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Level N Teichmüller TQFT and Complex Chern–Simons Theory

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Abstract

In this manuscript we review the construction of the Teichmüller TQFT due to Andersen and Kashaev. We further upgrading it to a theory dependent on an extra odd integer N using results developed by Andersen and Kashaev in their work on complex quantum Chern-Simons theory. We also describe how this theory is related with quantum Chern–Simons Theory at level N with gauge group $\mathrm{PSL}(2, \mathbb{C})$.

1 Introduction

In this paper we review Andersen and Kashaev's construction of the Teichmüller TQFT from [AK1] making it dependent on an extra odd integer N, called the level. The original work of [AK1] corresponds to the choice $N = 1$, and emerged as an extension to a 3–dimensional theory of the representations one obtains from Quantum Teichmüller Theory [K3]. In particular, it defines a class of quantum invariants for hyperbolic knots, dependent on a continuous parameter b. The level N Teichmüller TQFT is an analogously upgrade of representations in Quantum Teichmüller theory and it depends on a pair of parameters (b, N), one continuous and one discrete. Such quantum theory is related to the level N Chern–Simons theory with gauge group $\mathrm{PSL}(2, \mathbb{C})$ via the level N Weil-Gel'fand-Zak transform. Such a relation was proposed in [AK3], and here we show it in a more tight way for the four punctured sphere. One of the main ingredient in the construct of the Teichmüller TQFT is the *quantum dilogarithm*, that is a function $\mathrm{D_b} \colon \mathbb{R} \times \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, satisfying some particular properties. Such functions were introduced in [AK3], which for $N = 1$ is Faddeev's original quantum dilogarithm. The theory we get has different and interesting unitarity behaviour depending on the pair of parameters (b, N): for level $N = 1$ the theory is unitary whenever $\mathrm{b} > 0$ or $|\mathrm{b}| = 1$ while for higher level $N > 1$ the unitarity is only manifest when $|\mathrm{b}| = 1$ while in the case $\mathrm{b} > 1$ the behaviour is more exotic. We will consider

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both situations here and we will use the setting $b > 0$ to present some asymptotic properties in the limit $b \to 0$. The Teichmüller TQFT can be used to define knot invariants starting from triangulations of their complements. In this presentation we update the examples presented in [AK1] to the level N setting together with their asymptotic analysis. For the simplest hyperbolic knot we show the appearance of the Baseilhac–Benedetti invariant from [BB] in such a limit.

It is an interesting challenge how the TQFT's which are reviewed in this paper are related to the Witten-Reshetikhin-Turaev TQFT's [W1, RT1, RT2, BHMV1, BHMV2, B] and in particular how they are related to the geometric construction of these TQFT's [ADW, Hit2, Las, TUY, AU1, AU2, AU3, AU4] and to Witten's proposal for the construction of the complex quantum Chern-Simons theory [W2], which can actually be constructed from a purely mathematical point of view [AG], resulting also in the mathematically well-defined Hitchin-Witten connection in the bundle of quantizations of the moduli space of flat $\mathrm{SL}(n, \mathbb{C})$-connections over Teichmüller space. In the classical case of compact groups, the description of the representations of the mapping class groups via the monodromy of the Hitchin connections turned out to be very useful to prove deep properties about these representations [A1, A2, A3, A4, AH, AHJMMc], some of which also uses the theory of Toeplitz operators [BMS, KS]. Understanding how these kinds of results can be extended to the complex quantum theory discussed in this paper will be very interesting and most likely involve using Higgs bundles techniques [Hit1]. Certainly we have already seen the start of this with the Verlinde formula for Higgs bundle moduli space [AGP].

The paper is organised as follows. In section 2 we recall the definition of the (decorated) Ptolemy groupoid of punctured surfaces, which is the combinatorial foundation over which Quantum Teichmüller Theory is defined. In section 3 we recall the quantum dilogarithm $\mathrm{D_b}$, and we list some of its properties. The function $\mathrm{D_b}$ was introduced in [AK3] for the first time, but some of its properties that we list here are not present in the literature. In section 5 we carry out the quantization of the moduli space of $\mathrm{PSL}(2, \mathbb{C})$ flat connections over a four punctured sphere with unipotent holonomies around the punctures. We follow the prescriptions of geometric quantisation, together with a choice of real polarisation, and we connect the resulting algebra of observables to the $\mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z})$ representations of quantum Teichmüller theory of the previous section. Finally in section 6 we construct the the Teichmüller TQFT functor $F_{\mathrm{b}}^{(N)}$ mirroring the construction in [AK1] and we study some examples and their asymptotic behaviour. It would of course be interesting to go through all the examples treated in [AN] in this volume for the level N theory as well.

Acknowledgements

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2 Ptolemy Groupoid

Let $\Sigma_{g,s}$ be a surface of genus g with s punctures, such that $s > 0$ and $2-2g+s < 0$.

Definition 2.1. An *ideal arc* α is the homotopy class relative endpoints of the embedding of a path in $\Sigma_{g,s}$, such that the endpoints are punctures of the surface. An *ideal triangle* is a triangle with the vertices removed, such that the edges are ideal arcs.
An *ideal triangulation* τ of $\Sigma_{g,s}$ is a collection of disjoint ideal arcs such that $\Sigma_{g,s} \setminus \tau$ is a collection of interiors of ideal triangles.

Given an ideal triangulation τ, $\Delta_j(\tau)$ will denote the set of its j-dimensional cells.

Definition 2.2. A *decorated ideal triangulation* of $\Sigma_{g,s}$ is an ideal triangulation τ up to isotopy relative to the punctures, together with the choice of a distinguished corner in each ideal triangle and a bijective ordering map

$$\overline{\tau} : \{1,\ \ldots,\ s\} \ni j \mapsto \overline{\tau}_j \in \Delta_2(\tau).$$

We denote the set of all decorated ideal triangulation as $\dot{\Delta} = \dot{\Delta}(\Sigma_{g,s})$.

When we say that τ is a decorated ideal triangulation we mean that τ is the set of decorated ideal triangles in the triangulation.

One of the main interests in quantizing moduli spaces is the consequent construction of representations of (central extensions of) the mapping class group of the surfaces [W2, Hit1, Las, A1, AU4, AG]. Quantum Teichmüller theory produce instead representations of a bigger object called the (decorated) *Ptolemy Groupoid* that we are going to introduce now.

Recall that, given a group G acting freely on a set X, we can define an associated groupoid $\mathcal{G}$ as follows. The objects of $\mathcal{G}$ are G-orbits in X while morphisms are G-orbits in $X \times X$ with respect to the diagonal action. Then for any $x \in X$ we can consider the object $[x]$ and for any pair $(x, y) \in X \times X$ we can consider the morphism $[x, y]$. When $[y] = [u]$ there will be a $g \in G$ so that $gu = y$ and we can define the composition $[x, y][u, v] = [x, gv]$. The unit for $[x]$ is given by $[x, x]$. If the action of G is transitive, we would get an actual group. We will abbreviate $[x_1, x_2][x_2, x_3] \cdots [x_{n-1}, x_n]$ with $[x_1, x_2, \ldots, x_n]$.
We define the *decorated Ptolemy groupoid* $\mathcal{G}(\Sigma_{g,s})$ of a punctured surface $\Sigma_{g,s}$ following the above recipe. The set we consider is the set $\dot{\Delta}$ of decorated triangulations τ of $\Sigma_{g,s}$. The free group action is the one of the mapping class group $\mathrm{MCG}_{g,s}$ acting on $\dot{\Delta}$. This action is not transitive, meaning that not all pairs of decorated ideal triangulations can be related by a mapping class group element. However in the language of groupoids, we can still describe generators and relations for the morphism groups. For $\tau \in \dot{\Delta}$ there are three kind of generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i \tau]$ and $[\tau, \omega_{i,j} \tau]$, where τ^σ is obtained by applying the permutation $\sigma \in \mathbb{S}_{|\tau|}$

to the ordering of triangles in τ, $\rho_i\tau$ is obtained by changing the distinguished corner in the triangle $\bar{\tau}_i \in \tau$ as in Figure 1, and $\omega_{i,j}$ is obtained by applying a decorated diagonal exchange to the quadrilateral made of the two decorated ideal triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ as in Figure 2.

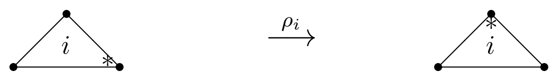


Figure 1: Transformation ρ_i.

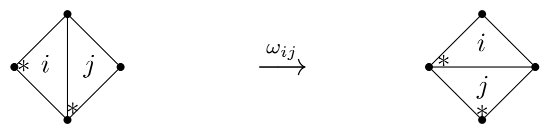


Figure 2: Transformation ω_{ij}.

The relations are usually grouped in two sets, the first being

$$[\tau, \tau^{\alpha}, (\tau^{\alpha})^{\beta}] = [\tau, \tau^{\alpha\beta}], \quad \alpha, \beta \in \mathbb{S}_{\tau}, \tag{2.1}$$

$$[\tau, \rho_i\tau, \rho_i\rho_i\tau, \rho_i\rho_i\rho_i\tau] = \mathrm{id}_{[\tau]}, \tag{2.2}$$

$$[\tau, \omega_{i,j}\tau, \omega_{i,k}\omega_{i,j}\tau, \omega_{j,k}\omega_{i,k}\omega_{i,j}\tau] = [\tau, \omega_{j,k}\tau, \omega_{i,j}\omega_{j,k}\tau] \tag{2.3}$$

$$[\tau, \omega_{i,j}\tau, \rho_i\omega_{i,j}\tau, \omega_{j,i}\rho_i\omega_{i,j}\tau] = [\tau, \tau^{(i,j)}, \rho_j\tau^{(i,j)}, \rho_i\rho_j\tau^{(i,j)}] \tag{2.4}$$

The first two relations are obvious, the third is called the Pentagon Relation and the fourth is called the Inversion Relation.

The second set of relations, are commutation relations

$$[\tau, \rho_i\tau, \rho_i\tau^{\sigma}] = [\tau, \tau^{\sigma}, \rho_{\sigma^{-1}(i)}\tau^{\sigma}], \tag{2.5}$$

$$[\tau, \omega_{i,j}\tau, (\omega_{i,j}\tau)^{\sigma}] = [\tau, \tau^{\sigma}, \omega_{\sigma^{-1}(i)\sigma^{-1}(j)}\tau^{\sigma}], \tag{2.6}$$

$$[\tau, \rho_j\tau, \rho_j\rho_i\tau] = [\tau, \rho_i\tau, \rho_i\rho_j\tau], \tag{2.7}$$

$$[\tau, \rho_i\tau, \omega_{j,k}\rho_i\tau] = [\tau, \omega_{j,k}\tau, \rho_i\omega_{j,k}\tau], \; i \notin \{j,k\}, \tag{2.8}$$

$$[\tau, \omega_{i,j}\tau, \omega_{k,l}\omega_{i,j}\tau] = [\tau, \omega_{k,l}\tau, \omega_{i,j}\omega_{k,l}\tau], \; \{i,j\} \cap \{k,l\} = \emptyset, \tag{2.9}$$

To every decorated ideal triangulation $\tau \in \dot{\Delta}$ it is possible to associate a simple symplectic space $\mathcal{R}(\tau)$, called the space of *Ratio Coordinates*. We summarize here its relation with the Ptolemy groupoid and refer to [K3] for a detailed introduction to ratio coordinates and their relation to the Teichmüller space. Let $M \equiv 2g - 2 + s = |\tau|$ be the number of ideal triangles, then $\mathcal{R}(\tau) \equiv (\mathbb{R}_{>0} \times \mathbb{R}_{>0})^M$. Let $x^j \equiv (x_1^j, x_2^j) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$, for $j = 1, \ldots, M$ be the coordinates associated to the ideal triangle $\bar{\tau}_j \in \Delta_2(\tau)$. The symplectic form that we consider on $\mathcal{R}(\tau)$ is

(2.10) $$\omega_\tau \equiv \sum_{j=1}^{M} \frac{\mathrm{d}x_1^j}{x_1^j} \wedge \frac{\mathrm{d}x_2^j}{x_2^j}$$

Now we want to describe the action of $\mathcal{G}(\Sigma_{g,s})$ as symplectomorphisms between these spaces. The morphisms $[\tau, \tau^\sigma]$ act by permuting the coordinates in $\mathcal{R}(\tau)$. The morphism $[\tau, \rho_i\tau]$ acts as the identity on any pair $x = (x_1, x_2)$ corresponding to ideal triangles different from $\bar{\tau}_i$ and as $(x_1, x_2) = x \mapsto y = (\frac{x_2}{x_1}, \frac{1}{x_1})$ for the pair of coordinates corresponding to $\bar{\tau}_i$. Finally the action of $[\tau, \omega_{i,j}\tau]$ is the identity on $\bar{\tau}_k$ for $k \neq i, j$ while letting $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be the coordinates corresponding to the triangles $\bar{\tau}_i$ and $\bar{\tau}_j$ respectively, and letting $u = (u_1, u_2)$ and $v = (v_1, v_2)$ be the coordinates of the triangles $\overline{\omega_{i,j}\tau}_i$ and $\overline{\omega_{i,j}\tau}_j$, then we have $u = x \bullet y$ and $v = x * y$ where

(2.11) $$\begin{aligned} x \bullet y &:= (x_1 y_1, x_1 y_2 + x_2) \\ x * y &:= \left(\frac{y_1 x_2}{x_1 y_2 + x_2}, \frac{y_2}{x_1 y_2 + x_2} \right). \end{aligned}$$

Let $\widetilde{\Delta}$ be the set of pairs $(\tau, \mathcal{R}(\tau))$, $\tau \in \dot{\Delta}$. Then the space $\mathcal{R}(\Sigma_{g,s})$ is defined as the quotient of $\widetilde{\Delta}$ by the action of $\mathcal{G}(\Sigma_{g,s})$ as described above. This space of coordinates is now independent of the triangulation. For more details on the (decorated or not) Ptolemy groupoid see [P],[FK],[K6][K4].

3 Quantum Dilogarithm

In this section we recall the quantum dilogarithm $\mathrm{D_b}$ over $\mathbb{A}_N$ and we state some of their properties.

Definition 3.1 (q-Pochammer Symbol)**.** Let $x, q \in \mathbb{C}$, such that $|q| < 1$. Define the *q-Pochammer Symbol* of x as

$$(x;q)_\infty := \prod_{i=0}^{\infty} (1 - xq^i)$$

Theorem 3.2. *Let X, Y satisfying $XY = qYX$. Then the following five-term relation holds true*

(3.1) $$(Y;q)_\infty (X;q)_\infty = (X;q)_\infty (-YX;q)_\infty (Y;q)_\infty .$$

Definition 3.3 (Faddeev's Quantum Dilogarithm [F])**.** Let $z, \mathrm{b} \in \mathbb{C}$ be such that $\mathrm{Re}\,\mathrm{b} \neq 0$, $|\,\mathrm{Im}(z)| < |\,\mathrm{Im}(c_\mathrm{b})|$, where $c_\mathrm{b} := i(\mathrm{b}+\mathrm{b}^{-1})/2$. Let $C \subset \mathbb{C}$, $C = \mathbb{R}+i0$ be a contour equal to the the real line outside a neighborhood of the origin that avoid the singularity in 0 going in the upper half plane. *Faddeev's quantum dilogarithm* is defined to be

$$\Phi_\mathrm{b}(z) = \exp\left(\int_C \frac{e^{-2izw}\mathrm{d}w}{4\sinh(w\mathrm{b})\sinh(w\mathrm{b}^{-1})w}\right). \tag{3.2}$$

It is evident that Φ_b is invariant under the following changes of parameter

$$\mathrm{b} \leftrightarrow \mathrm{b}^{-1} \leftrightarrow -\mathrm{b}, \tag{3.3}$$

so that our choice of b can be restricted to the first quadrant

$$\mathrm{Re}\,\mathrm{b} > 0, \qquad \mathrm{Im}\,\mathrm{b} \geq 0 \tag{3.4}$$

which implies

$$\mathrm{Im}(\mathrm{b}^2) \geq 0. \tag{3.5}$$

Faddeev's quantum dilogarithm has a lot of other interesting properties and applications, see for example [F],[FK],[FKV] and [V].

