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Optimal Income Taxation for the Alleviation of Working-Poverty When Domestic Work is Rewarded

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Optimal Income Taxation for the Alleviation of Working-Poverty When Domestic Work is Rewarded *

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Abstract

The increase in income needed for working households to escape relative poverty may be achieved either by households supplying more hours of paid work on the labor market, or by policy makers adjusting income taxation, minimum wages and social transfers targeted at working households. While past work has given considerable attention to the latter policy instruments, theoretical work on how income taxation could minimize working-poverty is scarce. Our study aims to fill this gap. Unlike the traditional optimal income taxation literature, which considers that households allocate time only between consumption and leisure, we explicitly model households' decisions as including domestic work, which is a social contribution and should be rewarded. This new framework highlights (i) the importance of emphasizing the difference between domestic work and real leisure and, (ii) the policy implications of non-market time allocation.

Keywords: Working-poverty, time allocation, optimal income taxation, household-public goods.

JEL classification: H21, I32, H24

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1 Introduction

In 2003, the European Union set the alleviation of in-work poverty as a policy aim. According to the national in-work poverty risk indicators published by the European Commission, progress in achieving this objective remains very limited[1]. These indicators identify the working poor as those individuals who have been working either in employment or self-employment for at least 7 months during the reference year and whose household equivalized disposable income does not exceed 60 per cent of the national median disposable household income. The attempt to understand how public policies (and specifically tax policy) may reduce in-work poverty is hindered by the lack of theoretical economic analysis on the matter. Only few works have been devoted to tax policy aimed at reducing general income poverty (Kanbur et al. 1994; Wane 2001; Pritilla and Tuomala 2004; Boone and Bovenberg 2004, among others).

This is striking because an estimated 9.3% of the members of the European Union workforce, or more than 15 million people, were affected by in-work poverty in 2012. In other words, one-third of adults (aged 18-64 years) who are at risk of poverty are employed (*Employment and Social Developments in Europe 2013*). An even worse finding is that households with children always fare worse than their childless counterparts with comparable employment status. For example, in the U.K., the *Institute for Fiscal Studies annual poverty and inequality 2015 report* states that nearly two-thirds of British children in poverty live in working families (Belfield et al. 2015). Thus, working poverty cannot be dismissed as a negligible phenomenon that concerns a marginal population.

While public policies to reduce working poverty are inevitably part of policies to reduce general income poverty, working poverty cannot be assumed to mirror general income poverty. Generally, households containing employed persons are less likely to be poor; thus, securing employment is suggested as a key pathway to escaping poverty. However, whether obtaining employment is sufficient for lifting a household out of poverty depends on both the level of the wage earned net of income tax (and plus social transfers) and the composition of the household. The *OECD policy brief September 2009* states that "*working full-time does not always provide a solid pathway out of poverty*" and "*employment is not a panacea: on average across OECD countries, 7% of individuals living in households with at least one worker are poor, and more than 10% of the working pop-*

[1]See the data from Eurostat at http://ec.europa.eu/eurostat/data/database

ulation in Japan, Mexico, Poland, Portugal, Turkey and the United States are poor." Even in Ireland, which has relatively low levels of in-work poverty, "*somewhat less than a half of those at risk of poverty in 2008 lived in households where at least one person was in employment*" (Daly, 2010, Page 4). A worker living alone may be protected against the risk of poverty even when he ir she has a low net wage, whereas in a household with a large number of dependent persons, even a substantial net wage could be inadequate. This example illustrates that the limited overlap between working poverty and general poverty stems from the use of two levels of analysis in defining working poverty: work is the attribute of an individual, whereas poverty is a household characteristic. Thus, theoretical results regarding reducing general income poverty cannot be readily applied to lifting households from working poverty.

At face value, it may seem that increasing the labor supply (such as obtaining full-time jobs instead of part-time jobs) of in-work poor households could reduce or eliminate working poverty. This view has been disputed by the existing empirical work[2]. Two flaws in this argument are at the core of our paper. First, households with dependents (especially children or aging parents) need to allocate at least some time to domestic work and, thus, cannot entirely devote their time to market work. Second, even when individuals work full time, low-earning households might still be poor, as mentioned above.

Drawing on these considerations and inspired by Becker's (1965) classical study "*A theory of allocation of time*" and the *American Time Use Survey* and *EU Time Use Survey*, we attempt to derive the optimal taxation of labor income assuming that working families must satisfy constraints when allocating their time to market work, domestic work and leisure.[3] We study a poverty-averse society that aggregates individuals' welfare into social welfare. The individual's contribution to social welfare consists of not only market work but also domestic work.[4]

Our study has two main contributions to the literature: (i) households' decisions in-

[2]For example, as early as 2003, Gerfin and Leu (2003) conclude that "... *adding a minimum hours requirement to the current social assistance system is the most cost-efficient reform*" in the study of working poor of Switzerland.

[3]See, http : //www.bls.gov/tus/ and https : //www.h2.scb.se/tus/tus/

[4]Obviously, the time that parents spend monitoring their young children's completion of homework, engagement in sports and so on, instead of drinking, smoking or getting into trouble on the street, has its own social value, although engagement in these activities reduces parents' market working time.

clude domestic work, which represents a social contribution and should be rewarded; (ii) the household is considered as one social taxation unit, which differs from the classical taxation of couples.

The above mentioned surveys clearly document that domestic work, family care and informal help provided to other households and so on, differ from leisure, which typically includes sports, outdoor activities, games, reading, sleeping and so on. Taking household's concerns into consideration, we explicitly model domestic work as part of households' decision by assuming that domestic work occupies part households' time, which is not assumed in most of the literature on optimal income taxation [5]. As Boadway and Tremblay (2013, page 103) mention, "*the standard model assumes times is devoted either to work or to pure leisure. In fact, non-work time takes a variety of forms*". One exception to the work-leisure pattern in the optimal income taxation literature is Beaudryy et al. (2009), who assume that workers can divide their time between market and non-market activities with different productivities. However, they do not consider leisure, which is traditionally examined in the literature.

Following the literature, such as Atkinson and Stiglitz (2015), we label domestic work as *household production* and label the corresponding outcome as *household-public goods*, which includes taking care of aged parents, nursing young children, cleaning, cooking, and so on and, can be completed only through family members' time contribution. Notably, Beaudry et al. (2009) illustrate some situations in which social programs use information on time worked; therefore, it is relevant to consider time as an observable variable. This is especially true for the case of working-poverty.

The second contribution is related to the taxation of households. Specially, we do not distinguish the difference between the first and second earners in one family as in the standard taxation of couples literature (Boskin and Sheshinski, 1983; Apps and Rees, 2009, 2011; Kleven et al. 2009). More generally, in their survey, Boadway and Tremblay (2013, Page 113) discuss four particular problems when considering the taxation of couples. Their second problem states "*some family goods and services are provided through household production. In principle, this should be treated like market income for tax purpose, but this is obviously difficult because of the absence of market prices* ". We take this kind of household production and its social value into consideration.

[5]See Boadway (2012) for the recent and complete survey.

From a technical point of view, we employ a modified version of Mirrlees' (1971) optimal taxation model, which consider poverty as a public bad and as,hence, having negative externality to the aggregate utility (Wane, 2001). In this paper, we propose an extension of this model by explicitly taking into account individuals' time allocation between market work, household production and leisure. Compared with the standard Mirrlees' (1971) model, our model includes two additional features: poverty alleviation and domestic work. In our model, individuals differ not only in their innate ability to earn an income from labor market, but also in their social statuses (being poor or rich) and in their valuation of non-market time. The additional dimensions of individual heterogeneity affect the social weights in the government's objective function. The analytic and numerical results documented in this paper show that the optimal tax rates of different types of individuals differ dramatically. For instance, the lowest-skilled working population benefits from a significant negative marginal tax rate. In this respect, our paper is closely related to a growing body of literature focusing on the implications of multiple heterogeneity and negative marginal taxation (Diamond, 1980; Laroque, 2005; Choné and Laroque 2010; Lockwood and Weinzierl, 2015, among others).

The remainder of this paper is organized as follows. We present the theoretical model in Sections 2 and 3 by modifying the Wane (2001) model in the following ways: individualss utility includes real leisure and two goods, consumption goods and household-public goods, where consumption goods come from labor income and household-public goods come from household production. To obtain explicit results and useful insights for policy recommendations, we devote Section 4 to numerical exercises and the presentation of simulation results. Concluding remarks are presented in Section 5.

2 Household production and individual behavior

We consider a nonlinear optimal taxation model *à la* Mirrlees (1971), which results from an imperfect information game between taxpayers and the social planner. In this model, individuals differ in their innate ability to earn an income. The planner can observe individuals' incomes but cannot directly observe their ability. Given this unobserved heterogeneity, the policy maker maximizes a social welfare function subject to the budget constraint and guarantees sufficient incentive for individuals to maintain their productivity. In this section, we focus on the individual's incentive concerns, and the next section presents the government's optimal taxation policy.

Formally, we assume that this economy consists of a continuum of working population over the support $[\underline{n}, \overline{n}]$ with distribution function $F(n)$ and density function $f(n)$. An individual with skills n chooses to work for $y(n)$ units of time and, hence, gains $z(n) = ny(n)$ units of final output, which we use as numeraire. The consumption level is equal to the after-tax income, $x(n) = z(n) - T(z(n))$, where $T(\cdot)$ denotes tax liability.

2.1 Household production

Our main departure from the classical optimal income taxation literature is that we introduce a new dimension of individuals' time allocation – domestic production or non-market labor. The classical theory considers that individuals maximize their utilities, which depend on only consumption goods and leisure. However, the Time Use Surveys data from various countries suggest a clear distinction between domestic works and real leisure. Typically, domestic work implies more responsibilities and duties than real leisure and could have a social value that differs from that of leisure. For instance, the social values of domestic work could include educating young children, monitoring teenagers instead of allowing them play on the street and become a social problem, taking care of aged parents given the shortage of social long-term care and so on.

