1) > 0$ or that $c(1) < 0$.

Suppose $c(1) < 0$.

Given that $c(-1) > 0$ the properties of a cubic equation imply that $\lambda_2 \in (0, 1)$, and $\lambda_3 > 1$ if and only if $c(1) < 0$. Hence, no additional information is required, and the method familiar from planar systems is sufficient.Thus, we may state that if $c(-1) > 0$ and $c(1) < 0$, then the steady-state equilibrium is a one-dimensional saddle.

Suppose $c(1) > 0$. Given that $c(1) > 0$ the properties of a cubic equation imply that, in general, $c(1) > 0$ is compatible with b) and c). To distinguish between these two alternatives consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one negative and two positive roots, (3.4) must have one positive and two negative roots.

But then $\mathcal{C}(1) > 0$ implies that the only positive root of (3.4) must be smaller than one. Since this root is the product of the two positive roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with b). Thus, we may state that if $c(-1) > 0$, $c(1) > 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 \in (-\infty, -1)$ and $\lambda_i \in (0, 1)$, $i = 2, 3$, i. e., the steady-state equilibrium is a two-dimensional saddle.

On the other hand, $\mathcal{C}(1) < 0$ implies that the only positive root of (3.4) must be greater than one. Since this root is the product of the two positive roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with c). Thus, we may state that if $c(-1) > 0$, $c(1) > 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 \in (-\infty, -1)$ and $\lambda_i > 1$, $i = 2, 3$, i. e., the steady-state equilibrium is an unstable node. ■

5.5 Proof of Proposition 4

Let λ_1 be the only positive root. Notice that λ_1 may either be greater or smaller than one. We will consider these two cases in turn.

1.) CASE 1: $\lambda_1 < 1$ Consider the characteristic polynomial (2.3). Suppose two of its roots are negative and one is positive. Let λ_1 denote the positive root and let λ_2 and λ_3 denote the negative roots. Then, irrespective of the negative roots, for $\lambda_1 < 1$ to hold it must be that $c(1) > 0$.

Next, notice that the negative roots can arise in three different combinations:

a) $\lambda_2 \in (-\infty, 1)$, and $\lambda_3 \in [-1, 0]$

b) $\lambda_i \in (-1, 0)$, $i = 2, 3$

c) $\lambda_i \in (-\infty, -1)$, $i = 2, 3$

We would like to pin down the conditions under which either of the three above alternatives occur. To do that we start by evaluating the characteristic polynomial at $\lambda = -1$. Notice that it may either be that $c(-1) > 0$ or that $c(-1) < 0$.

Suppose $c(-1) > 0$. Given that $c(1) > 0$ the properties of a cubic equation imply that $\lambda_2 \in (-\infty, 1)$, and $\lambda_3 \in [-1, 0]$ if and only if $c(-1) > 0$. Hence, no additional information is required, and the method familiar from planar systems is sufficient.Thus, we may state that if $c(1) > 0$ and $c(-1) > 0$, then the steady-state equilibrium is a two-dimensional saddle.

Suppose $c(-1) < 0$.

Given that $c(1) > 0$ the properties of a cubic equation imply that, in general, $c(-1) < 0$ is compatible with $b)$ and $c)$. To distinguish between these two alternatives consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one positive and two negative roots, (3.4) must have one positive and two negative roots.

But then $\mathcal{C}(1) > 0$ implies that the only positive root of (3.4) must be smaller than one. Since this root is the product of the two negative roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with $b)$. Thus, we may state that if $c(1) > 0$, $c(-1) < 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 < 1$ and $\lambda_i \in (-1, 0)$, $i = 2, 3$, i. e., the steady-state equilibrium is a stable node.

On the other hand, $\mathcal{C}(1) < 0$ implies that the only positive root of (3.4) must be greater than one. Since this root is the product of the two negative roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with $c)$. Thus, we may state that if $c(1) > 0$, $c(-1) < 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 < 1$ and $\lambda_i \in (-\infty, -1)$, $i = 2, 3$, i. e., the steady-state equilibrium is a one-dimensional saddle.

2.) CASE 2: $\lambda_1 > 1$ Consider the characteristic polynomial (2.3). Suppose two of its roots are negative and one is positive. Let λ_1 denote the positive root and let λ_2 and λ_3 denote the negative roots. Then, irrespective of the negative roots, for $\lambda_1 > 1$ to hold it must be that $c(1) < 0$.

As above the negative roots can arise in the three different combinations, repeated here for convenience:

a) $\lambda_2 \in (-\infty, 1)$, and $\lambda_3 \in [-1, 0]$

b) $\lambda_i \in (-1, 0)$, $i = 2, 3$

c) $\lambda_i \in (-\infty, -1)$, $i = 2, 3$

We would like to pin down the conditions under which either of the three above alternatives occur. To do that we start by evaluating the characteristic polynomial at $\lambda = -1$. Notice that it may either be that $c(-1) > 0$ or that $c(-1) < 0$.

Suppose $c(-1) > 0$.

Given that $c(1) < 0$ the properties of a cubic equation imply that $\lambda_2 \in (-\infty, 1)$, and $\lambda_3 \in [-1, 0]$ if and only if $c(-1) > 0$. Hence, no additional information is required, and the method familiar from planar systems is sufficient.Thus, we may state that if $c(1) < 0$ and $c(-1) > 0$, then the steady-state equilibrium is a one-dimensional saddle.

Suppose $c(-1) < 0$. Given that $c(1) < 0$ the properties of a cubic equation imply that, in general, $c(-1) < 0$ is compatible with b) and c). To distinguish between these two alternatives consider the auxiliary characteristic polynomial (3.4). Notice that since (2.3) has one positive and two negative roots, (3.4) must have one positive and two negative roots.

But then $\mathcal{C}(1) > 0$ implies that the only positive root of (3.4) must be smaller than one. Since this root is the product of the two negative roots of $c(\lambda)$, $\mathcal{C}(1) > 0$ is only compatible with b). Thus, we may state that if $c(1) < 0$, $c(-1) < 0$ and $\mathcal{C}(1) > 0$, then it must hold that $\lambda_1 > 1$ and $\lambda_i \in (-1, 0)$, $i = 2, 3$, i. e., the steady-state equilibrium is a two-dimensional saddle.

On the other hand, $\mathcal{C}(1) < 0$ implies that the only positive root of (3.4) must be greater than one. Since this root is the product of the two negative roots of $c(\lambda)$, $\mathcal{C}(1) < 0$ is only compatible with c). Thus, we may state that if $c(1) < 0$, $c(-1) < 0$ and $\mathcal{C}(1) < 0$, then it must hold that $\lambda_1 > 1$ and $\lambda_i \in (-\infty, -1)$, $i = 2, 3$, i. e., the steady-state equilibrium is an unstable node. ■

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