Let $N \geq 1$ be a positive *odd* integer. Then, following [AK3], we can define a *quantum dilogarithm* over $\mathbb{A}_N$ as follows

$$\mathrm{D}_\mathrm{b}(x,n) := \prod_{j=0}^{N-1} \Phi_\mathrm{b}\left(\frac{x}{\sqrt{N}} + (1-N^{-1})c_\mathrm{b} - i\mathrm{b}^{-1}\frac{j}{N} - i\mathrm{b}\left\{\frac{j+n}{N}\right\}\right) \tag{3.6}$$

where $\{p\}$ is the fractional part of p, and Φ_b is the Faddeev's quantum dilogarithm. Of course for $N = 1$ we have just $\Phi_\mathrm{b}(x)$. The function D_b was introduced in [AK3] only for $|\mathrm{b}| = 1$. It satisfies a series properties that we are going to list.

Lemma 3.4 (Inversion Relation [AK3])**.**

$$\mathrm{D}_\mathrm{b}(x,n)\mathrm{D}_\mathrm{b}(-x,-n) = e^{\pi i x^2} e^{-\pi i n(n+N)/N} \zeta_{N,\,inv}^{-1},$$

where

$$\zeta_{N,\,inv} = e^{\pi i (N + 2c_\mathrm{b}^2 N^{-1})/6}. \tag{3.7}$$

Unitarity properties are different in the two situations $|\mathrm{b}| = 1$ or $\mathrm{b} \in \mathbb{R}$.

Lemma 3.5 (Unitarity)**.**

$$\overline{\mathrm{D}_\mathrm{b}(x,n)} = \mathrm{D}_\mathrm{b}(\bar{x},n)^{-1} \qquad \textit{if } |\mathrm{b}| = 1, \tag{3.8}$$

$$\overline{\mathrm{D}_\mathrm{b}(x,n)} = \mathrm{D}_\mathrm{b}(\bar{x},-n)^{-1} \qquad \textit{if } \mathrm{b} \in \mathbb{R}_{>0}. \tag{3.9}$$

Remark 3.6. One can see that

$$\mathrm{D_b}(x,-n) = \mathrm{D_{b^{-1}}}(x,n) \tag{3.10}$$

just by the Definition 3.6 for $\mathrm{D_{b^{-1}}}$ and carefully substituting $j+n \mapsto j'$. In particular the unitarity for $\mathrm{b}>0$ can be re-expressed as

$$\overline{\mathrm{D_b}(x,n)} = \left(\mathrm{D_{b^{-1}}}(x,n)\right)^{-1} \tag{3.11}$$

Lemma 3.7 (Faddeev's difference equations)**.** *Let*

$$\chi^{\pm}(x,n) \equiv e^{2\pi \frac{\mathrm{b}^{\pm 1}}{\sqrt{N}} x} e^{\pm \frac{2\pi i n}{N}}, \tag{3.12}$$

for every x, $\mathrm{b}\in\mathbb{C}$, $\mathrm{Im(b)}\neq 0$ n, $N\in\mathbb{Z}$ we have

$$\mathrm{D_b}\left(x + i\frac{\mathrm{b}^{\pm 1}}{\sqrt{N}}, n\pm 1\right) = \mathrm{D_b}\left(x,n\right)\left(1+\chi^{\pm}(x,n)e^{-\pi i \frac{N-1}{N}} e^{\pi i \frac{\mathrm{b}^{\pm 2}}{N}}\right)^{-1} \tag{3.13}$$

$$\mathrm{D_b}\left(x - i\frac{\mathrm{b}^{\pm 1}}{\sqrt{N}}, n\mp 1\right) = \mathrm{D_b}(x,n)\left(1+\chi^{\pm}(x,n)e^{\pi i \frac{N-1}{N}} e^{-\pi i \frac{\mathrm{b}^{\pm 2}}{N}}\right) \tag{3.14}$$

Proposition 3.8. *If* $\mathrm{Im(b)}>0$ *and* $\mathrm{Re(b)}>0$ *we have*

$$\mathrm{D_b}(x,n) = \frac{\left(\chi^{+}(x+\frac{c_\mathrm{b}}{\sqrt{N}},n);q^2\omega\right)_\infty}{\left(\chi^{-}(x-\frac{c_\mathrm{b}}{\sqrt{N}},n);\tilde{q}^2\overline{\omega}\right)_\infty} \tag{3.15}$$

where $q=e^{i\pi\frac{\mathrm{b}^2}{N}}$, $\tilde{q}=e^{-\pi i \frac{\mathrm{b}^{-2}}{N}}$, $\omega=e^{\frac{2\pi i}{N}}$ *and* $\chi^{\pm}(x,n)=e^{2\pi\frac{\mathrm{b}^{\pm1}}{\sqrt{N}}x}e^{\pm\frac{2\pi i n}{N}}$.

Proposition 3.9. *The quantum dilogarithm* $\mathrm{D_b}(x,n)$, *for* $\mathrm{Im(b)}>0$ *has poles*

$$\begin{cases} x = \frac{c_\mathrm{b}}{\sqrt{N}} + i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l + i\frac{\mathrm{b}}{\sqrt{N}}m \\ n = m-l \mod N \end{cases}$$

and zeros

$$\begin{cases} x = -\frac{c_\mathrm{b}}{\sqrt{N}} - i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l - i\frac{\mathrm{b}}{\sqrt{N}}m \\ n = l-m \mod N \end{cases}$$

for $l,m\in\mathbb{Z}_{>0}$. *Moreover its residue at* $(x_{l,m},n_{l,m}) = \left(\frac{c_\mathrm{b}}{\sqrt{N}} + i\frac{\mathrm{b}^{-1}}{\sqrt{N}}l + i\frac{\mathrm{b}}{\sqrt{N}}m, m-l\right)$ *is*

$$\frac{\sqrt{N}}{2\pi\mathrm{b}^{-1}}\frac{(q^2\omega;q^2\omega)_\infty}{(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_\infty}\frac{(-\tilde{q}^2\overline{\omega})^l(\tilde{q}^2\overline{\omega})^{l(l-1)/2}}{(q^2\omega;q^2\omega)_m(\tilde{q}^2\overline{\omega};\tilde{q}^2\overline{\omega})_l} \tag{3.16}$$

The following Summation Formula can be shown by a residue computation. It is well known for $N=1$, i.e. for Φ_b, see [FKV] for example.

Theorem 3.10 (Summation Formula)**.** *Suppose* $\mathrm{Im}(\mathrm{b}) > 0$ *and* N *odd, and let* $u, v, w \in \mathbb{C}$ *and* $a, b, c \in \mathbb{Z}/N\mathbb{Z}$ *satisfy*

$$\mathrm{Im}\left(v + \frac{c_\mathrm{b}}{\sqrt{N}}\right) > 0, \qquad \mathrm{Im}\left(-u + \frac{c_\mathrm{b}}{\sqrt{N}}\right) > 0, \qquad \mathrm{Im}(v-u) < \mathrm{Im}(w) < 0. \tag{3.17}$$

Define

$$\Psi(u, v, w, a, b, c) \equiv \int_{\mathbb{A}_N} \frac{\mathrm{D_b}(x+u, a+d)}{\mathrm{D_b}(x+v, b+d)} e^{2\pi i w x} e^{-2\pi i \frac{cd}{N}} \mathrm{d}(x, d) \tag{3.18}$$

Then we have that

$$\begin{aligned}\Psi(u, v, w, a, b, c) &\\ &= \zeta_0 \frac{\mathrm{D_b}\left(v - u - w + \frac{c_\mathrm{b}}{\sqrt{N}}, b - a - c\right)}{\mathrm{D_b}\left(-w - \frac{c_\mathrm{b}}{\sqrt{N}}, -c\right) \mathrm{D_b}\left(v - u + \frac{c_\mathrm{b}}{\sqrt{N}}, b - a\right)} e^{2\pi i w\left(\frac{c_\mathrm{b}}{\sqrt{N}} - u\right)} \omega^{ac} \\ &= \zeta_0^{-1} \frac{\mathrm{D_b}\left(w + \frac{c_\mathrm{b}}{\sqrt{N}}, c\right) \mathrm{D_b}\left(-v + u - \frac{c_\mathrm{b}}{\sqrt{N}}, -b + a\right)}{\mathrm{D_b}\left(-v + u + w - \frac{c_\mathrm{b}}{\sqrt{N}}, -b + a + c\right)} e^{2\pi i w\left(-\frac{c_\mathrm{b}}{\sqrt{N}} - v\right)} \omega^{bc}\end{aligned}$$

where $\zeta_0 = e^{-\pi i (N - 4c_\mathrm{b}^2 N^{-1})/12}$.

Remark 3.11. Assumptions (3.17) even though sufficient are not optimal. Indeed they guarantee the theorem to hold true when the integration is performed along the real line, however we can deform the integration contour as long as

$$|\arg(iz)| < \pi - \arg \mathrm{b} \qquad z \text{ being one of } \left\{w, v - u - w, u - v - 2\frac{c_\mathrm{b}}{\sqrt{N}}\right\} \tag{3.19}$$

Using the notation for the Fourier Kernels from (A.13) in Appendix A.2 we have that

Proposition 3.12 (Fourier Transformation Formula, [AK3])**.** *For* N *odd we have that*

$$\begin{aligned}\int_{\mathbb{A}_N} \mathrm{D_b}(x, n) \langle (x, n); (w, c) \rangle \mathrm{d}(x, n) &= \frac{e^{2\pi i w \frac{c_\mathrm{b}}{\sqrt{N}}}}{\mathrm{D_b}\left(-w - \frac{c_\mathrm{b}}{\sqrt{N}}, -k\right)} e^{-\pi i (N - 4c_\mathrm{b}^2 N^{-1})/12} \\ &= \mathrm{D_b}\left(w + \frac{c_\mathrm{b}}{\sqrt{N}}, c\right) \overline{\langle (w, c) \rangle} e^{\pi i (N - 4c_\mathrm{b}^2 N^{-1})/12} \\ \int_{\mathbb{A}_N} (\mathrm{D_b}(x, n))^{-1} \langle (x, n); (w, c) \rangle \mathrm{d}(x, n) &= \frac{\langle (w, c) \rangle}{\mathrm{D_b}\left(-w - \frac{c_\mathrm{b}}{\sqrt{N}}, -k\right)} e^{-\pi i (N - 4c_\mathrm{b}^2 N^{-1})/12} \\ &= \mathrm{D_b}\left(w + \frac{c_\mathrm{b}}{\sqrt{N}}, c\right) e^{-2\pi i w \frac{c_\mathrm{b}}{\sqrt{N}}} e^{\pi i (N - 4c_\mathrm{b}^2 N^{-1})/12}\end{aligned}$$

Proposition 3.13 (Integral Pentagon Relation)**.** *Let* $\widetilde{\mathrm{D}_{\mathrm{b}}}(x,n) \equiv \mathsf{F}_N \circ \mathcal{F}^{-1}(\mathrm{D}_{\mathrm{b}})(x,n)$. *We have the following integral relation*

$$\begin{aligned}&\langle (x,n);(y,m)\rangle \widetilde{\mathrm{D}_{\mathrm{b}}}(x,n)\widetilde{\mathrm{D}_{\mathrm{b}}}(y,m)\\ &\quad = \int_{\mathbb{A}_N} \widetilde{\mathrm{D}_{\mathrm{b}}}(x-z,n-k)\widetilde{\mathrm{D}_{\mathrm{b}}}(z,k)\widetilde{\mathrm{D}_{\mathrm{b}}}(y-z,m-k)\langle (z,k)\rangle \mathrm{d}(z,k).\end{aligned}$$

Before we look at the asymptotic behavior of Φ_{b} let us recall the classical dilogarithm function, defined on $|z|<1$ by

$$\mathrm{Li}_2(z) = \sum_{n\geq 1} \frac{z^n}{n^2} \tag{3.20}$$

and recall that it admits analytic continuation to $\mathbb{C}\setminus[1,\infty]$ through the following integral formula

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\log(1-u)}{u}\mathrm{d}u. \tag{3.21}$$

Proposition 3.14. *We have the following behaviour when* $\mathrm{b}>0$*,* $\mathrm{b}\to 0$ *and* x*,* n*,* N *are fixed*

$$\mathrm{D}_{\mathrm{b}}(\frac{x}{2\pi\mathrm{b}},n) = \mathrm{Exp}\left(\frac{\mathrm{Li}_2(-e^{x\sqrt{N}})}{2\pi i b^2 N}\right)\phi_x(n)(1+\mathcal{O}(b^2)) \tag{3.22}$$

where $\phi_x(n)$ *is defined by* $\begin{cases}\phi_x(n+1) = \phi_x(n)\frac{(1-e^{x/\sqrt{N}}\overline{\omega}^{n+\frac{1}{2}})}{(1+e^{x\sqrt{N}})^{1/N}}\\ \phi_x(0) = (1+e^{x\sqrt{N}})^{-\frac{N-1}{2N}}\prod_{j=0}^{N-1}(1-e^{xN^{-\frac{1}{2}}}\overline{\omega}^{j+\frac{1}{2}})^{\frac{j}{N}}\end{cases}$
whenever N *is odd.*

Remark 3.15. The function ϕ_x on the finite set $\mathbb{Z}/N\mathbb{Z}$ is a cyclic quantum dilogarithm [FK],[K3], [K1]. Precisely $\frac{1}{\phi_x}$ corresponds to the function Ψ_λ from Proposition 10 in [K3] with $\lambda = e^{x/\sqrt{N}}$.

The Hilbert space $\mathrm{L}^2(\mathbb{A}_N)$ is naturally isomorphic to the tensor product $\mathrm{L}^2(\mathbb{R})\otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \cong \mathrm{L}^2(\mathbb{R})\otimes\mathbb{C}^N$, see Appendix A.2. Let p and q two self-adjoint operators on $\mathrm{L}^2(\mathbb{R})$ satisfying

$$[\mathsf{p},\mathsf{q}] = \frac{1}{2\pi i} \tag{3.23}$$

and let X and Y unitary operators satisfying

$$YX = e^{2\pi i/N}XY, \quad X^N = Y^N = 1, \tag{3.24}$$

together with the cross relations

$$[\mathsf{p}, X] = [\mathsf{p}, Y] = [\mathsf{q}, X] = [\mathsf{q}, Y] = 0. \tag{3.25}$$

The equations in (3.24) imply that X and Y will have finite and the same spectrum, and this will be a subset of the set $\mathbb{T}_N$ of all N-th complex roots of unity. Let

$$\mathrm{L}_N : \mathbb{T}_N \longrightarrow \mathbb{Z}/N\mathbb{Z}$$

be the natural group isomorphism. We can define $\mathrm{L}_N(A)$, by the spectral theorem, for any operator A of order N, such that it formally satisfies

$$A = e^{2\pi i\,\mathrm{L}_N(A)/N}.$$

One has that

$$\mathrm{L}_N(-e^{-\pi i/N} XY) = \mathrm{L}_N(X) + \mathrm{L}_N(Y). \tag{3.26}$$

For any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ recall the definition of $\tilde{f}$ and the operator function $\not{f}(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ from Appendix A.2. The following Pentagon Identity for D_b was first proved in [AK3], where a projective ambiguity was undetermined and $|\mathrm{b}| = 1$.