Additionally, the outcomes of domestic work can be substituted by consumption, which is paid by the market labor income, such as placing aged parents in retirement houses, sending young children to kindergarten or to nannies, employing individuals to clean the house and dining out. The direct utilities obtained from real leisure are different in this respect; for example, enjoying a beer or a nap cannot be replaced by someone else's beer or nap. Hence, we should treat domestic work and real leisure differently in our taxation model.

Formally, in this economy, there are two goods: the consumption normal good, which can be purchased in the market, and the household-public good, which can be obtained only via domestic labor. The domestic production technology is assumed to be homogeneous and independent of an individual's market productivity n. The only input of this production is time (unlike Becker's (1965) model, our model does not consider domestic capital). Domestically produced goods are directly consumed within the household, cannot be resold in the market and, therefore, are not subject to any taxes. The time spend producing household public goods is denoted by c.

The utility of a $n-$household is given by $u(c,x,y)$. We assume that both income and consumption increase with skill: $z'(n) > 0$ and $x'(n) > 0$. [6] Assume that utility is at least twice continuously differentiable, increasing in consumption and household goods, $u_c \geqslant 0, \quad u_x > 0$, and decreasing in working time, $u_y < 0$. Furthermore, we assume that the utility function is concave in $c-$space and strictly concave in the (x,y)-space. Thus, an individual's problem is choosing working time y, contribution to family life c and normal consumption goods x to obtain maximum utility.

We use notation that is similar to that used by Kanbur et al. (1994) or Wane (2001). Specially, the marginal rate of substitution between gross income and consumption goods is defined as:

$$s(c,x,y,n) = -\frac{U_z(c,x,z,n)}{U_x(c,x,z,n)}(> 0),$$

with $U(c,x,z,n) = u(c,x,y) = u(c,x,z/n)$ and $U_z = \frac{U_y}{n} < 0$. Furthermore, we impose Seade's (1982) Agent Monotonicity Condition on the derivative of s with respect to n.[7] Formally, it follows:

$$s_n = \frac{\partial s(c,x,y,n)}{\partial n} < 0, \forall (c,x,y).$$

The marginal rate of substitution between household-public goods and consumption goods is defined as:

$$\varepsilon(c,x,y,n) = \frac{U_c(c,x,z,n)}{U_x(c,x,z,n)} > 0.$$

A similar condition as $s_n(n) < 0$ should be imposed on $\varepsilon_n(c,x,y,n)$. Following the arguments of Beaudry et al. (2009, Page 220), "*one may be very productive in the for-*

[6] As mentioned by Diamond (1980, page 104), "*in general there is no reason for consumption to necessarily increase with income*". However, this assumption could help us focus on the main issues of poverty alleviation and domestic production.

[7] As clearly stated by Kanbur et al. (1994), "*this is Assumption B of Mirrlees (1971) ... It implies that indifference curves in consumption-gross-income space become flatter the higher is an individual's wage rate, which in turn ensures that both consumption and gross earnings increase with the wage rate.*"

mal/market sector but have low productivity in the informal sector, or vice versa", we make the following assumption:

$$\varepsilon_n = \frac{\partial \varepsilon(c, x, y, n)}{\partial n} = 0, \forall (c, x, y).$$

That is, the marginal rate of substitution between household-public good and consumption goods is type independent. In other words, richer households do not enjoy playing football with their children, reading bedtime stories to their children, taking care their aged parents, or preparing home-cooked meal, and so on, more (or less) than poorer households.[8]

2.2 The individual's incentive concerns

The individual's problem can be written as

$$\max_{c,\, x,\, y} u(c, x, y) = \max_{c,\, z} U\left(c, z - T(z), z, n\right),$$

subject to

$$c + y = c + \frac{z}{n} \leq 1. \tag{1}$$

Noticing the difference between c and $1 - c - y$, the time spend on housework and leisure are not the same under the current setting, which is the key difference between this study and the previous literature. It may happen that $c + y = 1$, that is, leisure time is equivalent to housework, or there is no real leisure. However, if we consider that performing housework is not exactly leisure but, rather, responsibilities with social value, then we should assume that $c + y < 1$. In this case, a policy may help to alleviate some households' burden by reducing time spent on housework. To illustrate this point clearly, we perform the following mathematics exercise.

[8] Alternatively, following the law of diminishing marginal rate of substitution, we may also impose that

$$\varepsilon_n = \frac{\partial \varepsilon(c, x, y, n)}{\partial n} \leq 0, \forall (c, x, y),$$

which states that more productive individuals would like to give up more consumption goods to have more time taking care of their family than less productive individuals, but it may be too costly for them to do so.

For the above utility maximization problem with inequality constraint, we define Lagrangian:

$$\mathcal{L}(c,z,\lambda(n)) = U\left(c, z - T(z), z, n\right) + \lambda(n)(1 - c - \frac{z}{n}),$$

with λ Kuhn-Tucker multiplier which could be individual productivity dependent. The standard Kuhn-Tucker first order necessary and sufficient condition yields (See, Theorem 7.16 of Sundaram, 2009):

$$\begin{cases} U_c = \lambda(n), \\ U_x\left(1 - T'(z)\right) + U_z = \frac{\lambda(n)}{n}, \\ \lambda(n) \geq 0,\ 1 - c - \frac{z}{n} \geq 0,\ \lambda \cdot \left(1 - c - \frac{z}{n}\right) = 0. \end{cases} \tag{2}$$

Then, we have to distinguish between the effective constraint, $c + \frac{z}{n} = 1$ with $U_c > 0$, and, the ineffective constraint, $c + \frac{z}{n} < 1$ with $U_c = 0$.[9]

Fully effective constraint. In this case, time is spent fully on working and domestic production, and there is no real leisure time. Thus, if there is extra time for household-public goods, it will increase the marginal utility of the household. This implies that the marginal utility of domestic activities is strictly positive $U_c > 0$. In this case, the Kuhn-Tucker first order conditions become:

$$\begin{cases} U_x\left(1 - T'(z)\right) + U_z = \frac{\lambda(n)}{n} = \frac{U_c}{n}, \\ U_c > 0,\ 1 - c - \frac{z}{n} = 0. \end{cases} \tag{3}$$

The Kuhn-Tucker multiplier, $\lambda(n) = U_c$, measures the optimal individual utility gain due to one extra unit of time. In Appendix B.1, we demonstrate that $\frac{d\lambda(n)}{dn} \leq 0$ if and only if $U_{cx} \leq 0$ and $\frac{dc(\hat{n})/dn}{dz(n)/dn} \geq -\frac{1}{n}$, that is, $\frac{dc(n)}{d\ z} \geq -\frac{1}{n}$; or $U_{cx} > 0$ and $\frac{dc(n)}{dz} < -\frac{1}{n}$. The economic intuition is straightforward. Given that the shadow value measures the value added to the utility due to an extra unit of time, the marginal shadow value of productivity depends on the shape of the utility, U_{cx}, and the marginal rate of substitution between the two goods, c and z, that is, $\frac{dc(n)}{dz}$.

Given the Kuhn-Tucker conditions, it is easy to see that the marginal tax rate of individual n should be as follows:

$$t(z(n)) = T'(z(n)) = 1 + \frac{U_z}{U_x} - \frac{U_c}{U_x n} = 1 - s(n) - \frac{\varepsilon(n)}{n}. \tag{4}$$

[9]An inequality constraint is *effective* at some point if the constraint holds with equality at this special point.

Furthermore, Seade's (1982) Monotonicity condition and the positive substitution between c and x implies the following:

$$\frac{\partial t(z(n))}{\partial n} = -s_n + \frac{\varepsilon(n)}{n^2} > 0,$$

which confirms the classical intuition that the marginal tax rate increase as skill increases. Nonetheless, in this model, the marginal tax rate also depends on the marginal substitution between household goods and consumption. Finally, combining the conditions in (3), we can rewrite the individual's concern in a more compact manner (proof is given in Appendix B.2. step 1):

$$\frac{du(n)}{dn} = U_c \left(\frac{dc}{dn} + \frac{1}{n} \frac{dz}{dn} \right) + U_n = U_c \frac{z}{n^2} + U_n. \tag{5}$$

Fully ineffective constraint. If households engage in leisure, that is, we always have $c + \frac{z}{n} < 1$, then both choices c and z are independent strategic variables. Thus, the *constraint qualification* in the inequality constraint of the Kuhn-Tucker condition fails. Then, we can solve the utility optimization problem without constraint, and after obtaining the solution, we check the strictly inequality condition, which has to be satisfied. Mathematically, given the last equation in Kuhn-Tucker condition (2), from $c + \frac{z}{n} < 1$, we must have $\lambda(n) = 0$. Thus, we obtain the marginal utility of housework $U_c = 0$, which comes from the fact that given that there is no time constraint, no shadow value is added to the utility from extra time allocated to domestic work. In this case, the Kuhn-Tucker first order conditions become:

$$\begin{cases} U_x \left(1 - T'(z)\right) + U_z = 0, \\ 1 - c - \frac{z}{n} > 0. \end{cases} \tag{6}$$

And the marginal tax rate has a standard form as in Wane (2001):

$$t(z(n)) = 1 + \frac{U_z}{U_x} = 1 - s(n), \tag{7}$$

with

$$\frac{\partial t(z(n))}{\partial n} = -s_n > 0,$$

where the individual's concern is as follows:

$$\frac{du(n)}{dn} = U_n. \tag{8}$$

In reality, some individuals can afford to engage in leisure activities, while others cannot. In other words, a fraction of the population's utility concern checks (3), while the other fraction checks (6).

However, who should belong to which group? To answer this question, we rely on the Time Use Survey data. First, the American Time Use Survey 2012 shows that individuals differ in terms of how they use their time at home. [10] For instance, employed adults living in households with no children under age 18 engaged in leisure activities for 4.7 hours per day, approximately an hour more than employed adults living with a child under age 6 (See Table 8 of the survey). Second, the survey shows that less-qualified individuals have higher opportunity costs for participating in the labor market, because they lose the possibility of participating in household production. For example, on the days they worked, 38% of employed people aged 25 and above with a bachelor's degree or higher could work at home, compared with only 5% of those with less than a high school degree (see Table 6 in the survey report). This result shows that the high-skilled population has fewer time constraints. Third, the survey also shows that the choice of allocating time to leisure or to household production depends on individuals' market productivity. For example, the data suggest that high-income households or better-educated households worked less on Saturdays, Sundays and holidays and, hence, had more opportunities to enjoy pure leisure time(see, Appendix A). The intuition behind this observation is that because some household goods can be substituted by market goods, individuals with higher abilities (and, thus, richer) can reduce their household production and have more time for leisure. Combining this intuition and these facts together, we conclude the following.