Lemma 3.16 (Pentagon Equation)**.** *Let* p,q, X *and* Y *be as above, then the following five-term relation holds*

$$\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X)\not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y) = \not{\mathrm{D}}_\mathrm{b}(\mathsf{q}, Y)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}+\mathsf{q}, -e^{\pi i/N} XY)\not{\mathrm{D}}_\mathrm{b}(\mathsf{p}, X). \tag{3.27}$$

Proof. This is equivalent to the Integral Pentagon equation of Proposition 3.13. To see this we need to use equation (A.14) an all the five terms. Then we compare the coefficients of $e^{2\pi i y \mathsf{q}} Y^m e^{2\pi i x \mathsf{p}} X^{-n}$ and get exactly the integral pentagon equation.
An alternative proof follows from the q-Pochammer presentation of D_b from Proposition 3.8. □

3.1 Charges

We are going to define a *charged* version of the dilogarithm. The charges will assume geometrical meaning in the construction of the partition function, however they already satisfy the purpose of turning all the conditional convergent integral relations of the dilogarithm D_b (e.g. Proposition 3.13 and 3.12) into absolutely convergent integrals.
Let a, b and c be three real positive numbers such that $a+b+c = \frac{1}{\sqrt{N}}$. We define the charged quantum dilogarithm

$$\psi_{a,c}(x, n) := \frac{e^{-2\pi i c_\mathrm{b} a x}}{\mathrm{D}_\mathrm{b}(x - c_\mathrm{b}(a+c), n)}. \tag{3.28}$$

From the Fourier transformation formula, Proposition 3.12, and the inversion formula in Lemma 3.4, we can deduce the following transformation formulas for $\psi_{a,c}$ (recall notation (A.15) for the inverse Fourier transform)

Lemma 3.17. *Suppose* $\mathrm{Im}(\mathrm{b})(1-|\mathrm{b}|)=0$*, then*

$$\tilde{\psi}_{a,c}(x,k) = \psi_{c,b}(x,k)\langle x,k\rangle e^{-\pi i c_{\mathrm{b}}^2 a(a+2c)}\zeta_0 \tag{3.29}$$

$$\overline{\psi_{a,c}(x,k)} = \psi_{c,a}(-x,\epsilon k)\langle x,k\rangle e^{\pi c_{\mathrm{b}}^2(a+c)^2}\zeta_{N,inv} \tag{3.30}$$

$$\overline{\tilde{\psi}_{a,c}(x,k)} = \psi_{b,c}(-x,\epsilon k) e^{-2\pi i c_{\mathrm{b}}^2 ab}\zeta_0 \tag{3.31}$$

where $\zeta_0 = e^{-\pi i(N-4c_{\mathrm{b}}^2N^{-1})/12}$ *and* $\zeta_{N,inv} = \zeta_0^2 e^{-\pi i c_{\mathrm{b}}^2/N}$ *and* $\epsilon = +1$ *if* $\mathrm{b} > 0$ *or* $\epsilon = -1$ *if* $|\mathrm{b}| = 1$*.*

Remark 3.18. The hypothesis on positivity of a, b and c assure that the Fourier integral of $\tilde{\psi}_{a,c}$ is absolutely convergent.

Theorem 3.19 (Charged Pentagon Equation)**.** *Let* a_j*,* $c_j > 0$ *such that* $\frac{1}{\sqrt{N}} - a_j - c_j > 0$ *for* $j = 0, 1, 2, 3$ *or* 4*. Define* $\psi_j \equiv \psi_{a_j,b_j}$*. Suppose the following relations hold true*

(3.32)
$$a_1 = a_0 + a_2 \qquad a_3 = a_2 + a_4 \qquad c_1 = c_0 + a_4 \qquad c_3 = a_0 + c_4 \qquad c_2 = c_1 + c_3$$

and consider the operators on $\mathrm{L}^2(\mathbb{A}_N)$ *defined to satisfy (3.23 -3.24). We have the following charged pentagon relation*

$$\begin{aligned}\psi_1(\mathsf{q},\mathrm{L}_N(X))\psi_3(\mathsf{p},\mathrm{L}_N(Y))\xi(a,c) &= \\ = \psi_4(\mathsf{p},\mathrm{L}_N(X))\psi_2(\mathsf{p}+\mathsf{q},\mathrm{L}_N(X)&+\mathrm{L}_N(Y))\psi_0(\mathsf{q},\mathrm{L}_N(Y))\end{aligned} \tag{3.33}$$

where $\xi(a,c) = e^{2\pi i c_{\mathrm{b}}^2(a_0a_2+a_0a_4+a_2a_4)}e^{\pi i c_{\mathrm{b}}^2 a_2^2}$*.*

4 Quantum Teichmüller Theory

In this section we are going to quantize the space $\mathcal{R}(\Sigma_{g,s})$ following [K3]. For any fixed τ the quantization of $\mathcal{R}(\tau)$ is just the canonical quantization in exponential coordinates of the space $\mathbb{R}^M_{>0}\times\mathbb{R}^M_{>0}$, where $M = 2g-2+s$, with symplectic form $\omega_\tau = \sum_{j=1}^M \mathrm{d}\log u_j \wedge \mathrm{d}\log v_j$. Formally, following the expectations from canonical quantization of $\mathbb{R}^{2M}$, we can quantize $\mathcal{R}(\tau)$ and associate to it an algebra of operator

$$\mathcal{X}(\tau) \text{ generated by } \{\hat{u}_j,\, \hat{v}_j\}\text{, where } 0 \le j < M\text{, subject to the relations}$$
$$\hat{u}_j\hat{v}_l = q^{\delta(j-l)}\hat{v}_l\hat{u}_j \quad \hat{u}_j\hat{u}_l = \hat{u}_l\hat{u}_j \quad \hat{v}_j\hat{v}_l = \hat{v}_l\hat{v}_j$$

where $q\in\mathbb{C}^*$. The algebra we mean here is the associative algebra of non commutative fractions of non commutative polynomials generated by these generators.

In order to obtain a quantization of $\mathcal{R}(\Sigma_{g,s})$ (i.e. triangulation independent) we have to look at the action of the $\mathcal{G}(\Sigma_{g,s})$ generators on coordinates and translate it into an action on the algebras $\mathcal{X}(\tau)$. Precisely consider the set of the couples $(\tau, \mathcal{X}(\tau))$ and let the generators $[\tau, \tau^\sigma]$, $[\tau, \rho_i\tau]$ and $[\tau, \omega_{i,j}\tau]$ act on them. The action on the algebras is as follows. The elements $[\tau, \tau^\sigma]$ just permutes the indexes of the generators according to the permutation σ. The change of decoration $[\tau, \rho_i\tau]$ acts trivially on the operators $(\hat{u}_j, \hat{v}_j)$ such that $j \neq i$ and as follows on $(\hat{u}_i, \hat{v}_i)$

$$(\hat{u}_i, \hat{v}_i) \mapsto (q^{-1/2}\hat{v}_i\hat{u}_i^{-1}, \hat{u}_i^{-1}). \tag{4.1}$$

The most interesting generator $[\tau, \omega_{i,j}\tau]$, is again trivial in the triangles not involved in the diagonal exchange, but it maps the two couples of operators $(\hat{u}_i, \hat{v}_i)$ and $(\hat{u}_j, \hat{v}_j)$ to the two new couples (following formulas (2.11))

$$(\hat{u}_i, \hat{v}_i) \bullet (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_i\hat{u}_j, \hat{u}_i\hat{v}_j + \hat{v}_i) \tag{4.2}$$

$$(\hat{u}_i, \hat{v}_i) * (\hat{u}_j, \hat{v}_j) \equiv (\hat{u}_j\hat{v}_i(\hat{u}_i\hat{v}_j + \hat{v}_i)^{-1}, \hat{v}_j(\hat{u}_i\hat{v}_j + \hat{v}_i)^{-1}). \tag{4.3}$$

In order to get an actual quantization we need to provide a representation of $\mathcal{X}(\tau)$ by operators acting on some vector space $\mathcal{H}$. In the original paper [K3], Kashaev proposed representations on the vector spaces $\mathrm{L}^2(\mathbb{R})$ and $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ for N odd. The former was used to construct the Andersen-Kashaev invariants in [AK1], while the latter are related to the colored Jones polynomials ([K1], [MM]) and the Volume Conjecture [K2]. In the more recent work [AK3] a representation on the vector space $\mathrm{L}^2(\mathbb{A}_N) \equiv \mathrm{L}^2(\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{R}) \otimes \mathbb{C}^N$ was implicitly proposed, or at least all the basics elements to construct it were presented. Here we describe the representations in $\mathrm{L}^2(\mathbb{A}_N)$.

4.1 $\mathrm{L}^2(\mathbb{A}_N)$ Representations

Fix N positive odd integer, $\omega \equiv e^{\frac{2\pi i}{N}}$ and $\mathrm{b} \in \mathbb{C}^*$, $\mathrm{Re}(\mathrm{b}) > 0$. To each decorated ideal triangle $\overline{\tau}_j \in \tau$ we associate the Hilbert space $\mathrm{L}^2(\mathbb{A}_N)$. Then the Hilbert space associated to $\mathcal{R}(\tau)$ will be $\mathcal{H} = \mathrm{L}^2(\mathbb{A}_N)^{\otimes M} \cong \mathrm{L}^2(\mathbb{A}_N^M)$ where $M = 2g-2+s$ is the number of triangles in τ. For conventions and notation on the space $\mathrm{L}^2(\mathbb{A}_N)$ see Appendix A.2. For every $i = 0, \ldots, M$ let p_i, q_i be self adjoint operator in $\mathrm{L}^2(\mathbb{R})$ and X_i, Y_i unitary operators in $\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N$ such that

$$[p_i, q_j] = \frac{\delta_{ij}}{2\pi i}, \qquad Y_iX_j = \omega^{\delta_{ij}}X_jY_i, \qquad X_i^N = Y_i^N = 1. \tag{4.4}$$

We can define the operators

$$\mathsf{u}_i = e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{q}_i}Y_i \qquad \mathsf{u}_i^* = e^{2\pi\frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{q}_i}Y_i^{-1} \tag{4.5}$$

$$\mathsf{v}_i = e^{2\pi\frac{\mathrm{b}}{\sqrt{N}}\mathsf{p}_i}X_i \qquad \mathsf{v}_i^* = e^{2\pi\frac{\mathrm{b}^{-1}}{\sqrt{N}}\mathsf{p}_i}X_i^{-1} \tag{4.6}$$

satisfying

$$\mathsf{u}_i\mathsf{v}_j = q^{\delta_{ij}}\mathsf{v}_j\mathsf{u}_i \qquad \mathsf{u}_i^*\mathsf{v}_j^* = \tilde{q}^{\delta_{ij}}\mathsf{v}_j^*\mathsf{u}_i^* \tag{4.7}$$

$$\mathsf{u}_i\mathsf{v}_j^* = \mathsf{v}_j^*\mathsf{u}_i \qquad \mathsf{u}_i^*\mathsf{v}_j = \mathsf{v}_j\mathsf{u}_i^* \tag{4.8}$$

$$q = e^{2\pi i\frac{\mathrm{b}^2}{N}}\omega \qquad \tilde{q} = e^{2\pi i\frac{\mathrm{b}^{-2}}{N}}\omega. \tag{4.9}$$

The Quantum algebra $\mathcal{X}(\tau)$ is generated by the u_j, v_j for $j = 0, \dots M$, and has a $*$–algebra structure when extended to include u_j^* and v_j^*. We remark that the $*$ operator we are using here is the standard hermitian conjugation only if $|\mathrm{b}| = 1$. Explicitly let X_j, Y_j, p_j, q_j , $j = 1, 2$ be operators acting on $\mathcal{H} := \mathrm{L}^2(\mathbb{A}_N^2)$ as follow

$$\mathsf{p}_j f(\mathsf{x},\mathsf{m}) = \frac{1}{2\pi i}\frac{\partial}{\partial x_j} f(\mathsf{x},\mathsf{m}), \qquad \mathsf{q}_j f(\mathsf{x},\mathsf{m}) = x_j f(\mathsf{x},\mathsf{m}) \tag{4.10}$$

$$X_1 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1+1,m_2)), \quad X_2 f(\mathsf{x},\mathsf{m}) = f(\mathsf{x},(m_1,m_2+1)) \tag{4.11}$$

$$Y_j f(\mathsf{x},\mathsf{m}) = \overline{\omega}^{m_j} f(\mathsf{x},\mathsf{m}), \tag{4.12}$$

where $\mathsf{m} = (m_1, m_2) \in \mathbb{Z}_N^2$ and $\mathsf{x} = (x_1, x_2) \in \mathbb{R}^2$.
These operators satisfy conditions (4.4). Let $\psi_{\mathrm{b}}(x,n) \equiv \frac{1}{\mathrm{D}_{\mathrm{b}}(x,n)}$ and consider the operators

$$\mathsf{D}_{12} \equiv e^{2\pi i \mathsf{q}_2\mathsf{p}_1} \sum_{j,k=0}^{N-1} \omega^{jk} Y_2^j X_1^k \tag{4.13}$$

$$\Psi_{12} \equiv \Psi\!\!\!/_{\mathrm{b}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) \tag{4.14}$$

$$\mathsf{T}_{12} \equiv \mathsf{D}_{12}\Psi_{12} \tag{4.15}$$

One has

Lemma 4.1 (Tetrahedral Equations)**.**

$$\mathsf{T}_{12}\mathsf{u}_1 = \mathsf{u}_1\mathsf{u}_2\mathsf{T}_{12} \tag{4.16}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{v}_2 = \mathsf{v}_2\mathsf{T}_{12} \tag{4.17}$$

$$\mathsf{T}_{12}\mathsf{v}_1\mathsf{u}_2 = \mathsf{v}_1\mathsf{u}_2\mathsf{T}_{12} \tag{4.18}$$

$$\mathsf{T}_{12}\mathsf{v}_1 = (\mathsf{u}_1\mathsf{v}_2 + \mathsf{v}_1)\mathsf{T}_{12} \tag{4.19}$$

$$\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12} \tag{4.20}$$

Remark 4.2. If we define $\tilde{\mathsf{T}}_{12} = \mathsf{D}_{12}\tilde{\Psi}_{12}$ where

$$\tilde{\Psi} \equiv \Psi\!\!\!/_{\mathrm{b}^{-1}}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} \overline{Y}_1 \overline{X}_2 Y_2) \tag{4.21}$$

then $\tilde{\mathsf{T}}$ satisfies equations (4.16 – 4.20) with u_i and v_i substituted by u_i^* and v_i^*. However from Remark 3.6 we know that $\Psi_{\mathrm{b}^{-1}}(x,n) = \Psi_{\mathrm{b}}(x,-n)$, and

$$\left(-e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}\right)^{-1} = -e^{\frac{\pi i}{N}} \overline{Y}_1 Y_2 \overline{X}_2 = -e^{-\frac{\pi i}{N}} \overline{Y}_1 \overline{X}_2 Y_2$$

so that

$$\tilde{\mathsf{T}} = \mathsf{T}.$$

From Lemma 4.1 we have the following implementations of equations (4.1 – 4.3).

Proposition 4.3. *Let* $\mathsf{w}_i \equiv (\mathsf{u}_i, \mathsf{v}_i)$ *and* $\mathsf{w}_i^* = (\mathsf{u}_i^*, \mathsf{v}_i^*)$*. Then we have*

$$\mathsf{w}_1 \bullet \mathsf{w}_2 \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_1, \quad \mathsf{w}_1 * \mathsf{w}_2 \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_2, \tag{4.22}$$

$$\mathsf{w}_1^* \bullet \mathsf{w}_2^* \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_1^*, \quad \mathsf{w}_1^* * \mathsf{w}_2^* \mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{w}_2^*. \tag{4.23}$$

Proposition 4.4. *Let*

$$\mathsf{A} \equiv e^{3\pi i \mathsf{q}^2} e^{\pi i (\mathsf{p}+\mathsf{q})^2} \sum_{j=0}^{N-1} \overline{\langle j \rangle}^3 Y^{3j} \sum_{l=0}^{N-1} \overline{\langle l \rangle} (-e^{-\pi i/N} Y X)^l, \tag{4.24}$$

where $\langle n \rangle = e^{-\pi i n(n+N)/N}$ *and* Y *and* X *are as above. Then*

$$\mathsf{A}(\mathsf{u},\mathsf{v}) = (q^{-1/2}\mathsf{v}\mathsf{u}^{-1},\, \mathsf{u}^{-1}) \qquad \mathsf{A}(\mathsf{u}^*,\mathsf{v}^*) = (\tilde{q}^{-1/2}\mathsf{v}^*(\mathsf{u}^*)^{-1},\, (\mathsf{u}^*)^{-1}) \tag{4.25}$$

where q *and* $\tilde{q}$ *are defined by equation* (4.9).

5 Quantization of the Model Space for Complex Chern-Simons Theory

In this Section we want to quantize the space $\mathbb{C}^* \times \mathbb{C}^*$ with the complex symplectic form

$$\omega_{\mathbb{C}} = \frac{\mathrm{d}x \wedge \mathrm{d}y}{xy}.$$

We think of it as a model space for Complex Chern-Simons Theory because it is an open dense of the $\mathrm{PSL}(2,\mathbb{C})$ moduli space of flat connections on a four punctured sphere, with unipotent holonomy around the punctures, [AK3, K5, D, FG]. Tetrahedral operators are supposedly related to states in the quantization of the four punctured sphere. Since we want to construct knot invariants starting from tetrahedral ideal triangulations this is the space we need to quantize. We follow the ideas in Andersen and Kashaev [AK3] using a real polarization with contractible leaves. We will further show that the level N Weil-Gel'fand-Zak Transform relates this quantization with the $\mathrm{L}^2(\mathbb{A}_N)$ representations in Quantum Teichmüller theory. To use this transform to relates the Andersen–Kashaev invariants to complex Chern–Simons Theory was already proposed in [AK3]. However the relation between the two approaches was not as tight as the one present here.

Let $t = N + is \in \mathbb{C}^*$ be the quantization constant, for $N \in \mathbb{R}$ and $s \in \mathbb{R} \sqcup i\mathbb{R}$. Denote also $\tilde{t} = N - is$. Fix $\mathrm{b} \in \mathbb{C}$ such that $s = -iN\frac{1-\mathrm{b}^2}{1+\mathrm{b}^2}$ and $\operatorname{Re}\mathrm{b} > 0$. This substitution, for $s \in i\mathbb{R}$, is only possible when $-N < is < N$. Notice that

$$(5.1) \qquad s \in \mathbb{R} \iff |\mathrm{b}| = 1 \text{ and } \mathrm{b} \neq \pm i, \qquad s \in i\mathbb{R} \iff \operatorname{Im}\mathrm{b} = 0$$

and notice the following useful expressions

$$(5.2) \qquad t = \frac{2N}{1+\mathrm{b}^2}, \qquad \tilde{t} = \frac{2N}{1+\mathrm{b}^{-2}}.$$

Consider the covering maps

$$(5.3) \qquad \begin{aligned} \zeta^{\pm} \colon \mathbb{R}^2 &\longrightarrow \mathbb{C}^* \\ (z,n) &\mapsto \exp\left(2\pi\mathrm{b}^{\pm 1} z \pm 2\pi i n\right) \end{aligned}$$

and consequently

$$(5.4) \qquad \pi^{\pm} : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{C}^* \times \mathbb{C}^*, \qquad \pi^{\pm} = (\zeta^{\pm}, \zeta^{\pm})$$

such that

$$(5.5) \qquad \begin{aligned} &\mathbb{C}^* \times \mathbb{C}^* \ni (x,y) = \pi^{+}((z,n),(w,m)), \\ &\mathbb{C}^* \times \mathbb{C}^* \ni (\tilde{x},\tilde{y}) = \pi^{-}((z,n),(w,m)) \quad \text{for } ((z,n),(w,m)) \in \mathbb{R}^2 \times \mathbb{R}^2. \end{aligned}$$

We remark that

$$(5.6) \qquad \overline{\zeta^{+}(z,n)} = \zeta^{-}(z,n) \qquad \iff \qquad |\mathrm{b}| = 1$$

in this case $\pi^- = \overline{\pi^+}$ and $\tilde{x} = \overline{x}$, $\tilde{y} = \overline{y}$. In this sense x, y $\tilde{x}$ and $\tilde{y}$ are natural coordinate functions to quantize in $\mathbb{C}^* \times \mathbb{C}^*$. If $\mathrm{b} \in \mathbb{R}$ they are still coordinates functions for the underlying real manifold, but we lose the complex conjugate interpretation. We will first consider the quantization of the covering $\mathbb{R}^2 \times \mathbb{R}^2$. Define the form

$$(5.7) \qquad \omega_t \equiv \frac{t}{4\pi}(\pi^+)^*(\omega_{\mathbb{C}}) + \frac{\tilde{t}}{4\pi}(\pi^-)^*(\omega_{\mathbb{C}}).$$

Lemma 5.1.

$$(5.8) \qquad \omega_t = 2\pi N(\mathrm{d}z \wedge \mathrm{d}w - \mathrm{d}n \wedge \mathrm{d}m).$$

In particular it is a real symplectic 2 *form on* $\mathbb{R}^2 \times \mathbb{R}^2$*, independent of* b.

Over $\mathbb{R}^2 \times \mathbb{R}^2$ we take the trivial line bundle $\widetilde{\mathcal{L}} = \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{C}$. On the N-th tensor power of this line bundle $\widetilde{\mathcal{L}}^N$ we consider the connection

$$(5.9) \qquad \nabla^{(t)} \equiv \mathrm{d} - i\alpha_t$$

where

$$\alpha_t \equiv \frac{t}{4\pi}\alpha^+_{\mathbb{C}} + \frac{\tilde{t}}{4\pi}\alpha^-_{\mathbb{C}}, \tag{5.10}$$

$$\alpha^\pm_{\mathbb{C}} \equiv 2\pi^2(\mathrm{b}^{\pm 1}z \pm in)\mathrm{d}(\mathrm{b}^{\pm 1}w \pm im) - 2\pi^2(\mathrm{b}^{\pm 1}w \pm im)\mathrm{d}(\mathrm{b}^{\pm 1}z \pm in) \tag{5.11}$$

In analogy to Lemma 5.1 we have

$$\alpha_t = \pi N(z\mathrm{d}w - w\mathrm{d}z - n\mathrm{d}m + m\mathrm{d}n). \tag{5.12}$$

It is simple to see that

$$\mathrm{d}\alpha^\pm_{\mathbb{C}} = (\pi^\pm)^*(\omega_{\mathbb{C}}), \qquad \text{which implies} \tag{5.13}$$

$$F_{\nabla^{(t)}} = -i\omega_t. \tag{5.14}$$

Further, on $\mathbb{R}^2 \times \mathbb{R}^2$ we have an action of $\mathbb{Z} \times \mathbb{Z}$ compatible with the projection π^+, i.e.

$$(\mathbb{Z} \times \mathbb{Z}) \times \left(\mathbb{R}^2 \times \mathbb{R}^2\right) \longrightarrow \mathbb{R}^2 \times \mathbb{R}^2 \tag{5.15}$$

$$(\lambda_1, \lambda_2) \cdot ((z, n), (w, m)) \mapsto ((z, n + \lambda_1), (w, m + \lambda_2)) \tag{5.16}$$

that satisfies

$$\pi^\pm((z, n + \lambda_1), (w, m + \lambda_2)) = \pi^\pm((z, n), (w, m)) \tag{5.17}$$

This action can be lifted to an action $\widetilde{\mathcal{L}}^N$ in such a way that the quotient bundle $\mathcal{L}^N \equiv \widetilde{\mathcal{L}}^N\ /(\mathbb{Z})^2 \to \mathbb{R}^4/\mathbb{Z}^2$ has first Chern class $c_1(\mathcal{L}^N) = \frac{1}{2\pi}[\omega_t]$ (ω_t is evidently $\mathbb{Z}^2$–invariant). Such a condition gives the requirement (which is in fact the prequantum condition) $\frac{1}{2\pi}[\omega_t] \in H^2((\mathbb{R}^2 \times \mathbb{R}^2)/(\mathbb{Z}^2), \mathbb{Z})$, which boils down to the requirement $N \in \mathbb{Z}$. Explicitly the action of $\mathbb{Z}\times\mathbb{Z}$ on $\widetilde{\mathcal{L}}^N$ is given by the following two multipliers

$$e_{(1,0)} = e^{-\pi N i m}, \qquad e_{(0,1)} = e^{\pi N i n}. \tag{5.18}$$

That means that we consider the space of sections

$$(\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \tag{5.19}$$

of $\mathbb{Z}^2$–invariant, smooth sections of $\widetilde{\mathcal{L}}^N$. Explicitly

$$s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2} \text{ if and only if } s \in \mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N) \text{ and satisfies}$$

$$s((z, n + 1), (w, m)) = e^{-\pi i N m} s((z, n), (w, m)), \tag{5.20}$$

$$s((z, n), (w, m + 1)) = e^{\pi i N n} s((z, n), (w, m)) \tag{5.21}$$

Lemma 5.2.

$$\nabla^{(t)} s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}, \qquad \textit{for any } s \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$$

The following Hermitian structure on $\widetilde{\mathcal{L}}^N$ is $\mathbb{Z}^2$–invariant and parallel with respect to $\nabla^{(t)}$.

$$s \cdot s'(p) \equiv s(p)\overline{s'(p)}, \quad \text{for any } p \in \mathbb{R}^2 \times \mathbb{R}^2 \tag{5.22}$$

Being parallel here means that

$$\mathrm{d}(s \cdot s') = (\nabla^{(t)} s) \cdot s' + s \cdot (\nabla^{(t)} s'), \tag{5.23}$$

and this is a simple consequence of α_t being a real 1-form. It follows that the following is a well defined inner product in the completion of $\left(\left(\mathrm{L}^2 \cap \mathrm{C}^\infty\right)(\mathbb{R}^4, \widetilde{\mathcal{L}}^N)\right)^{\mathbb{Z}^2}$

$$(s,s') \equiv \int_{\mathbb{R}} \mathrm{d}z \int_{\mathbb{R}} \mathrm{d}w \left(\int_0^1 \mathrm{d}n \int_0^1 \mathrm{d}m \; s \cdot s' \right) \tag{5.24}$$

Lemma 5.3. *We have the following Hamiltonian vector field for the coordinates functions on* $\mathbb{R}^2 \times \mathbb{R}^2$

$$X_z = \frac{1}{2\pi N}\frac{\partial}{\partial w} \qquad X_w = -\frac{1}{2\pi N}\frac{\partial}{\partial z}$$
$$X_n = -\frac{1}{2\pi N}\frac{\partial}{\partial m} \qquad X_m = \frac{1}{2\pi N}\frac{\partial}{\partial n}$$

From the definition of the pre-quantum operator $\hat{f}$ associated to the observable f, we have

$$\hat{f} = -i\nabla_{X_f} + f \tag{5.25}$$

Lemma 5.4 (Pre–Quantum operators)**.** *The following are the pre–quantum operators for the coordinate functions on* $\mathbb{R}^2 \times \mathbb{R}^2$

$$\hat{z} = \frac{-i}{2\pi N}\nabla_w^{(t)} + z \qquad \hat{w} = \frac{i}{2\pi N}\nabla_z^{(t)} + w$$
$$\hat{n} = \frac{i}{2\pi N}\nabla_m^{(t)} + n \qquad \hat{m} = \frac{-i}{2\pi N}\nabla_n^{(t)} + m$$

and they satisfy the following canonical commutation relations

$$[\hat{z},\, \hat{w}] = \frac{1}{2\pi i N} \qquad [\hat{n},\, \hat{m}] = -\frac{1}{2\pi i N} \tag{5.26}$$

$$[\hat{z}, \hat{n}] = [\hat{z}, \hat{m}] = [\hat{w}, \hat{n}] = [\hat{w}, \hat{m}] = 0 \tag{5.27}$$

The Hermitian line bundle $\mathcal{L}^N \to (\mathbb{R}^2 \times \mathbb{R}^2/\mathbb{Z}^2)$ together with the connection $\nabla^{(t)}$ define a pre–Quantization of the theory. In order to finish the quantization program we need to choose a Lagrangian polarization.