Proposition 1 *Assume there is a threshold value of consumption $\widetilde{x}$. Then,*

- *if $x(n) < \widetilde{x}$, the marginal utility and optimal time allocation of household-n satisfies the first order conditions (3);*
- *if $x(n) \geqslant \widetilde{x}$, the marginal utility and optimal time allocation of household-n satisfies the first order conditions (6).*

Proposition 1 implies that there is an ability threshold $\widetilde{n}$ that corresponds to the skill

[10] The EU Time Use Surveys show similar results as those of the American surveys.

level of a household that has a consumption level of $\widetilde{x}$, i.e., $x(\widetilde{n}) = \widetilde{x}$. [11] In other words, the $\widetilde{n}-$individual has the lowest skill level and consumption among the group of individuals who can afford time for real leisure. Technically, Proposition 1 introduces an additional dimension of individual heterogeneity into the utility function. Depending on their ability, some members of the working population have a utility function that satisfies $U_c \neq 0$, while others have $U_c = 0$. In the previous literature on multiple heterogeneity (Chon and Laroque 2010 and Lockwood and Weinzierl, 2015), the additional individual heterogeneity is typically modeled as a continuous and exogenous variable; thus, it is independent of ability. We consider a different approach here; specially, the additional heterogeneity is *endogenous* (in the sense that it depends on ability) and *discrete* (in the sense that it divides the population into two types).

Now, we can study how the two types of working populations that have different utility concerns (3) and (6) will react to the government's income tax policy. It is straightforward that the individual's expected marginal tax rate follows:

$$t(z(n)) = \begin{cases} 1 - s(n) - \frac{\varepsilon(n)}{n} \text{ if } x(n) < \widetilde{x}, \\ 1 - s(n) \text{ if } x(n) \geqslant \widetilde{x}. \end{cases} \quad (9)$$

The advantage of introducing this new dimension of non-market time allocation is that it leaves space for policy interpretation. If we consider that taking care of aged parents and nursing young children are social contributions (and, thus, can be paid activities), then the tax transfer or tax reduction can be more helpful for the low-income working population. In the following section, we formally describe the government's optimization problem that aims to maximize social welfare and minimize aggregate poverty.

3 Poverty alleviation and optimal income taxation

Following the arguments of Wane (2001) that the poverty is kind of public bad and as we mentioned in the introduction that working-poverty is one of EU governments' main concerns, we assume that government's objective is to alleviate the income of working-poverty via income taxation.

[11]The threshold $\widetilde{n}$ corresponds to the cut-off individual in Beaudry et al. (2009).

3.1 Income poverty alleviation

Following the tradition, specially, Kanbur et al (1994), Wane (2001) and the references therein, we impose that the poverty is measured by function $P(x, x^*)$ with $\underline{x}$ the minimum income and $x^* \in (0, \infty)$ the income poverty threshold. This poverty functions is embodied the following properties:

$$\forall x \in [\underline{x}, x^*), \quad P(x, x^*) \geq 0, \quad P_x(x, x^*) < 0, \quad P_{xx}(x, x^*) > 0$$

and

$$\forall x \geq x^*, \quad P(x, x^*) = 0, \quad P(x^*, x^*) = P_x(x^*, x^*) = 0.$$

The aggregate poverty measure of the economy is:

$$\mathcal{P}(x^*) = \int_{\underline{n}}^{\overline{n}} P(x(n), x^*) f(n) dn.$$

Combining the above together, the government's objective is therefore choosing the optimal income tax rate to

$$\max_{t(z(n))} \int_{\underline{n}}^{\overline{n}} [u(n) - \beta P(x(n), x^*)] f(n) dn, \tag{10}$$

subject to the government tax revenue budget constraint:

$$\int_{\underline{n}}^{\overline{n}} [z(n) - x(n)] f(n) dn = \overline{R} \tag{11}$$

and the individual's utility concern i.e., (5) or (8).

Constant $\overline{R}$ is the desired level of tax revenue, which can serve as a public good or source of redistribution. The constant parameter β measures the social aversion of poverty. Obviously, if $\beta = 0$, poverty is not part of the government's concern. Thus, in the following, we assume $\beta > 0$ and large poverty population would lead to a larger social welfare lost.

Denote γ multiplier of the budget constraint and $\mu(n)$ costate variable of the individual's utility concern. In other words, γ measures the value added to the social welfare due to an extra unit of tax revenue; $\mu(n)$ represents how much extra social welfare increases if marginal type of individual's utility increases by one unit. Therefore, these two variables are the key variables in the "equity-efficiency" study of poverty alleviation.

For simplicity without losing generality, we follow two assumptions of Diamond (1980):

Assumption 1 *Assume that*

- *(Strict monotonicity) a household of skill n can only work at a job with marginal product n; she can not take on a job requiring lower skill level;*
- *(Additive utility) utility checks: $U_{cz} = 0, \quad U_{cx} = 0$.*

If there were no monotonicity assumption and mimicking the others were possible, there would be infinitely many different skilled individuals facing the equality constraint. However, "*Without this strong assumption the model would become more complicated*" (Diamond, 1980, Page 103).

3.2 Optimal marginal taxation

Given the maximization problem (10) with constraints (5) and (11), the first order condition with respect to z yields:

$$\left[\gamma\left(1 - s(n) - \frac{\varepsilon(n)}{n}\right) - \beta P_x\left(s(n) + \frac{\varepsilon(n)}{n}\right)\right] f(n) = -\mu(n)\left[U_{nz} + \frac{U_c}{n^2}\right]. \quad (12)$$

The left hand side is the net social lost –increase in budget net of poverty– due to allocating more income to individual n within the population $n < \widetilde{n}$. The right hand side is net social gain induced by income increases of individual n, which is measured by the individual's marginal utility gains from consumption net of sacrificing in domestic-work and multiplied by its social value $\mu(n)$. This costate variable checks:

$$\mu'(n) + \mu(n)\left[\frac{U_{nz} + U_c/n^2}{U_z - U_c/n}\right] + \left[1 + \frac{\gamma}{U_z - U_c/n}\right] f(n) = 0, \quad (13)$$

with transversality condition:

$$\mu(\underline{n}) = \mu(\overline{n}) = 0.$$

Under Assumption 1, the social value of individual n's utility in terms of income is:

$$\begin{aligned}\mu(n) &= \int_{n}^{\tilde{n}} \left(1+\frac{\gamma}{U_z - U_c/p}\right) \exp\left(\int_{n}^{p} \frac{U_{nz}+U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp \\ &+ \int_{\tilde{n}}^{\overline{n}} \left(1+\frac{\gamma}{U_z}\right) \exp\left(\int_{\tilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp.\end{aligned} \tag{14}$$

From the transversality condition $\mu(\underline{n}) = 0$, (14) yields the shadow value of government budget:

$$\gamma = -\frac{\int_{\underline{n}}^{\tilde{n}} \exp\left(\int_{\underline{n}}^{p} \frac{U_{nz}+U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp + \int_{\tilde{n}}^{\overline{n}} \exp\left(\int_{\tilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp}{\int_{\underline{n}}^{\tilde{n}} \frac{1}{U_z - U_c/p} \exp\left(\int_{\underline{n}}^{p} \frac{U_{nz}+U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp + \int_{\tilde{n}}^{\overline{n}} \frac{1}{U_z} \exp\left(\int_{\tilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp}, \tag{15}$$

which is, naturally, always positive given $U_z < 0$. Combining (14) and (15) together, we can prove that $\mu(n) < 0$ for $\underline{n} < n < \overline{n}$ with transversality condition $\mu(\underline{n}) = \mu(\overline{n}) = 0$. In other worlds, individuals' concerns always have negative social value in government's aggregate concerns. The details of the calculations above are reported in Appendix B.3.

Finally, it is straightforward that the marginal tax rate of individual n is

$$\begin{aligned}t(z(n)) &= 1 - s(n) - \tfrac{\varepsilon(n)}{n} \\ &= \frac{\beta P_x(\cdot, x^*)}{\gamma} \left(s(n) + \frac{\varepsilon(n)}{n}\right) - \frac{\mu(n)}{\gamma f(n)} \left(U_{nz} + \frac{U_c}{n^2}\right).\end{aligned} \tag{16}$$

This expression shows that government's optimal taxation policy is a trade-off between reducing aggregate poverty and maximizing utilitarian welfare. (16) states that if poverty is part of the government's concern, given $P_x(x, x^*) < 0$ for $x < x^*$, it would reduce the tax rate of low income individual (a negative marginal taxation). For the higher income individuals ($x > x^*$), the marginal poverty is $P_x(x, x^*) = 0$. Thus, the higher income individuals face a strictly positive tax rate of $-\frac{\mu(n)}{\gamma f(n)} \left(U_{nz} + \frac{U_c}{n^2}\right)$.

When the constraints of government's maximization problem follow (8) and (11), our model coincides with the standard situation in the literature. The first order conditions are identical to those in Wane (2001)'s model, except that we have one more first order condition (with respect to c) for determining the optimal household production. In other words, c is an additional control variable in our optimization problem.

When $n \geqslant \widetilde{n}$, the first component in (14) equals zero. Then, the costate equation in terms of income becomes:

$$\mu(n) = \int_n^{\overline{n}} \left(1 + \frac{\gamma}{U_z}\right) \exp\left(\int_n^p \frac{U_{nz}}{U_z} dm\right) f(p)dp. \tag{17}$$

The standard first order condition with respect to z yields:

$$[\gamma(1 - s(n)) - \beta P_x s(n)]f(n) = -\mu(n)U_{nz}. \tag{18}$$

The first order condition with respect to c yields:

$$(\gamma + \beta P_x)\varepsilon(n)f(n) + \mu(n)\varepsilon_n U_x = 0. \tag{19}$$

Given that $\varepsilon_n = 0$, the first order condition with respect to c is equivalent to $U_c = 0$. The marginal tax rate of individual n is:

$$t(z(n)) = 1 - s(n) \quad = \frac{\beta P_x(\cdot, x^*)s(n)}{\gamma} + \frac{-\mu(n)U_{nz}}{\gamma f(n)}. \tag{20}$$

Finally, the above analysis can be concluded in the following.

Proposition 2 *Suppose Assumption 1 holds and the government concerns (10). For any individual concern and government's budget constraint $(U(n), R)$ given by (5) (or (8)) and (11) that corresponds to an optimal choice, there exist piecewise absolutely continuous co-state variable $\mu(n)$ and a multiplier γ, such that, the optimal choice of marginal tax rate $t(z)$ is given by (16) (or (20)), and the costate is given by (14) (or (17)), and the multiplier is (15).*

To close this section, we discuss the impact of the income poverty minimization and the domestic work on the optimal income taxation. Then, in the next section with an explicit utility function, we present more results via numerical simulations.

3.3 Impacts of household production and poverty consideration on optimal taxation

Under Assumption 1, it is easy to see that there is a jump in the optimal marginal tax rate around the threshold ability value, $\widetilde{n}$, due to the binding constraint. Denote n^* as $x(n^*) = x^*$. Thus, depending on the relative location of n^* and $\widetilde{n}$, three cases appear.

First, if the time constraint binding individuals are exactly the ones on the poverty threshold, that is, $\widetilde{x} = x^*$ or $n^* = \widetilde{n}$, then given $P(x, x^*) = P_x(x, x^*) = 0$ for $x > x^*$, the optimal marginal tax rate would be simply as

$$t(z(n)) = \begin{cases} \dfrac{\beta P_x}{\gamma}\left(s(n) + \dfrac{\varepsilon(n)}{n}\right) - \dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_c}{n^2}\right) & \underline{n} \leq n < n^*; \\ -\dfrac{\mu(n)}{\gamma f(n)} U_{nz} & n^* \leq n \leq \overline{n}. \end{cases}$$

Second, the time constraint binding individuals are the ones above the poverty threshold, that is, $\widetilde{x} > x^*$ or $n^* < \widetilde{n}$. In this case, the optimal taxation would be given by:

$$t(z(n)) = \begin{cases} \dfrac{\beta P_x}{\gamma}\left(s(n) + \dfrac{\varepsilon(n)}{n}\right) - \dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_c}{n^2}\right) & \underline{n} \leq n < n^*; \\ -\dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_c}{n^2}\right) & n^* \leq n < \widetilde{n}; \\ -\dfrac{\mu(n)}{\gamma f(n)} U_{nz} & \widetilde{n} \leq n \leq \overline{n}. \end{cases}$$

The last case is that the time constraint binding individuals are the ones below poverty threshold, $n^* > \widetilde{n}$. Then, the optimal taxation would be

$$t(z(n)) = \begin{cases} (i) \quad \dfrac{\beta P_x}{\gamma}\left(s(n) + \dfrac{\varepsilon(n)}{n}\right) - \dfrac{\mu(n)}{\gamma f(n)}\left(U_{nz} + \dfrac{U_c}{n^2}\right) & \underline{n} \leq n < \widetilde{n}; \\ (ii) \quad \dfrac{\beta P_x}{\gamma} s(n) - \dfrac{\mu(n)}{\gamma f(n)} U_{nz} & \widetilde{n} \leq n < n^*; \\ (iii) \quad -\dfrac{\mu(n)}{\gamma f(n)} U_{nz} & n^* \leq n \leq \overline{n}. \end{cases}$$

The first case only can happen by accident. The second case states that even some households, who are not under income poverty threshold, can not afford to engage in leisure. Nevertheless, the Europe- and American-Time-Use-Survey suggest that even the poor households take some time off to enjoy some pure leisure, although may be less than the more productive individuals. In other words, the last case is closer to the reality in most of west European countries and the United States. Thus, we will only focus on the last case where $\underline{n} < \widetilde{n} < n^* < \overline{n}$.

The expression in the last case shows that the three groups of individuals face different tax regimes. This results is due to the multiple heterogeneity. In our model, both household-production and poverty consideration introduce additional dimensions to individual heterogeneity. First, our model distinguishes the poor and the rich populations by their time allocation. Second, it distinguishes the poor and the rich populations by their consumption. On the one hand, the low-skilled individuals with $n < \widetilde{n}$ have a strictly positive marginal utility, $U_c > 0$, thus, the outcome of their household-production contributes to the social welfare. On the other hand, the low-skilled individuals with $n < n^*$, also cause higher income poverty, which results as a negative externality on the social welfare. In response, government assigns different social weights to different types of individuals.

Formally, the individuals below and above $\widetilde{n}$ differ in their marginal value of labor or in their valuation of non-market time. The $n < \widetilde{n}$ populations's marginal utility (or disutility) of work is $-U_z + \frac{U_c}{n}$ (or $U_z - \frac{U_c}{n}$). The $n \geq \widetilde{n}$ populations' marginal utility (or disutility) is $-U_z$ (or U_z). As consequences of this difference, all other things being equal, the tax rate in (i) differs from (ii) in two factors: the substitution rate between labor and consumption, i.e., $s(n) + \frac{\varepsilon(n)}{n} = \frac{-U_z + U_c/n}{U_x}$; the derivative of disutility of labor with respect to n, i.e., $U_{nz} + \frac{U_c}{n^2} = \frac{\partial(U_z - U_c/n)}{\partial n}$. When it comes to compare individuals below and above n^*, the main difference is that of the value of marginal poverty: for the high-skilled population, $P_x(x, x^*) = 0$. Thus, the tax rate for the high-skilled population in (iii) is the standard welfarist result, which is non-negative.

The main policy concerns from the above analysis is that, considering working-poverty alleviation, it is important to emphasize on the difference between domestic works and real leisure, especially their social value differences. These results show that optimal tax policy should help to alleviate some burden of poorest working households by proving more generous tax credits, such as offering more or free child care and/or old age supports, thus free individuals from domestic works to have more time to work outside of the house (or obtain some extra training to improve their ability) and earn some extra income.

4 Explicit solution and numerical analysis

As we notice from last section that due to the complexity, it is impossible to obtain an explicit solution for the general model. In this section, we use special functions as an example to show some explicit results. Following Mirrlees (1971) and many after him, especially Kanbur et al (1994), Wane (2001) and Boone and Bovenberg (2004), we consider quasi-linear utility: logarithm function of x and linear in z and c. The implication of this setting is, as mentioned by Boone and Bovenberg (2004) that, *"the strictly concavity of logarithm function implies that a utilitarian government aims to fight poverty. Such a government thus aims at an equal distribution of consumption (i.e. the alleviation of poverty)"*. The utility function is given by:

$$U(c,x,z,n) = \begin{cases} \ln x(n) - \frac{z}{n} + \sigma c, & \text{for } x(n) < \widetilde{x}, \\ \ln x(n) - \frac{z}{n} + \sigma \widetilde{c}, & \text{for } x(n) \geqslant \widetilde{x}. \end{cases} \tag{21}$$

The economic interpretation of this function form is that: for the relative high-income population ($x(n) \geqslant \tilde{x}$), the household production ($\widetilde{c}$, a fixed value) is neutral and there is no incentive to allocate more time to the household production, $U_c = 0$. For this part of population, the model is solved in the same way as Wane (2001). The relative high-income individual simply set their $c = \widetilde{c}$, which is time needed for household work and, hence, the marginal utility $U_c = 0$ (Proposition1).

For the relative low-income population ($x(n) < \widetilde{x}$), the household production, c, is non-neutral with a positive marginal utility $U_c = \sigma > 0$. In this case, the relative low-skill individual will set their household production to the maximum level given the time constraint, namely, $1 - \frac{z}{n}$.

Following Kanbur et al. (1994) and Wane (2001), we take the poverty function as:

$$P(x(n), x^*) = \left(\frac{x(n) - x^*}{x^*}\right)^{\alpha}, \ \alpha > 1.$$

It is straightforward, the three marginal utility are:

$$U_x = \frac{1}{x}, \quad U_z = -\frac{1}{n}, \ \text{ and } \ U_c = \begin{cases} \sigma & \text{for } x(n) < \widetilde{x}; \\ 0 & \text{for } x(n) \geqslant \widetilde{x}. \end{cases}$$

The second order derivatives check: $U_{nz} = 1/n^2, \quad U_{nx} = 0, \quad U_{nc} = 0$, and the two elasticity functions are given by:

$$s = \frac{x}{n}; \qquad s_n = -\frac{x}{n^2} \text{ and } \varepsilon = \begin{cases} \sigma x & \text{for } x(n) < \widetilde{x} \\ 0 & \text{for } x(n) \geqslant \widetilde{x} \end{cases}; \quad \varepsilon_n = 0.$$

4.1 Solving the model

First, we determine the corresponding ability threshold $\widetilde{n}$ by solving the following equation:

$$\widetilde{x}\left[\gamma + \beta P_x(\widetilde{x}, x^*)\right] = \frac{\widetilde{n}\gamma}{1+\sigma} + \frac{\mu(\widetilde{n})}{f(\widetilde{n})\widetilde{n}},$$

where the costate variable, $\mu(n)$, and the multiplier, γ, take the following forms:

$$\mu(n) = \int_n^{\widetilde{n}} \left(1 - \frac{\gamma p}{1+\sigma}\right)\left(\frac{n}{p}\right) f(p)dp + \int_{\widetilde{n}}^{\overline{n}} (1-\gamma p)\left(\frac{\widetilde{n}}{p}\right) f(p)dp.$$