Choose the following real Lagrangian polarization

$$\widetilde{\mathcal{P}} \equiv \mathrm{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial w} + \frac{\partial}{\partial n}, \frac{\partial}{\partial z} - \frac{\partial}{\partial m} \right\}. \tag{5.28}$$

The leaves of this polarization are all contractible after quotient by the action of $\mathbb{Z}^2$ on $\mathbb{R}^2 \times \mathbb{R}^2$, so we do have polarized global sections. In particular the space $T \subset \mathbb{R}^2 \times \mathbb{R}^2$

$$T \equiv \{z = w = 0\} \tag{5.29}$$

is a transversal for the polarization. For any $\psi \in (\mathrm{C}^\infty(\mathbb{R}^4, \widetilde{\mathcal{L}}^N))^{\mathbb{Z}^2}$ polarized by $\widetilde{\mathcal{P}}$, the following two differential equations will determine $\psi \equiv \psi((z,n),(w,m))$ by its value in (n,m)

$$\nabla^{(t)}_w \psi = -\nabla^{(t)}_n \psi \qquad\qquad \nabla^{(t)}_z \psi = \nabla^{(t)}_m \psi. \tag{5.30}$$

The space $T/\mathbb{Z}^2$ is of course $\mathbb{T} \times \mathbb{T}$, and the line bundle $\mathcal{L}^N$ will restrict to a non trivial line bundle over $\mathbb{T} \times \mathbb{T}$ that we shall call $\mathcal{L}^N$ again. The quantum space that we obtain is then

$$\mathcal{H}^{(N)} \equiv \mathrm{C}^\infty \left(\mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right). \tag{5.31}$$

We consider the obvious inner product on $\mathcal{H}^{(N)}$

$$(\psi, \phi) = \int_0^1 \int_0^1 \psi \overline{\phi} \, \mathrm{d}n \mathrm{d}m \tag{5.32}$$

that is the standard inner product on the completion $\mathrm{L}^2 \left(\mathbb{T} \times \mathbb{T}, \mathcal{L}^N \right)$. Finally the quantum operators acts on polarized sections as

$$\hat{x} \equiv \exp\left(2\pi \mathrm{b}\hat{z} + 2\pi i \hat{n}\right) = \exp\left(i\frac{\mathrm{b}}{N} \nabla^{(t)}_n - \frac{1}{N}\nabla^{(t)}_m + 2\pi i n \right) \tag{5.33}$$

$$\hat{y} \equiv \exp\left(2\pi \mathrm{b}\hat{w} + 2\pi i \hat{m}\right) = \exp\left(i\frac{\mathrm{b}}{N} \nabla^{(t)}_m + \frac{1}{N}\nabla^{(t)}_n + 2\pi i m \right) \tag{5.34}$$

$$\hat{\tilde{x}} \equiv \exp\left(2\pi \mathrm{b}^{-1}\hat{z} - 2\pi i \hat{n}\right) = \exp\left(i\frac{\mathrm{b}^{-1}}{N} \nabla^{(t)}_n + \frac{1}{N}\nabla^{(t)}_m - 2\pi i n \right) \tag{5.35}$$

$$\hat{\tilde{y}} \equiv \exp\left(2\pi \mathrm{b}^{-1}\hat{w} - 2\pi i \hat{m}\right) = \exp\left(i\frac{\mathrm{b}^{-1}}{N} \nabla^{(t)}_z - \frac{1}{N}\nabla^{(t)}_n - 2\pi i m \right) \tag{5.36}$$

Now we are going to connect the quantization with the Quantum Teichmüller theory. Recall the operators $\mathsf{u} = \mathsf{u}(\mathrm{b})$ and $\mathsf{v} = \mathsf{v}(\mathrm{b})$ from equations (4.4 – 4.12), and recall that they depend on a parameter b. Define the rescaling operator

$$\mathcal{O}_{\sqrt{N}} \colon \mathrm{L}^2(\mathbb{A}_N) \longrightarrow \mathrm{L}^2(\mathbb{A}_N)$$

$$\mathcal{O}_{\sqrt{N}}(\mathsf{f})(z,n) = \mathsf{f}\left(\sqrt{N}z, n\right) \tag{5.37}$$

and the following rescaled analogues of the Quantum Teichmüller Theory operators

$$\hat{u} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \qquad \hat{v} = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \tag{5.38}$$

$$\hat{u}^* = \mathcal{O}_{\sqrt{N}} \circ \mathsf{u}^*(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \qquad \hat{v}^* = \mathcal{O}_{\sqrt{N}} \circ \mathsf{v}^*(\mathrm{b}^{-1}) \circ \mathcal{O}^{-1}_{\sqrt{N}} \tag{5.39}$$

which acts on $\mathsf{f} \in \mathrm{L}^2(\mathbb{A}_N)$ as

$$\hat{u}^*\mathsf{f}(z,l) = e^{2\pi \mathrm{b} z} e^{2\pi i l/N}\mathsf{f}(z,l) \qquad \hat{u}\mathsf{f}(z,l) = e^{2\pi \mathrm{b}^{-1} z} e^{-2\pi i l/N}\mathsf{f}(z,l) \tag{5.40}$$

$$\hat{v}^*\mathsf{f}(z,l) = f(z - i\frac{\mathrm{b}}{N}, l-1) \qquad \hat{v}\mathsf{f}(z,l) = f(z - i\frac{\mathrm{b}^{-1}}{N}, l+1) \tag{5.41}$$

We make use of the level-N Weil-Gel'fand-Zak Transform, [AK3].

Theorem 5.5. *Recall the line bundle $\mathcal{L}^N$. The following map $Z^{(N)} \colon \mathcal{S}(\mathbb{A}_N) \longrightarrow \mathrm{C}^\infty(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$ is a an isomorphism*

$$Z^{(N)}(\mathsf{f})(n,m) = \frac{1}{\sqrt{N}} e^{\pi i N mn} \sum_{p\in\mathbb{Z}} \sum_{l=0}^{N-1} \mathsf{f}\left(n + \frac{p}{N}, l\right) e^{2\pi i m p} e^{2\pi i l p/N} \tag{5.42}$$

with inverse

$$\overline{Z}^{(N)}(s)(x,j) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} e^{-2\pi i \frac{lj}{N}} \int_0^1 s\left(x - \frac{l}{N}, v\right) e^{-\pi i N (x + \frac{l}{N}) v} \mathrm{d}v.$$

which preserves the inner products $\mathrm{L}^2(\mathbb{A}_N)$ and $(\cdot,\cdot)$, i.e.

$$\left(Z^{(N)}(\mathsf{f}),\, Z^{(N)}(\mathsf{g})\right) = \langle \mathsf{f},\, \mathsf{g}\rangle$$

and so extends to an isometry between $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathrm{L}^2(\mathbb{T}\times\mathbb{T}, \mathcal{L}^N)$.

Proposition 5.6. *We have*

$$Z^{(N)} \circ \hat{u}^* \circ (Z^{(N)})^{-1} = \hat{y}^{-1} \qquad Z^{(N)} \circ \hat{v}^* \hat{\circ} (Z^{(N)})^{-1} = \hat{x}^{-1}$$

$$Z^{(N)} \circ \hat{u} \circ (Z^{(N)})^{-1} = \hat{\tilde{y}}^{-1} \qquad Z^{(N)} \circ \hat{v} \circ (Z^{(N)})^{-1} = \hat{\tilde{x}}^{-1}$$

All together we have showed that the quantization for the model space of complex Chern-Simons theory is equivalent to the $\mathrm{L}^2(\mathbb{A}_N)$ representations of the quantum algebra defined from Quantum Teichmüller Theory. In the following section we will extend the $\mathrm{L}^2(\mathbb{A}_N)$ representations to knots invariants following the recipe given by Andersen and Kashaev in [AK1]. The previous discussion on the different quantizations serves to link such invariants to Complex Quantum Chern–Simons Theory.

6 The Andersen–Kashaev Teichmüller TQFT at Level N

6.1 Angle Structures on 3-Manifolds

We present here shaped triangulated pseudo 3-manifolds, which are the combinatorial data underlying the Andersen-Kashaev construction of their invariant. Following strictly [AK1] we will describe the *categroid* of *admissible* oriented triangulated pseudo 3-manifolds, where the words admissible and categroid go together because admissibility is what will obstruct us to have a full category. See Appendix B for a definition of categroids.

Definition 6.1 (Oriented Triangulated Pseudo 3-manifold)**.** An *Oriented Triangulated Pseudo* 3*-manifold* X is a finite collection of 3-simplices (tetrahedra) with totally ordered vertices together with a collection of *gluing homeomorphisms* between some pairs of codimension 1 faces, so that every face is in, at most, one such pairs. By gluing homeomorphism we mean a vertex order preserving, orientation reversing, affine homeomorphism between the two faces.
The quotient space under the glueing homeomorphisms has the structure of CW-complex with oriented edges.

For $i \in \{0, 1, 2, 3\}$ we denote by $\Delta_i(X)$ the collection of i-dimensional simplices in X and, for $i > j$, we denote

$$\Delta_i^j(X) = \{(a, b) | a \in \Delta_i(X), b \in \Delta_j(a)\}.$$

We have projection maps

$$\phi_{i,j} : \Delta_i^j(X) \longrightarrow \Delta_i(X), \quad \phi^{i,j} : \Delta_i^j(X) \longrightarrow \Delta_j(X),$$

and boundary maps

$$\partial_i : \Delta_j(X) \longrightarrow \Delta_{j-1}(X), \quad \partial_i[v_0, \dots, v_j] \mapsto [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_j]$$

where $[v_0, \dots, v_j]$ is the j-simplex with vertices $v_0, \dots, v_j$ and $i \leq j$.

Definition 6.2 (Shape Structure)**.** Let X be an oriented triangulated pseudo 3-manifold. A *Shape Structure* is a map

$$\alpha_X : \Delta_3^1(X) \longrightarrow \mathbb{R}_{>0},$$

so that, in every tetrahedron, the sum of the values of α_X along three incident edges is π.
The value of the map α_X in an edge e inside a tetrahedron T is called the *dihedral angle* of T at e. If we allow α_X to take values in $\mathbb{R}$ we talk about a *Generalized Shape Structure*.

The set of shape structures supported by X is denoted $\mathrm{S}(X)$. The space of generalized shape structures is denoted by $\widetilde{\mathrm{S}}(X)$. X together with α_X is called *Shaped Pseudo* 3*-manifold*.

Remark 6.3 (Ideal Tetrahedron)**.** A shape structure on a simplicial tetrahedron T as above defines an embedding of $T \setminus \Delta_0(T)$ in the hyperbolic 3–space $\mathbb{H}^3$ which extends to a map of T to $\overline{\mathbb{H}^3}$. In fact we can change a given embedding, so that it send the four vertices (v_0, v_1, v_2, v_3) to the four points $(\infty, 0, 1, z) \in \mathbb{CP}^1 \simeq \partial\mathbb{H}^3$, where

$$z = \frac{\sin\alpha_T([v_0, v_2])}{\sin\alpha_T([v_0, v_3])} \exp\left(i\alpha_T([v_0, v_1])\right).$$

This four points in $\partial\mathbb{H}^3$ extend to a unique ideal tetrahedron in $\mathbb{H}^3$, by taking the geodesic convex hull, that has dihedral angles defined by α_T.

Remark 6.4. In every tetrahedron, its orientation induces a cyclic ordering of all triples of edges meeting in a vertex. Such a cyclic ordering descends to a cyclic ordering of the pairs of opposite edges of the whole tetrahedron. Moreover, it follows from the definition that opposite edges share the same dihedral angle. Hence, we get a well defined cyclic order preserving projection $p : \Delta_3^1(X) \longrightarrow \Delta_3^{1/p}(X)$ which identifies opposite edges. α_X descends to a map from $\Delta_3^{1/p}(X)$ and we can consider the following skew-symmetric functions

$$\varepsilon_{a,b} \in \{0, \pm 1\}, \quad \varepsilon_{a,b} = -\varepsilon_{b,a}, \quad a, b \in \Delta_3^{1/p}(X),$$

defined to be $\varepsilon_{a,b} = 0$ if the underlying tetrahedra are distinct, and $\varepsilon_{a,b} = +1$ if the underlying tetrahedra coincides and b cyclically follows a in the order induced on $\Delta_3^{1/p}(X)$.

Definition 6.5. To any shaped pseudo 3-manifold X, we associate a *Weight* function

$$\omega_X : \Delta_1(X) \longrightarrow \mathbb{R}_{>0}, \quad \omega_X(e) = \sum_{a \in (\phi^{3,1})^{-1}(e)} \alpha_X(a).$$

An edge e in X is called *balanced* if e is internal and $\omega_X(e) = 2\pi$. A shape structure is fully balanced if all its edges are balanced.

The shape structures of closed fully balanced 3-manifolds are called *Angle Structures* in the literature. For more details on them and their geometric admissibility see [Lac] and [LT].

Definition 6.6. A *leveled* (generalised) shaped pseudo 3-manifold is a pair (X, l_X) consisting of a (generalized) shaped pseudo 3-manifold X and a real number $l_X \in \mathbb{R}$, called the *level*. The set of all leveled (generalised) shaped pseudo 3-manifolds is denoted by $\mathrm{LS}(X)$ (respectively $\widetilde{\mathrm{LS}}(X)$).

There is a gauge action of $\mathbb{R}^{\Delta_1(X)}$ on $\widetilde{\mathrm{LS}}(X)$.

Definition 6.7. Let (X, l_X) and (Y, l_Y) be two (generalized) leveled shaped pseudo 3-manifolds. They are said to be *gauge equivalent* if there exists an isomorphism $h : X \longrightarrow Y$ of the underlying cellular structures, and a function $g : \Delta_1(X) \longrightarrow \mathbb{R}$ such that

$$\Delta_1(\partial X) \subset g^{-1}(0),$$
$$\alpha_Y(h(a)) = \alpha_X(a) + \pi \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} g(\phi^{3,1}(b)), \quad \forall a \in \Delta_3^1(X), \text{ and}$$
$$l_Y = l_X + \sum_{e \in \Delta_1(X)} g(e) \sum_{a \in (\phi^{3,1})^{-1}(e)} \left(\frac{1}{3} - \frac{\alpha_X(a)}{\pi}\right).$$

We remark that $\omega_X = \omega_Y \circ h$.

Definition 6.8. Let (α_X, l_X) and $(\alpha_{X'}, l_{X'})$ be two (generalized) leveled shape structures of the oriented pseudo 3-manifold X. They are said *based gauge equivalent* if they are gauge equivalent as in Definition 6.7 if the isomorphism $h : X \longrightarrow X$ is the identity.

Based gauge equivalence is an equivalence relation in the sets $\mathrm{S}(X)$, $\mathrm{LS}(X)$, $\widetilde{\mathrm{S}}(X)$, $\widetilde{\mathrm{LS}}(X)$ and the quotient sets are denoted (resp.) $\mathrm{S}_r(X)$, $\mathrm{LS}_r(X)$, $\widetilde{\mathrm{S}}_r(X)$, $\widetilde{\mathrm{LS}}_r(X)$. We remark that $\mathrm{S}_r(X)$ is an open convex (possibly empty) subset of the space $\widetilde{\mathrm{S}}_r(X)$. We will return to existence of shape structures later. Let us focus on $\widetilde{\mathrm{S}}(X)$ for now. Let

$$\tilde{\Omega}_X : \widetilde{\mathrm{S}}(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

be the map which sends the shape structure α_X to the corresponding weight function ω_X. This map is gauge invariant, so it descends to a map

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

For fixed $a \in \Delta_3^{1/p}(X)$ we can think of $\alpha_a := \alpha_X(a)$ as an element of $C^\infty\left(\widetilde{\mathrm{S}}(X)\right)$.

Definition 6.9 ([NZ])**.** The Neumann-Zagier symplectic structure on $\widetilde{\mathrm{S}}(X)$ is the unique symplectic structure which induces the Poisson bracket $\{\cdot,\cdot\}$ satisfying

$$\{\alpha_a, \alpha_b\} = \varepsilon_{a,b}$$

for all a, $b \in \Delta_3^{1/p}(X)$.

For a triangulated pseudo 3-manifold we have a symplectic decomposition

$$\widetilde{\mathrm{S}}(X) = \prod_{T \in \Delta_3(X)} \widetilde{\mathrm{S}}(T).$$

Theorem 6.10 ([AK1])**.** *The gauge action of* $\mathbb{R}^{\Delta_1(X)}$ *on* $\widetilde{\mathrm{S}}(X)$ *is symplectic and* $\tilde{\Omega}_X$ *is a moment map for this action. It follows that* $\widetilde{\mathrm{S}}_r(X) = \widetilde{\mathrm{S}}(X)/\mathbb{R}^{\Delta_1(X)}$ *is a Poisson manifold with symplectic leaves corresponding to the fibers of* $\tilde{\Omega}_{X,r}$.

Let $N_0(X)$ be a sufficiently small neighbourhood of $\Delta_0(X)$, then $\partial N_0(X)$ is a surface which inherits a triangulation from X, with a shape structure, if X has a shape structure. Notice that this surface can have boundary if $\partial X \neq \emptyset$.