Combining the above expression with the transversality condition $\mu(\underline{n}) = 0$, we obtain:

$$\gamma = \frac{\underline{n}\displaystyle\int_{\underline{n}}^{\widetilde{n}} \frac{f(p)}{p}dp + \widetilde{n}\int_{\widetilde{n}}^{\overline{n}} \frac{f(p)}{p}dp}{\displaystyle\frac{\underline{n}}{1+\sigma}\int_{\underline{n}}^{\widetilde{n}} f(p)dp + \widetilde{n}\int_{\widetilde{n}}^{\overline{n}} f(p)dp}.$$

Given the value of ability threshold, the first order conditions (12) and (18) can be rewritten for obtaining $x(n)$:

$$\begin{cases} x(n)(\gamma + \beta P_x) = \dfrac{n\gamma}{1+\sigma} + \dfrac{\mu(n)}{f(n)n} & \underline{n} \leq n < \widetilde{n}; \\ x(n)(\gamma + \beta P_x) = n\gamma + \dfrac{\mu(n)}{f(n)n} & \widetilde{n} \leq n < n^*; \\ x(n) = n + \dfrac{\mu(n)}{f(n)n\gamma} & n^* \leq n \leq \overline{n}. \end{cases} \tag{22}$$

Then, using γ, $\mu(n)$ and $x(n)$, we can calculate the marginal tax rate as:

$$t(z(n)) = \begin{cases} (1+\sigma)\left[\dfrac{\beta P_x x(n)}{\gamma n} - \dfrac{\mu(n)}{\gamma f(n) n^2}\right] & \underline{n} \le n < \widetilde{n}; \\ \dfrac{\beta P_x x(n)}{\gamma n} - \dfrac{\mu(n)}{\gamma f(n) n^2} & \widetilde{n} \le n < n^*; \\ -\dfrac{\mu(n)}{\gamma f(n) n^2} & n^* \le n \le \overline{n}. \end{cases} \tag{23}$$

Obviously, the marginal tax rate in our model differs from Wane (2001) because of σ and $\widetilde{n}$. When $\sigma = 0$, that is, the margianl household public good is equivalent to leisure, our model is identical to Wane (2001). But with $\sigma \neq 0$, the outcomes are quite different. We conclude the above findings in the following.

Proposition 3 *Given the assumption in this section and define $x(\widetilde{n}) = \widetilde{x} < x^*$. With quasi-linear utility function defined as in (21), the optimal marginal tax rate is given by (23). Furthermore, there is jump at the marginal tax rate.*

Another variable of interest is the labor supply $z(n)$, which can be obtained by combing the individual's utility concern (5) (or (8)) and the government's budget constraint (11). It is straightforward from the utility function that the marginal labor supply of household n follows:

$$z'(n) = \begin{cases} n\left[U(n) - \ln(x(n) + \sigma c)\right] & x(n) < \widetilde{x}, \\ n\left[U(n) - \ln(x(n) + \sigma \widetilde{c})\right] & x(n) \ge \widetilde{x}. \end{cases}$$

Proposition 4 *Given the assumption in this section. With quasi-linear utility function defined as in (21), the optimal aggregate labor supply is given by*

$$z(n) = \int_{\underline{n}}^{n} z'(m)dm - K$$

with

$$z'(n) = \begin{cases} \dfrac{nU_x x'(n)}{1+U_c}, & n < \widetilde{n}; \\ nU_x x'(n), & \widetilde{n} \le n, \end{cases} \tag{24}$$

The calculation of labor supply and the expression of constant term K are reported in Appendix B.4. (24) shows that household labor supply depends on the individual's utility concern, given it changes before and after the ability threshold point, so does the aggregate labor supply. It is easy to see that due to $U_c > 0$, the low skilled working households has a smaller marginal labor supply.

4.2 Numerical findings

In order to compare with literature, we follow Wane's (2001) numerical example with density function $f(n) = \frac{5}{6} - \frac{n}{5}, n \in [1, 4]$. We assume that government is targeting in a balanced budget $R = 0$. The calibration of other parameters follows: $\beta = 0.5$, $\sigma = 0.2$, $x^* = 2$ and $\widetilde{x} = 1.7$. This calibration implies that $n^* = 2.32$ and $\widetilde{n} = 1.75$.

In each of Figure 1, 2 and 3, we have two groups of results. The left figures show the case where there is no poverty consideration at all. Thus, we study, from a welfarist planner's point of view, the impacts of household production on the optimal income taxation. In the right figures, we compared our model with the poverty alleviation literature. In both left and right figures, the black lines are Wane (2001)'s numerical results under the welfarist planner without poverty consideration (solid line) and the poverty-as-public-bad (in short PPB) planner (dash line). The red line corresponds the outcomes when $n < \widetilde{n}$. The blue line corresponds the outcomes when $n \geqslant \widetilde{n}$. Table 1 and 2 summarize the numerical results for a selection of individual types, in particular the threshold point $\widetilde{n} = 1.75$.

Figure 1 invites at least three comments. First, there is a jump in terms of consumption before and after $\widetilde{n}$ under both the welfarist planner and the PPB planner. This jump prevents the high-ability household to mimic the low-ability ones. The households with $n < \widetilde{n}$ consume less than the higher-skilled households. This is because the market purchased goods and the domestic goods are substitutes. Thus, the low-skilled households can substitute their needs by relying on domestic production. Second, after $\widetilde{n}$, our model does not significantly differ from Wane's (2001). The small differences are due to the values of γ and $\mu(n)$. Third, under the PPB planner, the consumption of individuals with $n < n^*$ is higher than the consumption under the welfarist planner. This is because the PPB planner is more concerned about simulating the consumption

Table 1: Simulation results with welfarist planner ($\beta = 0$)

	Our model			Welfarist model		
n	$x(n)$	$t(n)$	$z(n)$	$x(n)$	$t(n)$	$z(n)$
1	0.83	0	0.90	1	0	0.87
1.50	1.02	0.18	1.12	1.24	0.17	1.15
1.75	1.18 to 1.47	0.1 to 0.16	1.31	1.44	0.18	1.39
2	1.69	0.16	1.57	1.66	0.17	1.66
2.50	2.19	0.13	2.15	2.17	0.13	2.27
3	2.75	0.09	2.78	2.74	0.09	2.91
3.50	3.35	0.04	3.43	3.34	0.05	3.57
4	4	0	4.11	4	0	4.26

Table 2: Simulation results with PPB planner ($\beta = 0.5$)

	Our model			Wane (2001)'s model		
n	$x(n)$	$t(n)$	$z(n)$	$x(n)$	$t(n)$	$z(n)$
1	1.27	-0.52	1.38	1.38	-0.38	1.37
1.50	1.42	-0.13	1.50	1.56	-0.04	1.52
1.75	1.52 to 1.70	-0.04 to 0.03	1.60	1.68	0.03	1.65
2	1.84	0.09	1.74	1.81	0.09	1.79
2.50	2.19	0.13	2.14	2.17	0.13	2.20
3	2.75	0.09	2.77	2.74	0.09	2.84
3.50	3.35	0.04	3.41	3.34	0.05	3.49
4	4	0	4.09	4	0	4.18

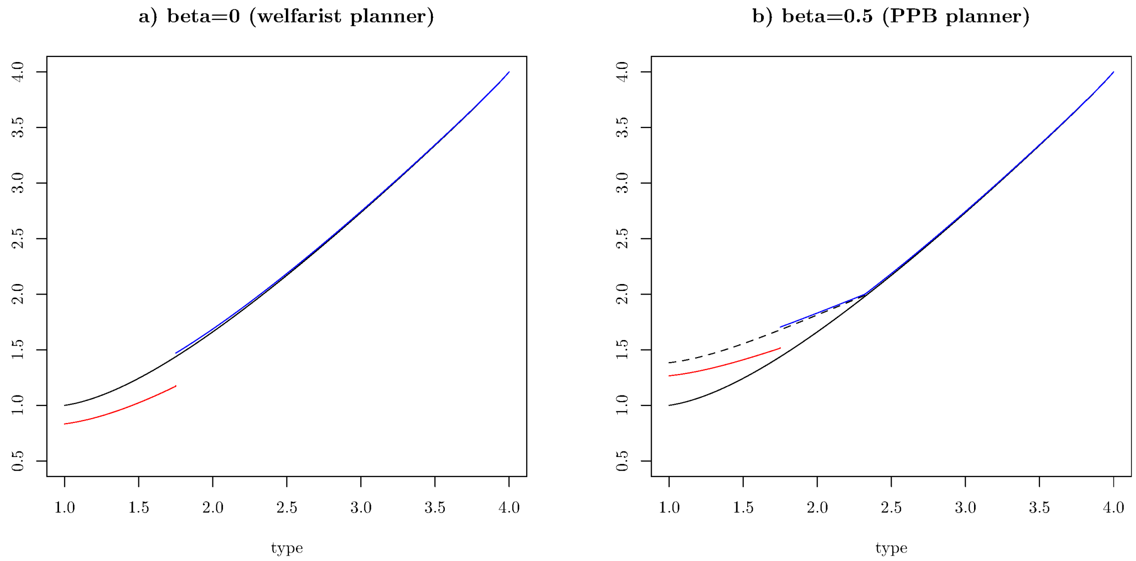

Figure 1: Consumption

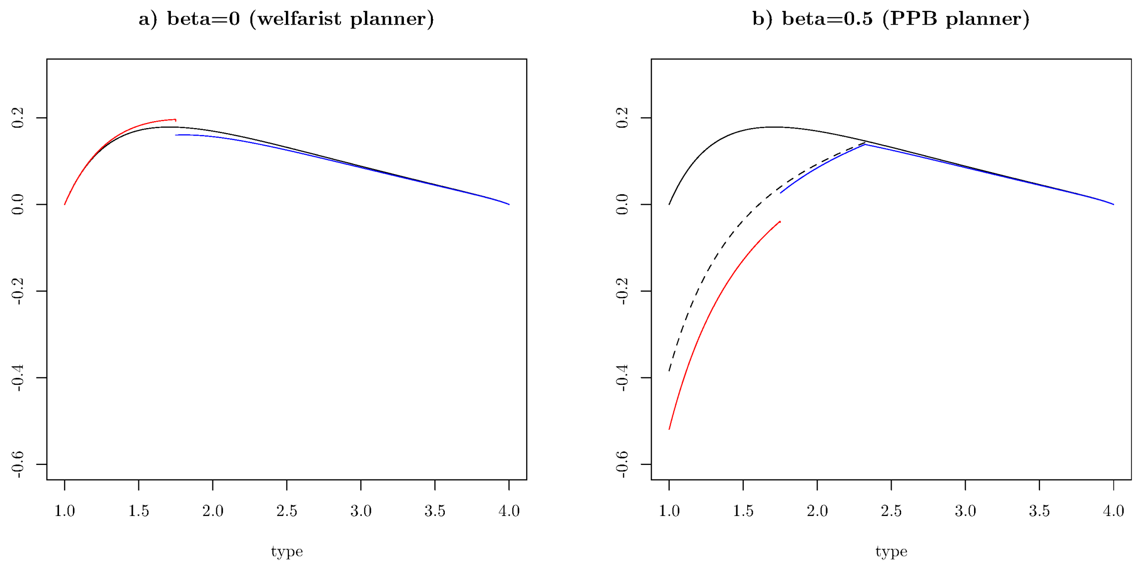

Figure 2: Marginal tax rate

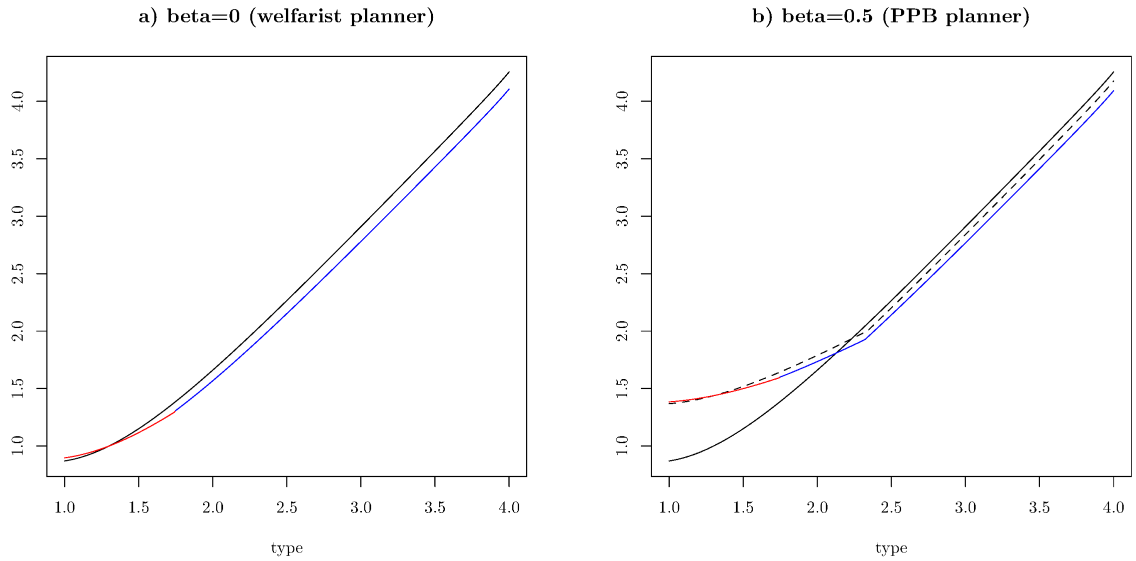

Figure 3: Labor supply

of low-skilled population.

Figure 2 depicts our main variable of interest - the marginal tax rate. Similar to the consumption pattern, there is a jump in the marginal tax rate around the point of $\widetilde{n}$. However, depending on the government's objective, the nature of this jump is quit different. Under the welfarist planner, there is an upward jump for the low-skilled households, and a downward jump for high skilled households. The opposite situation is occurred under the PPB planner. This observation can be explained by examining (23). Under the welfarist planner, the government's sensibility to the income poverty is null, $\beta = 0$. In this case, with other things being equal, the marginal tax rate faced by the $n < \widetilde{n}$ households is higher than the one faced by the $n \geq \widetilde{n}$ households. Intuitively, this is because the $n < \widetilde{n}$ households has a higher opportunity cost to work. Thus, in order to maximize the social welfare, there is no need to encourage the $n < \widetilde{n}$ households to work longer time. As a consequence, it is optimal to set a higher tax rate for the $n < \widetilde{n}$ households. The situation is dramatically different for the PPB planner with a poverty consideration, $\beta = 0.5$. In this case, the government also wishes to help the low-skilled households by providing them a subsidize (or negative tax rate). The size of this subsidize for the $n < \widetilde{n}$ households is magnified by $(1 + \sigma)$, which draws the marginal tax rate downward in Figure 2 b).

Figure 3 presents the outcome of labor supply. Given our calibration of the model, the labor supply is relatively smooth around the point $\widetilde{n}$. However, The marginal labor supply $z'(n)$ are slightly different before and after $\widetilde{n}$, before of U_c as showed in Proposition 4. We also observe that the high-skilled households are slightly decreases their market work compared to the case of Wane's (2001) model.

5 Conclusion

In this paper, we investigate how income taxation policy could alleviate working poverty when domestic work is considered. Formally, we extend Wane's (2001) optimal taxation model by explicitly taking into account households' time allocation between market labor, domestic work and pure leisure. In our model, households differ not only in their innate ability to earn an income from labor market participation but also in their social statuses (being poor or rich) and their valuation of non-market time. Thus, this paper contributes to the debate on the optimal design of income taxation by combining two strands of literature: (i) alleviation of working-poverty (Kanbur et al. 1994; Wane 2001; Pritilla and Tuomala 2004; Boone and Bovenberg 2004) and (ii) multiple heterogeneity (Diamond, 1980; Laroque, 2005; Choné and Laroque 2010; Lockwood and Weinzierl, 2015). This new approach also provides some useful insights for policy makers.

The analytic and numerical results documented in this paper show that the taxation treatments of different types of populations should differ according to their social statuses; thus, the design of an optimal taxation policy should consider not only household income level but also other characteristics. From the planner's perspective, this indicates that the social weights depend not only on labor market outcomes but also on the valuation of domestic work. The optimal taxation policy is a result of the trade-off between different driving forces. For instance, our model shows that the outcomes of household production could directly benefit the poorest working population by allowing them to substitute their consumption. Thus, the tax policy does not need to encourage the poorest working population to work even harder on the market. In the case that, the planner is willing to alleviate working-poverty, which is measured in terms of consumption, the optimal tax policy should help the poorest working households by providing them with more generous tax credits or social supports to free individuals from domestic work to have more time to work on the market (or obtain more training to improve their market productivity), earn extra income and, in turn, consume more goods. Our findings are based on theoretical analysis and numerical simulations. Arguably, empirical tests and calibration are needed before any policy recommendations are made.

A Appendix: American Time Use Survey

Table 3: Employed persons (25 years and over) who worked on an average Saturday, Sunday, and holiday periods–by education levels

Education	2008	2012
Less than high school diploma	6.82	6.30
High School graduate no college	6.44	6.76
Some college or associate degree	5.88	6.08
Bachelors degree or higher	4.31	4.21

Table 4: Employed persons (25 years and over) who worked on an average Saturday, Sunday, and holiday periods–by income level

Income	2008	2012
\$0 –\$530	6.70	7.26
\$531 –\$830	6.10	7.02
\$831 –\$1290	6.37	5.52
\$1291 and higher	4.10	3.60

Table 3 and 4 are extracted from the American Time Use Survey 2012 report.

B Appendix: calculations

B.1 Properties of $c(n)$

Differential the first equation (3) with respect to $\hat{n}$, it yields that the marginal shadow value change over type is

$$\frac{d\lambda(\hat{n})}{d\hat{n}} = u_{cc}\frac{d\hat{c}(\hat{n})}{d\hat{n}} + u_{cx}(1-T'(z))\frac{d\hat{z}(\hat{n})}{d\hat{n}} + u_{cy}\frac{\frac{d\hat{z}}{d\hat{n}}\hat{n}-\hat{z}}{\hat{n}^2},$$

where the last terms can be replaced by total differential of the effective constraint,

$$\frac{\frac{d\hat{z}}{d\hat{n}}\hat{n}-\hat{z}}{\hat{n}^2} = \frac{d\hat{c}}{d\hat{n}}.$$

Thus, we have

$$\frac{d\lambda(\hat{n})}{d\hat{n}} = (u_{cc}-u_{cy})\frac{d\hat{c}(\hat{n})}{d\hat{n}} + u_{cx}(1-T'(z))\frac{d\hat{z}(\hat{n})}{d\hat{n}}. \tag{25}$$

Taking partial derivatives on both sides of the second equation (3) with respect to $\hat{c}$, via applying envelope theorem, it follows

$$u_{cx}(1-T'(\hat{z})) = \frac{u_{cc}-u_{cy}}{\hat{n}}.$$

Substituting into (25) and rearranging terms, it yields

$$\frac{d\lambda(\hat{n})}{d\hat{n}} = u_{cx}(1-T'(z))\left(\hat{n}\frac{d\hat{c}(\hat{n})}{d\hat{n}} + \frac{d\hat{z}(\hat{n})}{d\hat{n}}\right).$$

Therefore, with assumption $u_{cx} < 0$ and income increases with productivity $\frac{d\hat{z}(\hat{n})}{d\hat{n}} > 0$, thus $\frac{d\lambda(\hat{n})}{d\hat{n}} < 0$ if and only if $\left(\hat{n}\frac{d\hat{c}(\hat{n})}{d\hat{n}} + \frac{d\hat{z}(\hat{n})}{d\hat{n}}\right) > 0$. The last condition can be rewritten as

$$\frac{d\hat{c}(\hat{n})/d\hat{n}}{d\hat{z}(\hat{n})/d\hat{n}} > -\frac{1}{\hat{n}}, \quad \text{or} \quad \frac{d\hat{c}(\hat{n})}{d\hat{z}} > -\frac{1}{\hat{n}}.$$

That finishes the proof.

B.2 Solving the optimization problem

The proof is completed in several steps.