Theorem 6.11 ([AK1])**.** *The map*

$$\tilde{\Omega}_{X,r} : \widetilde{\mathrm{S}}_r(X) \longrightarrow \mathbb{R}^{\Delta_1(X)}$$

is an affine $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$*-bundle. The Poisson structure of* $\widetilde{\mathrm{S}}_r(X)$ *coincide with the one induced by the* $\mathrm{H}^1(\partial N_0(X), \mathbb{R})$*-bundle structure.*
If $h : X \longrightarrow Y$ *is an isomorphisms of cellular structure, the induced morphism* $h^* : \widetilde{\mathrm{S}}_r(Y) \longrightarrow \widetilde{\mathrm{S}}_r(X)$ *is compatible with all this structures, i.e. it is a Poisson affine bundle morphism which fiberwise coincide with the naturally induced group morphism* $h^* : \mathrm{H}^1(\partial N_0(Y), \mathbb{R}) \longrightarrow \mathrm{H}^1(\partial N_0(X), \mathbb{R})$*. Moreover* h^* *maps* $\mathrm{S}_r(Y)$ *to* $\mathrm{S}_r(X)$*.*

Definition 6.12 (Shaped $3-2$ Pachner moves)**.** Let X be a shaped pseudo 3 manifold and let e be a balanced internal edge in it, shared exactly by three distinct tetrahedra t_1, t_2 and t_3 with dihedral angles at e exactly α_1, α_2 and α_3. Then the triangulated pseudo 3-manifold X_e obtained by removing the edge e, and substituting the three tetrahedra t_1, t_2 and t_3 with other two new tetrahedra t_4 and t_5 glued along one face, is topologically the same space as X. In order to have the same weights of X on X_e, the dihedral angles of t_4 and t_5 are uniquely determined by the ones of t_1, t_2 and t_3 as follows

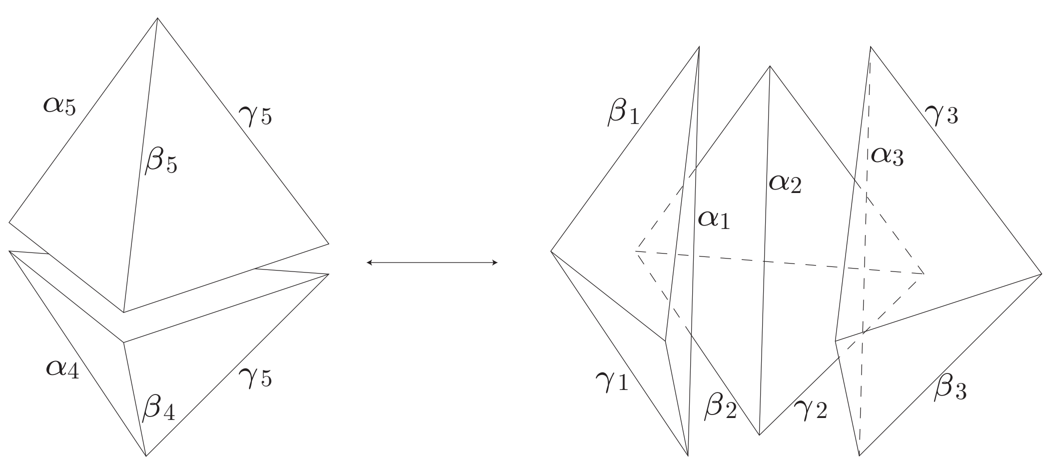


Figure 3: A 3–2 Pachner move.

$$\begin{aligned} &\alpha_4 = \beta_2 + \gamma_1 \quad \alpha_5 = \beta_1 + \gamma_2 \\ &\beta_4 = \beta_1 + \gamma_3 \quad \beta_5 = \beta_3 + \gamma_1 \\ &\gamma_4 = \beta_3 + \gamma_2 \quad \gamma_5 = \beta_2 + \gamma_3. \end{aligned} \tag{6.1}$$

where $(\alpha_i, \beta_i, \gamma_i)$ are the dihedral angles of t_i. In this situation we say that X_e is obtained from X by a *shaped* $3-2$ *Pachner move.*

We remark that the linear system, together with e being balanced, guarantees the positivity of the dihedral angles of t_4 and t_5 provided the positivity for t_1, t_2 and t_3 but it does not provide any guarantees on the converse, i.e. the positivity of a shaped $2-3$ Pachner moves. However, two different solutions for the angles for t_1, t_2 and t_3 from the same starting angles for t_4 and t_5 are always gauge equivalent.
The system (6.1) define a map $P^e : \mathrm{S}(X) \longrightarrow \mathrm{S}(X_e)$, that extends to a map

$$\tilde{P}^e : \widetilde{\mathrm{S}}(X) \longrightarrow \widetilde{\mathrm{S}}(X_e).$$

For a balanced edge e, the latter restricts to the map

$$\tilde{P}_r : \tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \longrightarrow \widetilde{\mathrm{S}}_r(X_e),$$

and it can be noticed that $\tilde{P}_r(\tilde{\Omega}_{X,r}(e)^{-1}(2\pi) \cap \mathrm{S}_r(x)) \subset \mathrm{S}_r(Y)$.
We also say that a leveled shaped pseudo 3-manifold (X, l_X) is obtained from (Y, l_Y) by a *leveled shaped* 3-2 *Pachner move* if, for some balanced $e \in \Delta_1(X)$, $Y = X_e$ as above and

$$l_Y = l_X + \frac{1}{12\pi} \sum_{a \in (\phi^{3,1})^{-1}(e)} \sum_{b \in \Delta_3^1(X)} \varepsilon_{p(a),p(b)} \alpha_X(b).$$

Definition 6.13. A (leveled) shaped pseudo 3-manifold X is called a *Pachner refinement* of a (leveled) shaped pseudo 3-manifold Y if there exists a finite sequence of (leveled) shaped pseudo 3-manifolds

$$X = X_1, X_2, \ldots, X_n = Y$$

such that for any $i \in \{1, \ldots, n-1\}$, X_{i+1} is obtained from X_i by a (leveled) shaped $3-2$ Pachner move. Two (leveled) shaped pseudo 3-manifolds X and Y are called *equivalent* if there exist gauge equivalent (leveled) shaped pseudo 3-manifolds X' and Y' which are respective Pachner refinements of X and Y.

For technical reasons, which we will discuss later, we will restrict the category of triangulated $2+1$ cobordisms, discussed so far, to a certain sub-categroid, as discussed below. This means that we will remove some morphisms as the following definition imposes.

Definition 6.14 (Admissibility)**.** An oriented triangulated pseudo 3-manifold is called *admissible* if

$$\mathrm{S}_r(X) \neq \emptyset,$$

and

$$\mathrm{H}_2(X - \Delta_0(X), \mathbb{Z}) = 0.$$

Definition 6.15. Two (leveled) admissible shaped pseudo 3-manifolds X and Y are said to be *admissibly equivalent* if there exists a gauge equivalence

$$h' : X' \longrightarrow Y'$$

of (leveled) shaped 3-manifolds X' and Y' which are respective Pachner refinements of X and Y such that $\Delta_1(X') = \Delta_1(X) \cup D_X$ and $\Delta_1(Y') = \Delta_1(Y) \cup D_Y$ and the following holds

$$\left[h(\mathrm{S}_r(X) \cap \tilde{\Omega}_{X',r}(D_X)^{-1}(2\pi))\right] \cap \left[\tilde{\Omega}_{Y',r}(D_Y)^{-1}(2\pi)\right] \neq \emptyset.$$

Theorem 6.16 ([AK1])**.** *Suppose two (leveled) shaped pseudo 3-manifolds X and Y are equivalent. Then there exist $D \subset \Delta_1(X)$ and $D' \subset \Delta_1(Y)$ and a bijection*

$$i : \Delta_1(X) - D \to \Delta_1(Y) - D'$$

and a Poisson isomorphism

$$R : \tilde{\Omega}_{X,r}(D)^{-1}(2\pi) \to \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi),$$

which is covered by an affine $\mathbb{R}$-bundle isomorphism from $\widetilde{\mathrm{LS}}_r(X)|_{\tilde{\Omega}_{X,r}(D)^{-1}(2\pi)}$ to $\widetilde{\mathrm{LS}}_r(Y)_{\tilde{\Omega}_{X,r}(D')^{-1}(2\pi)}$ and such that we get the following commutative diagram

$$\begin{array}{ccc} \tilde{\Omega}_{X,r}(D)^{-1}(2\pi) & \xrightarrow{R} & \tilde{\Omega}_{Y,r}(D')^{-1}(2\pi) \\ \Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{X,r} & & \Big\downarrow \mathrm{proj} \circ \tilde{\Omega}_{Y,r} \\ \mathbb{R}^{\Delta_1(X)-D} & \xrightarrow{i^*} & \mathbb{R}^{\Delta_1(Y)-D'}. \end{array}$$

Moreover, if X and Y are admissible and admissibly equivalent, the isomorphism R takes an open convex subset U of $S_r(X) \cap \tilde{\Omega}_{X,r}(D)^{-1}(2\pi)$ onto an open convex subset U' of $S_r(Y) \cap \tilde{\Omega}_{Y,r}(D)^{-1}(2\pi)$.

We remark that in the previous notation $D = \Delta_1(X) \cap h^{-1}(D_Y)$ and $D' = \Delta_1(Y) \cap h(D_X)$.

For a tetrahedron $T = [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ with ordered vertices v_0, v_1, v_2, v_3, we define its sign

$$\mathrm{sign}(T) = \mathrm{sign}(\det(v_1 - v_0, v_2 - v_0, v_3 - v_0)),$$

as well as the signs of its faces

$$\operatorname{sign}(\partial_i T) = (-1)^i \operatorname{sign}(T), \text{ for } i \in \{0, \dots, 3\}.$$

For a pseudo 3-manifold X, the signs of faces of the tetrahedra of X induce a sign function on the faces of the boundary of X, $\operatorname{sign}_X : \Delta_2(\partial X) \to \{\pm 1\}$, which permits us to split the components of the boundary of X into two sets, $\partial X = \partial_+ X \cup \partial_- X$, where $\Delta_2(\partial_\pm X) = \operatorname{sign}_X^{-1}(\pm 1)$. Notice that $|\Delta_2(\partial_+ X)| = |\Delta_2(\partial_- X)|$.

Definition 6.17 (Cobordism Categroid)**.** The category $\mathcal{B}$ is the category that has triangulated surfaces as objects,equivalence classes of (leveled) shaped pseudo 3-manifolds X as morphisms (so that $X \in \operatorname{Hom}_{\mathcal{B}}(\partial_- X, \partial_+ X)$) and the composition given by glueing along boundary components, through edge orientation preserving and face orientation reversing CW-homeomorphisms.
The Categroid $\mathcal{B}_a$ is the subcategroid of $\mathcal{B}$ whose morphisms are restricted to be admissible equivalence classes of admissible (leveled) shaped pseudo 3-manifolds. In particular composition is possible only if the gluing gives an (leveled) admissible pseudo 3-manifold.

Remark 6.18. *Admissible Shaped Pseudo* 3*-Manifolds in the real world.*
Even though we will discuss the whole Andersen Kashaev construction of the Teichmüller TQFT functor in the general setting of the above defined cobordism categroid, the the main parts of this construction, we want to put our hands on in this paper, are invariants of links. We interpret Triangulated Pseudo 3-manifolds X as ideal triangulations of the (non closed) manifold $X \setminus \Delta_0(X)$. This interpretation is enlighten in Remark 6.3. We shall ask ourself when a cusped 3-manifold (cusped means non compact with finite volume here) admits a positive fully balanced shape structure. This requirement is weaker than asking for a full geometric structure on the manifold and in our language, this can be expressed by the fact that we did not required a precise gauge to be fixed. The problem of finding positive or generalized angle structures has been studied in [LT], where necessary and sufficient conditions for their existence are given. In the work [HRS] it is proved, among other things, that a particular class of manifolds M supporting positive shape structures are complements in S^3 of hyperbolic links. However the admissibility conditions kicks in here and further restrict us to just complements of hyperbolic knots. So, at the least, we know that the Andersen Kashaev construction will work on complements of hyperbolic knots, and that are the examples we will look a bit closer at below. Now we should clarify the equivalence relation in $\mathcal{B}_a$, in the context of knot complements. Combinatorially speaking, any two ideal triangulations of a knot complement are related by finite sequences of 3–2 or 2–3 Pachner moves. On the other hand it is not known (at least to the authors) if any such sequence can be realised as a sequence of shaped Pachner moves. For sure we know that 3–2 shaped Pachner moves are

well defined in the category $\mathcal{B}_a$ as we remarked when we defined them, and if a shaped Pachner 2–3 move is possible in some particular case, then it is an equivalence in the category $\mathcal{B}_a$. So the knot invariants that we will define starting from $\mathcal{B}_a$ are not guaranteed to be topological invariants. There is however another construction of the Andersen–Kashaev invariant [AK2], that avoid this problem with analytic continuation properties of the partition function. The equivalence of the two constructions is still conjectural though.
In [AK1] another way to define knot invariants is suggested, by taking one vertex Hamiltonian triangulations of knots, that is, one vertex triangulations of S^3 (or a general manifold M) where the knot is represented by a unique edge with a degenerating shape structure, meaning that we take a limit on the shapes, sending all the weights to be balanced except the weight of the knot that is sent to 0. The partition function is actually divergent but a residue can be computed as an invariant. We will show this in a couple of examples in subsection 6.4.

6.2 The target Categroid $\mathcal{D}_N$

W recall all the relevant things regarding tempered distributions and the space $\mathcal{S}(\mathbb{A}_N)$ in Appendix A. As always, here N is an odd positive integer and $\mathrm{b} \in \mathbb{C}$ is fixed to satisfy $\mathrm{Re}(\mathrm{b}) > 0$ and $\mathrm{Im}\, \mathrm{b}(1 - |\mathrm{b}|) = 0$.

Definition 6.19. The categroid $\mathcal{D}_N$ has as objects finite sets and for two finite sets n, m the set of morphisms from n to m is

$$\mathrm{Hom}_{\mathcal{D}_N}(n, m) = \mathcal{S}'(\mathbb{A}_N^{n \sqcup m}) \simeq \mathcal{S}'(\mathbb{R}^{n \sqcup m}) \otimes \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^{n \sqcup m}).$$

Definition 6.20. For $\mathcal{A} \otimes A_N \in \mathrm{Hom}_{\mathcal{D}_N}(n, m)$ and $\mathcal{B} \otimes B_N \in \mathrm{Hom}_{\mathcal{D}_N}(m, l)$, such that $\mathcal{A}$ and $\mathcal{B}$ satisfy condition (A.3) and $\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})$ continuously extends to $S(\mathbb{R}^{n \sqcup m \sqcup l})_m$, we define

$$(\mathcal{A} \otimes A_N) \circ (\mathcal{B} \otimes B_N) = (\pi_{n,l})_*(\pi^*_{n,m}(\mathcal{A})\pi^*_{m,l}(\mathcal{B})) \otimes A_N B_N \in \mathrm{Hom}_{\mathcal{D}_N}(n, l).$$

Where the product $A_N B_N$ is just the matrix product.
We will frequently use the following notation in what follows. For any $a \in \mathbb{A}_N$, $a = (x, n) \in \mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ we will consider the b–dependent operator

$$\varepsilon \equiv \varepsilon(\mathrm{b})\colon \mathbb{A}_N \to \mathbb{A}_N$$

defined by

$$\varepsilon(x, n) \equiv \begin{cases} (x, n) & \text{if } |\mathrm{b}| = 1, \\ (x, -n) & \text{if } \mathrm{b} \in \mathbb{R} \end{cases} \tag{6.2}$$

For any $\mathcal{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, we have a unique adjoint $\mathcal{A}^* \in \mathcal{L}(\mathcal{S}(\mathbb{R}^m), \mathcal{S}'(\mathbb{R}^n))$ defined by the formula

$$\mathcal{A}^*(f)(g) = \overline{\bar{f}(\mathcal{A}(\bar{g}))}$$

for all $f \in \mathcal{S}(\mathbb{R}^m)$ and all $g \in \mathcal{S}(\mathbb{R}^n)$.