Step 1. Kuhn-Tucker Condition: Incentive of the individuals

As argued in the previous subsection, the consumption is a function of income, $x = \xi(z)$. Thus, the individual's problem can be rewritten as

$$\max_{c,x,y} u(c,x,y) = \max_{c,x,z} U(c,x,z,n) = \max_{c,z} U(c,\xi(z),z,n),$$

subject to

$$c + y = c + \frac{z}{n} \leq 1.$$

Define the Lagrangian as

$$\mathcal{L}(c(n),z(n),\lambda(n)) = U(c,\xi(z),z,n) + \lambda(n)(1 - c - \frac{z}{n}).$$

Then the Kuhn-Tucker necessary condition is

$$\begin{cases} U_c = \lambda(n), \\ U_x\xi'(z) + U_z - \frac{\lambda}{n} = 0, \\ \lambda(n) \geq 0, \;\; 1 - c - \frac{z}{n} \geq 0, \;\; \lambda(n)\left(1 - c - \frac{z}{n}\right) = 0. \end{cases}$$

Furthermore, total differential $u(n) = u(c(n),x(n),y(n)) = U(c,\xi(z),z,n)$ with respect to n reads

$$\frac{du(n)}{dn} = U_c\frac{dc}{dn} + (U_x\xi'(z) + U_z)\frac{dz}{dn} + U_n = U_c\left(\frac{dc}{dn} + \frac{1}{n}\frac{dz}{dn}\right) + U_n. \quad (26)$$

It is easy to see if the constraint is not binding, then $U_c = 0$, thus the individual's incentive constraint is the same as the one in Wane (2001). However, when the constraint is binding, that is, $c + \frac{z}{n} = 1$, it yields, by total differential that

$$\frac{dc}{dn} + \frac{1}{n}\frac{dz}{dn} = \frac{z}{n^2}.$$

Substituting into (26), it follows that

$$\frac{du(n)}{dn} = U_c\frac{z}{n^2} + U_n = \frac{U_c}{n}\frac{z}{n} + U_n,$$

where in the last term $\frac{z}{n} = y$ states the time contribute to work, while $\frac{U_c}{n}$ reads *kind of average marginal utility from household-public good*.

Step 2. Preparation for Government's problem

In this step, we only provide the calumniation for the case when $n < \widetilde{n}$. When $n < \widetilde{n}$, the calculations are similar to Wane (2001). Due the assumption that utility of individual n, $u(n) = u(c(n), x(n), z(n))$, is strictly concave in the $c - x$ and z space, and by implicit function theorem, we can rewrite the consumption of individual n, $x(n)$, as a function of $u(n), c(n), z(n)$: $x(n) = \Psi(u(n), c(n), z(n))$. Thus, total differential $u(n) = U(c(n), x(n), z(n))$ reads

$$\begin{aligned} du &= U_c dc + U_x dx + U_z dz \\ &= (U_c + U_x \Psi_c) dc + U_x \Psi_u du + (U_z + U_x \Psi_z) dz. \end{aligned}$$

In other words, $\forall dx, dc, dz$,

$$(1 - U_x \Psi_u) du = (U_c + U_x \Psi_c) dc + (U_z + U_x \Psi_z) dz,$$

which is true if and only if

$$1 - U_x \Psi_u = U_c + U_x \Psi_c = U_z + U_x \Psi_z = 0,$$

that is,

$$\begin{cases} \Psi_u = \frac{1}{U_x}, \\ \Psi_c = -\frac{U_c}{U_x} = -\varepsilon(n)(< 0), \\ \Psi_z = -\frac{U_z}{U_x} = s(n)(> 0). \end{cases}$$

Step 3. Government's problem

$$\max_{c(n), z(n)} \int_{\underline{n}}^{\overline{n}} \left(u(n) - \beta P(x(n), x^*)\right) f(n) dn,$$

subject to government budget constraint

$$\int_{\underline{n}}^{\overline{n}} \left(z(n) - x(n)\right) f(n) dn = \overline{R}$$

and individual's incentive

$$\frac{du(n)}{dn} = U_c \left(\frac{dc}{dn} + \frac{1}{n}\frac{dz}{dn}\right) + U_n = U_c \frac{z}{n^2} + U_n.$$

Denote γ and $\mu(n)$ as multipliers for government budget and individual's incentive constraints, respectively, the Hamiltonian can then be defined as

$$\mathcal{H}(c(n), z(n), u(n), \mu(n); \gamma) \quad = [(u(n) - \beta P(x(n), x^*)) + \gamma\,(z(n) - x(n))]\, f(n) \\ +\mu(n)\left(U_c\, \tfrac{z}{n^2} + U_n\right),$$

in which, the choice variable is $c(n), z(n)$ and state variable is $u(n)$. The first order conditions yields as the followings. The costate equation is given by

$$-\mu'(n) = \frac{\partial \mathcal{H}}{\partial u(n)} = [1 - \beta P_x \Psi_u - \gamma \Psi_u]\, f(n) + \mu(n)\left[\frac{\partial U_x}{\partial x}\Psi_u + \frac{\partial U_c}{\partial x}\Psi_u\, \frac{z}{n^2}\right].$$

Rearranging the term, it yields:

$$\mu'(n) + \mu(n)\left[\frac{U_{nx}}{U_x} + \frac{U_{cx}}{U_x}\,\frac{z}{n^2}\right] + \left(1 - \frac{\beta P_x + \gamma}{U_x}\right) f(n) = 0,$$

with transversality condition

$$\mu(\underline{n}) = \mu(\overline{n}) = 0.$$

First order condition with respect to $z(n)$ reads

$$\tfrac{\partial \mathcal{H}}{\partial z(n)} = 0 = \left[-\beta P_x \tfrac{\partial x(n)}{\partial z} + \gamma(1 - \tfrac{\partial x(n)}{\partial z})\right] f(n) \\ +\mu(n)\left[\left(U_{nz} + U_{nx}\Psi_z - \tfrac{U_{nc}}{n}\right) + \left(U_{cz} + U_{cx}\Psi_z - \tfrac{U_{cc}}{n}\right)\tfrac{z}{n^2} + \tfrac{U_c}{n^2}\right].$$

When $n < \widetilde{n}$, the binding constraint $c = 1 - z/n$, implies that $\partial x(n)/\partial z = \Psi_z - \Psi_c/n$. This is because $x(n) = \Psi(u(n), c(n), z(n)) = \Psi(u(n), 1 - z(n)/n, z(n))$. The binding constraint also implies two additional terms for the first order condition: $-U_{nc}/n$ and $-U_{cc}/n$, which come from the fact that $U(x(n), c(n), z(n)) = U(x(n), 1 - z(n)/n, z(n))$. Given the assumption that the second-order derivatives of U w.r.t $c(n)$ equal zero, we can simplify the previous expression as:

$$\left[-\beta P_x(\Psi_z - \frac{\Psi_c}{n}) + \gamma(1 - (\Psi_z - \frac{\Psi_c}{n}))\right] f(n) + \mu(n)\left[(U_{nz} + U_{nx}\Psi_z) + \frac{U_c}{n^2}\right] = 0.$$

Replacing Ψ_z and Ψ_c by $s(n)$ and $-\epsilon(n)$, respectively, yields:

$$\left[-\beta P_x(s(n) + \frac{\epsilon(n)}{n}) + \gamma(1 - (s(n) + \frac{\epsilon(n)}{n}))\right] f(n) + \mu(n)\left[(U_{nz} + U_{nx}s(n)) + \frac{U_c}{n^2}\right] = 0.$$

From the fact that $s(n) = \Psi_z = -\frac{U_z}{U_x}$, that is, $s(n)U_x = -U_z$, taking partial derivative with respect to n, we have

$$U_{nz} + s(n)U_{nx} = -s_n U_x.$$

Substituting into the first order condition and simplifying terms, it follows:

$$[\gamma(1 - s(n) - \frac{\epsilon(n)}{n}) - \beta P_x(s(n) + \frac{\epsilon(n)}{n})]f(n) - \mu(n)s_n U_x + \mu(n)\frac{U_c}{n^2} = 0.$$

Finally, the marginal tax rate is:

$$\begin{aligned} t(n) &= 1 - s(n) - \tfrac{\varepsilon(n)}{n} \\ &= \tfrac{\beta P_x(\cdot, x^*))}{\gamma}\left[s(n) + \tfrac{\epsilon(n)}{n}\right] + \tfrac{\mu(n)}{\gamma f(n)}\left[-s_n U_x + \tfrac{U_c}{n^2}\right]. \end{aligned}$$

B.3 Solution of $\mu(n)$ and γ

In order to rewritten the costate equational term of income, we use the condition $\partial H/\partial u = -\mu(n)'$:

$$\frac{\partial H}{\partial u} = \left(1 + \gamma\frac{\partial z}{\partial u}\right) f(n) + \mu(n)\left(U_{cz}\frac{\partial z}{\partial u} + U_{cc}\frac{\partial z}{\partial u} + \frac{U_c}{n^2}\frac{\partial z}{\partial u} + U_{nz}\frac{\partial z}{\partial u} + U_{nc}\frac{\partial z}{\partial u}\right) = -\mu'(n).$$

Then, assuming that the second derivatives of U_c are null, we can simplify this expression:

$$\frac{\partial H}{\partial u} = \left(1 + \gamma\frac{\partial z}{\partial u}\right) f(n) + \mu(n)\left(\frac{U_c}{n^2}\frac{\partial z}{\partial u} + U_{nz}\frac{\partial z}{\partial u}\right) = -\mu'(n).$$

Now the question is how to calculate $\partial z/\partial u$. Similar to $\partial x/\partial z$ in Step 2 of Appendix B.2, we need to define an inverse function for $z(n)$. However, the inversion is slightly different here, because of $c(n) = 1 - z(n)/n$. The variable $z(n)$ appears in two arguments of the utility function: $U\left(c(n), x(n), z(n)\right) = U\left(1 - \frac{z(n)}{n}, x(n), z(n)\right)$. This implies that the inverse function should look like this:

$$z(n) = \Gamma(u(n), x(n)).$$

This is just to say that we should inverse out all the z components from the utility function. Using the same technique as in Step 2 of Appendix B.2:

$$\begin{aligned} du &= U_c dc + U_x dx + U_z dz \\ &= U_x dx + \left(U_z - \frac{U_c}{n}\right) dz \\ &= U_x dx + \left(U_z - \frac{U_c}{n}\right)(\Gamma_u du + \Gamma_x dx) \\ &= \left(U_x + \left(U_z - \frac{U_c}{n}\right)\Gamma_x\right) dx + \left(U_z - \frac{U_c}{n}\right)\Gamma_u du = 0 \end{aligned}$$

Then, we obtain: $\Gamma_u = \frac{1}{\left(U_z - \frac{U_c}{n}\right)}$ and $\Gamma_x = -\frac{U_x}{\left(U_z - \frac{U_c}{n}\right)}$. Replacing $\frac{\partial z}{\partial u}$ with Γ_u yields:

$$\mu'(n) + \mu(n)\left[\frac{U_{nz} + U_c/n^2}{U_z - U_c/n}\right] + \left[1 + \frac{\gamma}{U_z - U_c/n}\right] f(n) = 0.$$

It is straightforward that the most general form of solution is:

$$\mu(n) = \int_n^{\widetilde{n}} \left(1 + \frac{\gamma}{U_z - U_c/p}\right) \exp\left(\int_n^p \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp.$$

Noticing that $U_c(n) \neq 0$ when $n < \widetilde{n}$, thus the solution of $\mu(n)$ for low skilled population should be

$$\begin{aligned} \mu(n) &= \int_n^{\widetilde{n}} \left(1 + \frac{\gamma}{U_z - U_c/p}\right) \exp\left(\int_n^p \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp \\ &\quad + \underbrace{\int_{\widetilde{n}}^{\overline{n}} \left(1 + \frac{\gamma}{U_z}\right) \exp\left(\int_{\widetilde{n}}^p \frac{U_{nz}}{U_z} dm\right) f(p)dp}_{\widetilde{\mu}} \end{aligned}$$

where the second term of the this equation is a constant, i.e., $\widetilde{\mu} \equiv \mu(\widetilde{n})$. When $n \geqslant \widetilde{n}$, the costate variable becomes:

$$\mu(n) = \int_n^{\overline{n}} \left(1 + \frac{\gamma}{U_z}\right) \exp\left(\int_n^p \frac{U_{nz}}{U_z} dm\right) f(p)dp$$

Therefore, besides the usual transversality conditions $\mu(\underline{n}) = \mu(\overline{n}) = 0$, we have an additional transition condition, $\mu(\widetilde{n}) = \widetilde{\mu}$. Next, we can obtain an explicit solution of γ

by using the initial condition, $\mu(\underline{n}) = 0$.

$$\int_{\underline{n}}^{\widetilde{n}} \exp\left(\int_{n}^{p} \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp + \gamma \int_{\underline{n}}^{\widetilde{n}} \frac{1}{U_z - U_c/p} \exp\left(\int_{n}^{p} \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp$$

$$+ \int_{\widetilde{n}}^{\overline{n}} \exp\left(\int_{\widetilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp + \gamma \int_{\widetilde{n}}^{\overline{n}} \frac{1}{U_z} \exp\left(\int_{\widetilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp = 0.$$

The above equality yields

$$\gamma = -\frac{\int_{\underline{n}}^{\widetilde{n}} \exp\left(\int_{\underline{n}}^{p} \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp + \int_{\widetilde{n}}^{\overline{n}} \exp\left(\int_{\widetilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp}{\int_{\underline{n}}^{\widetilde{n}} \frac{1}{U_z - U_c/p} \exp\left(\int_{\underline{n}}^{p} \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp + \int_{\widetilde{n}}^{\overline{n}} \frac{1}{U_z} \exp\left(\int_{\widetilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp}.$$

Now, the government's problem can be interpreted as a two-stage optimization problem: one for the population with $n < \widetilde{n}$ and one for the population with $n \geqslant \widetilde{n}$. We can rewrite this problem as follow:

$$\max_{t(z(n))} \int_{\underline{n}}^{\widetilde{n}} [u_1(n) - \beta P(x_1(n), x^*)] f(n)dn + \int_{\widetilde{n}}^{\overline{n}} [u_2(n) - \beta P(x_2(n), x^*)] f(n)dn.$$

The two components are subject to the same budget constraint, but to different forms of individual's concern. However, the two maximization problem is not fully separated. They shares the same γ and are linked by the smoothness condition, i.e., $\mu(\widetilde{n}) = \widetilde{\mu}$.

B.4 Numerical section calculation

Calculate γ:

$$\begin{aligned}
\mu(n) &= \int_{n}^{\widetilde{n}} \left(1 + \frac{\gamma}{U_z - U_c/p}\right) \exp\left(\int_{n}^{p} \frac{U_{nz} + U_c/m^2}{U_z - U_c/m} dm\right) f(p)dp \\
&\quad + \int_{\widetilde{n}}^{\overline{n}} \left(1 + \frac{\gamma}{U_z}\right) \exp\left(\int_{\widetilde{n}}^{p} \frac{U_{nz}}{U_z} dm\right) f(p)dp \\
&= \int_{n}^{\overline{n}} (1 - \frac{\gamma p}{1+\sigma}) \exp\left(\int_{n}^{p} -\frac{1}{m} dm\right) f(p)dp \\
&\quad + \int_{\widetilde{n}}^{\overline{n}} (1 - \gamma p) \exp\left(\int_{\widetilde{n}}^{p} -\frac{1}{m} dm\right) f(p)dp \\
&= \int_{n}^{\widetilde{n}} (1 - \frac{\gamma p}{1+\sigma}) \left(\frac{n}{p}\right) f(p)dp + \int_{\widetilde{n}}^{\overline{n}} (1 - \gamma p) \left(\frac{\widetilde{n}}{p}\right) f(p)dp.
\end{aligned}$$

Given that $\mu(\underline{n}) = 0$, it follows

$$\int_{\underline{n}}^{\widetilde{n}} (1 - \frac{\gamma p}{1+\sigma}) \left(\frac{n}{p}\right) f(p)dp + \int_{\widetilde{n}}^{\overline{n}} (1 - \gamma p) \left(\frac{\widetilde{n}}{p}\right) f(p)dp = 0.$$

that is,

$$\underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(n)}{p} dp - \gamma \underline{n} \int_{\underline{n}}^{\widetilde{n}} \frac{f(n)}{1+\sigma} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} \frac{f(p)}{p} dp - \gamma \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} f(p)dp = 0.$$

Thus, we obtain:

$$\gamma = \frac{\underline{n} \displaystyle\int_{\underline{n}}^{\widetilde{n}} \frac{f(p)}{p} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} \frac{f(p)}{p} dp}{\underline{n} \displaystyle\int_{\underline{n}}^{\widetilde{n}} \frac{f(p)}{1+\sigma} dp + \widetilde{n} \int_{\widetilde{n}}^{\overline{n}} f(p)dp}.$$

Calculate $z(n)$:

$$z(n) = \int_{\underline{n}}^{n} z'(m)dm - K.$$

When $n < \widetilde{n}$, the individual's utility concern is:

$$U_x \frac{\partial x}{\partial n} + U_z \frac{\partial z}{\partial n} = \frac{\partial z}{\partial n} \frac{U_c}{n}$$

$$\Leftrightarrow \quad z'(n) = \frac{\partial z}{\partial n} = \frac{U_x x'(n)}{U_c/n - U_z} = \frac{n U_x x'(n)}{1 + U_c}.$$

Then,

$$\int_{\underline{n}}^{n} z'(m)dm = \int_{\underline{n}}^{n} \frac{m U_x x'(m)}{1 + U_c} dm$$

$$= \frac{1}{1+\sigma} \int_{\underline{n}}^{n} m \frac{x'(m)}{x(m)} dm$$

$$= \frac{1}{1+\sigma} \left(\int_{\underline{n}}^{n} \frac{\partial}{\partial m} m \ln x(m) dm - \int_{\underline{n}}^{n} \ln x(m) dm \right)$$

$$= \frac{1}{1+\sigma} \left(n \ln x(n) - \underline{n} \ln x(\underline{n}) - \int_{\underline{n}}^{n} \ln x(m) dm \right)$$

$$= v_1(n).$$

When $n \geqslant \widetilde{n}$, the marginal utility of c is null. Thus, in this case the following integral is calculated separately for two supports:

$$\int_{\underline{n}}^{n} z'(m)dm = \int_{\underline{n}}^{\widetilde{n}} \frac{m U_x x'(m)}{1 + U_c} dm + \int_{\widetilde{n}}^{n} m U_x x'(m) dm$$

$$= v_1(\widetilde{n}) + n \ln x(n) - \widetilde{n} \ln x(\widetilde{n}) - \int_{\widetilde{n}}^{n} \ln x(m) dm$$

$$= v_2(n).$$

Next, we determine the constant term K using the government's budget constraint with $\overline{R} = 0$:

$$\int_{\underline{n}}^{\overline{n}} [z(n) - x(n)] f(n) dn = 0$$

$$\Leftrightarrow \quad \int_{\underline{n}}^{\overline{n}} \left[\int_{\underline{n}}^{n} z'(m) dm - K - x(n) \right] f(n) dn = 0.$$

Then,

$$
\begin{aligned}
K &= \int_{\underline{n}}^{\overline{n}} \left[\int_{\underline{n}}^{n} z'(m)dm - x(n) \right] f(n)dn \\
&= \int_{\underline{n}}^{\widetilde{n}} \left[\int_{\underline{n}}^{n} z'(m)dm - x(n) \right] f(n)dn + \int_{\widetilde{n}}^{\overline{n}} \left[\int_{\underline{n}}^{n} z'(m)dm - x(n) \right] f(n)dn \\
&= \int_{\underline{n}}^{\widetilde{n}} \left[v_1(n) - x(n) \right] f(n)dn + \int_{\widetilde{n}}^{\overline{n}} \left[v_2(n) - x(n) \right] f(n)dn.
\end{aligned}
$$

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