Definition 6.21 ($\star_{\mathrm{b}}$ structure)**.** Consider $\mathrm{b} \in \mathbb{C}$ fixed as above and $N \in \mathbb{Z}_{>0}$ odd. Let $A_N \in \mathrm{Hom}(\mathcal{S}((\mathbb{Z}/N\mathbb{Z})^m), \mathcal{S}((\mathbb{Z}/N\mathbb{Z})^n)$. Recall the involution ε on $\mathbb{Z}/N\mathbb{Z}$ form equation (6.2). Define $A_N^{\star_{\mathrm{b}}}$ as

$$\langle j_1, \dots j_m | A_N^{\star_{\mathrm{b}}} | p_1, \dots, p_n \rangle = \overline{\langle \varepsilon p_1, \dots, \varepsilon p_n | A_N | \varepsilon j_1, \dots \varepsilon j_m \rangle} \tag{6.3}$$

We can finally define the $*_{\mathrm{b}}$ operator as

$$(\mathcal{A} \otimes A_N)^{*_{\mathrm{b}}} = \mathcal{A}^* \otimes A_N^{\star_{\mathrm{b}}} \tag{6.4}$$

6.3 Tetrahedral Partition Function

Recall the operators from Section 4.1, u_j, v_j, X_j, Y_j, p_j, q_j , $j = 1, 2$ acting on $\mathcal{H} := \mathcal{S}(\mathbb{A}_N^2)$. Define the *Charged Tetrahedral Operator* as follows

Definition 6.22. Let a, b, $c > 0$ such that $a+b+c = \frac{1}{\sqrt{N}}$. Recall the Tetrahedral operator T defined in (4.15). Define the charged tetrahedral operator $\mathsf{T}(a, c)$ as follows

$$\mathsf{T}(a,c) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2(a-c)+\frac{1}{\sqrt{N}}\right)/6} e^{2\pi i c_{\mathrm{b}}(c\mathsf{q}_2 - a\mathsf{q}_1)} \mathsf{T}_{12} e^{-2\pi i c_{\mathrm{b}}(a\mathsf{p}_2 + c\mathsf{q}_2)} \tag{6.5}$$

Lemma 6.23. *We have that*

$$\mathsf{T}(a,c) = e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2(a-c)+\frac{1}{\sqrt{N}}\right)/6} e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \mathsf{D}_{12} \psi_{a,c}(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2, -e^{-\frac{\pi i}{N}} Y_1 X_2 \overline{Y_2}) \tag{6.6}$$

were $\psi_{a,c}(x,n)$ *is the charged quantum dilogarithm from* (3.28)

Extra Notation. Recall the notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$. For $a = (x, n)$ and $a' = (y, m)$ in $\mathbb{A}_N$ we write

$$\langle a, a' \rangle \equiv e^{2\pi i x y} e^{-2\pi i n m/N} \qquad \langle a \rangle \equiv e^{\pi i x^2} e^{-\pi i n(n+N)/N}$$

For $a = (x, n) \in \mathbb{A}_N$, define $\delta(a) \equiv \delta(x)\delta(n)$ where $\delta(x)$ is Dirac's delta distribution while $\delta(n)$ is the Kronecker delta $\delta_{0,n}$ between 0 and n mod N. Define

$$\varphi_{a,c}(x,n) \equiv \psi_{a,c}(x,-n). \tag{6.7}$$

Denote, for $x, y \in \mathbb{R}$ and $z \in \mathbb{A}_N$

$$\nu(x) \equiv e^{-\pi i \frac{c_{\mathrm{b}}^2}{\sqrt{N}}\left(2x+\frac{1}{\sqrt{N}}\right)/6} \qquad \nu_{x,y} = \nu(x-y) e^{\pi i c_{\mathrm{b}}^2 x(x+y)} \tag{6.8}$$

The equations from Lemma 3.17 can be upgraded to

$$\nu_{a,c}\tilde{\varphi}_{a,c}(z) = \nu_{c,b}\varphi_{c,b}(z)\langle z \rangle e^{-\pi i N/12} \tag{6.9}$$

$$\nu_{a,c}\overline{\varphi_{a,c}}(z) = \nu_{c,a}\varphi_{c,a}(-\varepsilon z)\langle z \rangle e^{-\pi i N/6} \tag{6.10}$$

$$\nu_{a,c}\overline{\tilde{\varphi}_{a,c}}(z) = \nu_{b,c}\varphi_{b,c}(-\varepsilon z) e^{-\pi i N/12} \tag{6.11}$$

Where ε was defined in (6.2). From the Charged Pentagon Equation (3.33) we get the following

Proposition 6.24 (Charged Tetrahedral Pentagon equation)**.** *Let a_j, $c_j > 0$ such that $\frac{1}{\sqrt{N}} - a_j - c_j > 0$ for $j = 0, 1, 2, 3$ and 4, which further satisfies the following relations*

(6.12)
$$a_1 = a_0 + a_2 \qquad a_3 = a_2 + a_4 \qquad c_1 = c_0 + a_4 \qquad c_3 = a_0 + c_4 \qquad c_2 = c_1 + c_3.$$

Then we have that

$$\mathsf{T}_{12}(a_4, c_4)\mathsf{T}_{1,3}(a_2, c_2)\mathsf{T}_{23}(a_0, c_0) = \mu \mathsf{T}_{23}(a_1, c_1)\mathsf{T}_{12}(a_3, c_3) \tag{6.13}$$

where
$$\mu = \exp \pi i \frac{c_{\mathrm{b}}^2}{6\sqrt{N}} \left(2(c_0 + a_2 + c_4) - \frac{1}{\sqrt{N}} \right)$$

We have an integral kernel description for the charged tetrahedral operator. We use the Bra-Ket notation to denote integral kernels, see Appendix A.1.

Proposition 6.25. *Let $\overline{\mathsf{T}}(a, c) \equiv (\mathsf{T}(a, c))^{-1}$.*

$$\begin{aligned}
&\langle a_0,\, a_2 |\, \mathsf{T}_{12}(a, c)\, | a_1,\, a_3 \rangle \\
&\qquad = \nu(a - c) e^{\pi i c_{\mathrm{b}}^2 a(a+c)} \langle a_3 - a_2,\, a_0 \rangle \overline{\langle a_3 - a_2 \rangle} \delta(a_0 + a_2 - a_1) \tilde{\varphi}_{a,c}(a_3 - a_2) \\
&\langle a_0,\, a_2 | \overline{\mathsf{T}}(a, c) | a_1,\, a_3 \rangle \\
&\qquad = \nu(b - c) e^{\pi i c_{\mathrm{b}}^2 b(b+c)} e^{-\pi i N/12} \langle a_3 - a_2, a_1 \rangle \langle a_3 - a_2 \rangle \delta(a_1 + a_3 - a_0) \varphi_{b,c}(a_3 - a_2).
\end{aligned}$$

The appearance of ε is due to the non-unitarity of the theory for $\mathrm{b} > 0$ and $N > 1$.

Let A and B two operators on $\mathrm{L}^2(\mathbb{A}_N)$ defined as bra-ket distributions by

$$\langle a_1, a_2 | \mathsf{A} \rangle = \delta(a_1 + a_2) \langle a_1 \rangle e^{\pi i N/12} \qquad \langle a_1, a_2 | \mathsf{B} \rangle = \langle a_1 - a_2 \rangle \tag{6.14}$$

$$\langle \overline{\mathsf{A}} | a_1, a_2 \rangle = \overline{\langle \varepsilon a_1, \varepsilon a_2 | \mathsf{A} \rangle} \qquad \langle \overline{\mathsf{B}} | a_1, a_2 \rangle = \overline{\langle \varepsilon a_1, \varepsilon a_2 | \mathsf{B} \rangle}. \tag{6.15}$$

Lemma 6.26 (Fundamental Lemma)**.** *We have the following three relations*

$$\int_{\mathbb{A}_N^2} \langle \overline{\mathsf{A}} | v, s \rangle \langle x, s | \mathsf{T}(a, c) | u, t \rangle \langle t, y | \mathsf{A} \rangle \mathrm{d}s\mathrm{d}t = \langle x, y | \overline{\mathsf{T}}(a, b) \langle u, v \rangle \tag{6.16}$$

$$\int_{\mathbb{A}_N^2} \langle \overline{\mathsf{A}} | u, s \rangle \langle s, x | \mathsf{T}(a, c) | v, t \rangle \langle t, y | \mathsf{B} \rangle \mathrm{d}s\mathrm{d}t = \langle x, y | \overline{\mathsf{T}}(b, c) \langle u, v \rangle \tag{6.17}$$

$$\int_{\mathbb{A}_N^2} \langle \overline{\mathsf{B}} | u, s \rangle \langle s, y | \mathsf{T}(a, c) | t, v \rangle \langle t, x | \mathsf{B} \rangle \mathrm{d}s\mathrm{d}t = \langle x, y | \overline{\mathsf{T}}(a, b) \langle u, v \rangle. \tag{6.18}$$

6.3.1 TQFT Rules, Tetrahedral Symmetries and Gauge Invariance

We consider oriented surfaces with cellular structure such that all 2-cells are either bigons or triangles. Not all the edge orientations will be admitted. We forbid cyclically oriented triangles. For the bigons, we consider only the *essential* ones, the others being contractible to an edge. These essential bigons are precisely the ones with cellular structure isomorphic to the unit disk with vertices $\pm 1 \in \mathbb{C}$ and edges $\{e_1 = e^{\pi it}; e_2 = -e^{\pi it}$, for $t \in [0,1]\}$ or $\{e_1 = -e^{-\pi it}; e_2 = e^{-\pi it}$, for $t \in [0,1]\}$. Given such an ideally triangulated surface Σ we will associate a copy of $\mathbb{C}$ to any bigon and a copy of $\mathcal{S}'(\mathbb{A}_N)$ to any triangle. Globally we associate to the surface the space $\mathcal{S}'(\mathbb{A}_N^{\Delta_2(\Sigma)})$. To a shaped tetrahedron T with ordered vertices $\{v_0, v_1, v_2, v_3\}$ we associate the partition function $Z_{\mathrm{b}}^{(N)}(T)$ through the Nuclear Theorem (A.6) as a ket distribution

$$(6.19)\quad \langle x|\widetilde{Z_{\mathrm{b}}^{(N)}(T)}\rangle = \begin{cases} \langle a_0, a_2|\mathsf{T}(c(v_0v_1), c(v_0v_3))|a_1, a_3\rangle & \text{if } \operatorname{sign}(T) = 1; \\ \langle a_1, a_3|\overline{\mathsf{T}}(c(v_0v_1), c(v_0v_3))|a_0, a_2\rangle & \text{if } \operatorname{sign}(T) = -1. \end{cases}$$

where

$$\mathbb{A}_N \ni a_i := a(\partial_i T), \quad i \in \{0,1,2,3\}$$

and

$$c := \frac{1}{\pi\sqrt{N}}\alpha_T : \Delta_1(T) \to \mathbb{R}_{>0}.$$

Having allowed bigons in triangulations of surfaces, we must also allow cones over such as cobordisms. From the 2 classes of bigons described above we have 4 isotopy classes of cellular structures of cones over them, described in the following as embedded in $\mathbb{R}^3 \simeq \mathbb{C} \times \mathbb{R}$. The bigon is identified with the unit disc embedded in $\mathbb{C}$. The apex of the cone will be the point $(0,1) \in \mathbb{C} \times \mathbb{R}$. The 1-cells will be either

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm(1-t), t)\}$$

or

$$\{e^1_{0\pm}(t) = (\pm e^{i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}$$

or

$$\{e^1_{0\pm}(t) = (\mp e^{-i\pi t}, 0),\ e^1_{1\pm}(t) = (\pm t, 1-t)\}.$$

We name these types of cones A_+, A_-, B_+ and B_- respectively. We need TQFT rules for the gluing of these cones. We just need to consider their gluing to a tetrahedra. We assign a partition function to the cones as follows

(6.20)

$$\langle a_1, a_2|\widetilde{Z_{\mathrm{b}}^{(N)}(A_\pm)}\rangle = \delta(a_1 + a_2)\langle a_1\rangle^{\pm 1} e^{\pm\pi iN/12}, \quad \langle a_1, a_2|\widetilde{Z_{\mathrm{b}}^{(N)}(B_\pm)}\rangle = \langle a_1 - a_2\rangle^{\pm 1}.$$

Tetrahedral symmetries are generated by permutation of the ordered vertices. Indeed the group of tetrahedral symmetries is identified with the symmetric group $\mathbb{S}_4$ and is generated by three transpositions. The three equations of the Fundamental Lemma 6.26 gain an interpretation as glueing of cones on the faces of a tetrahedron through definitions (6.20). These three glueing generates all the symmetries of a tetrahedron, and through this interpretation, the Fundamental Lemma assure that the partition function $Z_{\mathrm{b}}^{(N)}$ satisfies all the tetrahedral symmetries. For more details on tetrahedral symmetries and the cone's partition function see [AK1, GKT].

We can now formulate the main Theorem for the Teichmüller TQFT. This theorem was proved by Andersen and Kashaev for the case $N = 1$ in [AK1]. The statement that we have here is for every N odd, and it is strictly speaking not present as such in the literature.

Theorem 6.27 (Level N Teichmüller TQFT, Andersen and Kashaev)**.** *For any* $\mathrm{b} \in \mathbb{C}^*$ *such that* $\operatorname{Im}\mathrm{b}(|\mathrm{b}|-1) = 0$ *and* $\operatorname{Re}\mathrm{b} > 0$, *and for any* $N \in \mathbb{Z}_{>0}$ *odd there exists a unique* $*_{\mathrm{b}}$*-functor* $F_{\mathrm{b}}^{(N)}\colon \mathcal{B}_a \to \mathcal{D}_N$ *such that* $F_{\mathrm{b}}^{(N)}(A) = \Delta_2(A)$, $\forall A \in \operatorname{Ob}\mathcal{B}_a$, *and for any admissible leveled shaped pseudo 3-manifold* (X, l_X), *the associated morphism in* $\mathcal{D}_N$ *takes the form*

$$F_{\mathrm{b}}^{(N)}(X, l_X) = Z_{\mathrm{b}}^{(N)}(X) e^{-\pi i \frac{l_X c_{\mathrm{b}}^2}{N}} \in \mathcal{S}'\left(\mathbb{A}_N^{\Delta_2(\partial X)}\right), \tag{6.21}$$

where $Z_{\mathrm{b}}^{(N)}$ *is defined in* (6.19) *for a tetrahedron.*

Here $*_{\mathrm{b}}$-functor means that $F_{\mathrm{b}}^{(N)}(X^*) = F_{\mathrm{b}}^{(N)}(X)^{*_{\mathrm{b}}}$, where X^* is the oppositely oriented pseudo 3-manifold to X.

The discussion so far proves the theorem except for the gauge invariance and the convergence of the partition functions under glueings. We will not discuss the convergence here because it follows directly from the convergence in the case level $N = 1$, which was addressed in [AK1]. We just remark that the hypothesis of admissibility is used to prove the convergence of the partition function.

For the gauge invariance consider the suspension of an n-gone SP_n naturally triangulated into n tetrahedra sharing the only internal edge e. Every gauge transformation can be decomposed in a sequence of gauge transformations involving only one edge e, and every such gauge transformation can be understood in the example of the suspension. Suppose all the tetrahedra to be positive, and having vertex order such that the last two vertices are the endpoints of the internal common edge. After enumerating the tetrahedra in cyclic order, let a_i, c_i be the two shape parameter of T_i, $i = 0, \ldots, n$, and $\mathrm{a} = (a_0, \ldots, a_n)$, $\mathrm{c} = (c_0, \ldots, c_n)$. Notice that $\sqrt{N}\pi a_i$ is the dihedral angle corresponding to the edge e. So a gauge transformation corresponding to e will affect the partition function of SP_N

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) := \operatorname{Tr}_0(\mathsf{T}_{01}(a_1, c_1)\mathsf{T}_{02}(a_2, c_2)\cdots\mathsf{T}_{0n}(a_n, c_n))$$

by shifting c by an amount $\lambda = (\lambda, \ldots, \lambda)$ say. One can show from the definitions and the discussion above, that

$$\mathsf{T}(a, c+\lambda) = e^{-2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} \mathsf{T}(a, c) e^{2\pi i c_{\mathrm{b}} \lambda \mathsf{p}_1} e^{\pi i c_{\mathrm{b}}^2 \left(\frac{1}{\sqrt{N}} - 6a\right)\lambda/3}$$

which, after tracing, leads to the following

Proposition 6.28. *[AK1]*

$$Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}+\lambda) = Z_{\mathrm{b}}^{(N)}(SP_N)(\mathrm{a}, \mathrm{c}) e^{\pi i c_{\mathrm{b}}^2 \left(\frac{n}{\sqrt{N}} - 6Q_e\right)\lambda/3}$$

where

$$Q_e = a_1 + a_2 + \ldots a_n$$

6.4 Knot Invariants: Computations and Conjectures

In this section we update the examples computed in [AK1] to the level $N \geq 1$ setting. Similar results were obtained in [D].

Notation. In the examples we are going to use the following notation for quantum dilogarithms

$$\varphi_{\mathrm{b}}(x, n) \equiv \mathrm{D}_{\mathrm{b}}(x, -n). \tag{6.22}$$

Moreover we will often abuse notation in favor of readability in the following ways. For $z = (x, n) \in \mathbb{A}_N$ we will sometimes write $e^{2\pi i c_{\mathrm{b}} z \alpha}$ in place of $e^{2\pi i c_{\mathrm{b}} x \alpha}$. Moreover sums of the form $z + c_{\mathrm{b}} a$ will always mean $(x + c_{\mathrm{b}} a, n)$.

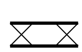
In the following examples we encode an oriented triangulated pseudo 3-manifold X into a diagram where a tetrahedron T is represented by an element

where the vertical segments, ordered from left to right, correspond to the faces $\partial_0 T, \partial_1 T, \partial_2 T, \partial_3 T$ respectively. When we glue tetrahedron along faces, we illustrate this by joining the corresponding vertical segments.

6.4.1 Figure–Eight Knot 4_1

Let X be represented by the diagram

(6.23)



Choosing an orientation, it consists of one positive tetrahedron T_+ and one negative tetrahedron T_- with four identifications

$$\partial_{2i+j} T_+ \simeq \partial_{2-2i+j} T_-, \quad i, j \in \{0, 1\}.$$

Combinatorially, we have $\Delta_0(X) = \{*\}$, $\Delta_1(X) = \{e_0, e_1\}$, $\Delta_2(X) = \{f_0, f_1, f_2, f_3\}$, and $\Delta_3(X) = \{T_+, T_-\}$ with the boundary maps

$$f_{2i+j} = \partial_{2i+j} T_+ = \partial_{2-2i+j} T_-, \quad i, j \in \{0, 1\},$$

$$\partial_i f_j = \begin{cases} e_0, & \text{if } j - i \in \{0, 1\}; \\ e_1, & \text{otherwise,} \end{cases}$$

$$\partial_i e_j = *, \quad i, j \in \{0, 1\}.$$

The topological space $X \setminus \{*\}$ is homeomorphic to the complement of the figure–eight knot, and indeed $X \setminus \{*\}$ is an ideal triangulation of such a cuspidal manifold. The set $\Delta_{3,1}(X)$ consists of the elements $(T_\pm, e_{j,k})$ for $0 \le j < k \le 3$. We fix a shape structure

$$\alpha_X \colon \Delta_{3,1}(X) \to \mathbb{R}_{>0}$$

by the formulae

$$\alpha_X(T_\pm, e_{0,1}) = \pi\sqrt{N} a_\pm, \quad \alpha_X(T_\pm, e_{0,2}) = \pi\sqrt{N} b_\pm, \quad \alpha_X(T_\pm, e_{0,3}) = \pi\sqrt{N} c_\pm,$$

where $a_\pm + b_\pm + c_\pm = \frac{1}{\sqrt{N}}$. The weight function

$$\omega_X \colon \Delta_1(X) \to \mathbb{R}_{>0}$$

takes the values

$$\omega_X(e_0) = \sqrt{N}\pi(2a_+ + c_+ + 2b_- + c_-) =: 2\pi w, \quad \omega_X(e_1) = 2\pi(2 - w).$$

As the figure–eight knot is hyperbolic, the completely balanced case $w = 1$ is accessible directly. We can state the balancing condition $w = 1$ as

$$2b_+ + c_+ = 2b_- + c_-. \tag{6.24}$$

The kernel representations for the operators $\mathsf{T}(a_+, c_+)$ and $\mathsf{T}(a_-, c_-)$ are as follows. Let $z_j \in \mathbb{A}_N$, $j = 0, 1, 2, 3,$

$$\begin{aligned} &\langle z_0, z_2 | \mathsf{T}(a_+, c_+) | z_1, z_3 \rangle \\ &\qquad = \nu_{a_+,c_+} \langle z_3 - z_2, z_0 \rangle \overline{\langle z_3 - z_2 \rangle} \delta(z_0 + z_2 - z_1) \tilde{\varphi}_{a_+,c_+}(z_3 - z_2) \end{aligned} \tag{6.25}$$

$$\begin{aligned} &\langle z_3, z_1 | \overline{\mathsf{T}}(a_+, c_-) | z_2, z_0 \rangle = \overline{\langle \varepsilon z_2, \varepsilon z_0 | \mathsf{T}(a_+, c_-) | \varepsilon z_3, \varepsilon z_1 \rangle} \\ &\qquad = \overline{\nu_{a_-,c_-}} \langle z_0 - z_1, z_2 \rangle \langle z_1 - z_0 \rangle \delta(z_0 + z_2 - z_3) \overline{\tilde{\varphi}_{a_-,c_-}(\varepsilon z_1 - \varepsilon z_0)} \end{aligned} \tag{6.26}$$

The Andersen–Kashaev invariant at level N for the complement of the figure–eight knot is then

$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N^4} \langle z_0, z_2 | \mathsf{T}(a_+, c_+) | z_1, z_3 \rangle \langle z_3, z_1 | \overline{\mathsf{T}}(a_+, c_-) | z_2, z_0 \rangle \mathrm{d}z_0 \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3$$

$$\begin{aligned}
&= \int_{\mathbb{A}_N^4} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_3 - z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1 - \varepsilon z_0)}\delta(z_0 + z_2 - z_1)\times \\
&\quad \times \delta(z_0 + z_2 - z_3)\langle z_3 - z_2, z_0\rangle\langle z_0 - z_1, z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2\mathrm{d}z_3 \\
&= \int_{\mathbb{A}_N^3} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_1 - z_2)\overline{\varphi_{c_-,b_-}(\varepsilon z_1 - \varepsilon z_0)}\delta(z_0 + z_2 - z_1)\times \\
&\quad \times \langle z_1 - z_2, z_0\rangle\langle z_0 - z_1, z_2\rangle \mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N^2} \nu_{c_+,b_+}\overline{\nu_{c_-,b_-}}\varphi_{c_+,b_+}(z_0)\overline{\varphi_{c_-,b_-}(\varepsilon z_2)}\langle z_0, z_0\rangle\langle -z_2, z_2\rangle \mathrm{d}z_0\mathrm{d}z_2 \\
&= \int_{\mathbb{A}_N} \nu_{c_+,b_+}\varphi_{c_+,b_+}(z_0)\langle z_0, z_0\rangle \mathrm{d}z_0 \overline{\int_{\mathbb{A}_N} \nu_{c_-,b_-}\varphi_{c_-,b_-}(\varepsilon z_2)\langle z_2, z_2\rangle \mathrm{d}z_2} \\
&= \sigma_{c_+,b_+}\overline{\sigma_{c_-,b_-}}
\end{aligned}$$

We can compute

$$\begin{aligned}
\sigma_{c_\pm,b_\pm} &= \nu_{c_\pm,b_\pm}\int_{\mathbb{A}_N} \frac{e^{-2\pi i c_{\mathrm{b}} z c_\pm}}{\varphi_{\mathrm{b}}(z - c_{\mathrm{b}}(b_\pm + c_\pm))}\langle z\rangle^2 \mathrm{d}z \\
&= \nu'_{c_\pm,b_\pm}\int_{\mathbb{A}_N + di} \frac{e^{4\pi i c_{\mathrm{b}} z(2b_\pm + c_\pm)}}{\varphi_{\mathrm{b}}(z)}\langle z\rangle^2 \mathrm{d}z
\end{aligned}$$

where

$$\nu'_{c_\pm,b_\pm} = \nu_{c_\pm,b_\pm} e^{4\pi i c_{\mathrm{b}}^2 (c_\pm b_\pm - b_\pm^2)} \tag{6.27}$$

and the domain of integration $\mathbb{A}_N + di = (\mathbb{R} + di) \times \mathbb{Z}/N\mathbb{Z})$. Note we have shifted the real integral to a contour integral in the complex plane, and $d \in \mathbb{R}$ is such that the integral converges absolutely. We sometimes omit the contour shift in the computations but we state it in the results. Defining

$$\lambda \equiv 2b_+ + c_+ = 2b_- + c_-$$

we have

$$\begin{aligned}
Z_{\mathrm{b}}^{(N)}(X) &= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{e^{4\pi i c_{\mathrm{b}}\lambda(z_0+z_2)}}{\varphi_{\mathrm{b}}(z_0)\overline{\varphi_{\mathrm{b}}(\varepsilon z_2)}}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_{\mathrm{b}}(z_2)}{\varphi_{\mathrm{b}}(z_0)} e^{4\pi i c_{\mathrm{b}}\lambda(z_0+z_2)}\langle z_0\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2 \\
&= \nu'_{c_+,b_+}\overline{\nu'_{c_-,b_-}}\int_{\mathbb{A}_N^2} \frac{\varphi_{\mathrm{b}}(z_2 - z_0)}{\varphi_{\mathrm{b}}(z_0)} e^{4\pi i c_{\mathrm{b}}\lambda z_2}\langle z_0, z_2\rangle^2\overline{\langle z_2\rangle^2}\mathrm{d}z_0\mathrm{d}z_2
\end{aligned}$$

that has the structure

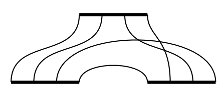
(6.28)
$$Z_{\mathrm{b}}^{(N)}(X) = e^{i\phi} \int_{\mathbb{A}_N + i0} \chi_{4_1}^{(N)}(x, \lambda) \mathrm{d}x,$$

(6.29)
$$\chi_{4_1}^{(N)}(x, \lambda) = \chi_{4_1}^{(N)}(x) e^{4\pi i c_{\mathrm{b}} \lambda x}, \qquad \chi_{4_1}^{(N)}(x) = \int_{\mathbb{A}_N - i0} \frac{\varphi_{\mathrm{b}}(x-y)}{\varphi_{\mathrm{b}}(y)} \langle x, y \rangle^2 \overline{\langle x \rangle^2} \mathrm{d}y$$

where ϕ is some constant quadratic combination of dihedral angles.

6.4.2 The Complement of the Knot 5_2

Let X be the closed S.O.T.P. 3-manifold represented by the diagram



This triangulation has only one vertex $*$ and $X \setminus \{*\}$ is topologically the complement of the knot 5_2. We denote T_1, T_2, T_3 the left, right, and top tetrahedra respectively. We choose the orientation so that all of them are positive. Balancing all the edges correspond to require the following equations to be true

$$2a_3 = a_1 + c_2, \quad b_3 = c_1 + b_2. \tag{6.30}$$

The three integral kernels reads

$$\begin{aligned}
&\langle z, w | \mathsf{T}(a_1, c_1) | u, x \rangle = \\
&\qquad = \nu_{a_1,c_1} \langle x - w, z \rangle \overline{\langle x - w \rangle} \delta(z + w - u) \tilde{\varphi}_{a_1,c_1}(x - w) \\
&\langle x, v | \mathsf{T}(a_2, c_2) | y, w \rangle = \\
&\qquad = \nu_{a_2,c_2} \langle w - v, x \rangle \overline{\langle w - v \rangle} \delta(x + v - y) \tilde{\varphi}_{a_2,c_2}(w - v) \\
&\langle y, u | \mathsf{T}(a_3, c_3) | v, z \rangle = \\
&\qquad = \nu_{a_3,c_3} \langle z - u, y \rangle \overline{\langle z - u \rangle} \delta(y + u - v) \tilde{\varphi}_{a_3,c_3}(z - u)
\end{aligned}$$

Carrying out the computations, defining $\lambda = -c_1 + b_2 - c_2 + a_3$, one gets that

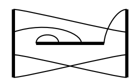
$$Z_{\mathrm{b}}^{(N)}(X) = \int_{\mathbb{A}_N + i0} \chi_{5_2}^{(N)}(x, \lambda) \mathrm{d}x, \qquad \chi_{5_2}^{(N)}(x, \lambda) = \chi_{5_2}^{(N)}(x) e^{2\pi i c_{\mathrm{b}} \lambda x} \tag{6.31}$$

$$\chi_{5_2}^{(N)}(x) = \int_{\mathbb{A}_N - i0} \frac{\overline{\langle x \rangle} \langle z \rangle}{\varphi_{\mathrm{b}}(z + x) \varphi_{\mathrm{b}}(z) \varphi_{\mathrm{b}}(z - x)} \mathrm{d}z \tag{6.32}$$

6.4.3 H-Triangulations

In this section we will look at one vertex H-triangulations of knots.

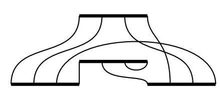
Let X be an H–Triangulation for the figure–eight knot, i.e. let X be given by the diagram



where the figure-eight knot is represented by the edge of the central tetrahedron connecting the maximal and the next to maximal vertices. If we choosing central tetrahedron (T_0) to be positive, the left tetrahedron (T_+) will be positive and the right one (T_-) negative.The shape structure, in the limit $a_0 \to 0$ satisfies $2b_+ + c_+ = 2b_- + c_- =: \lambda$ The partition function satisfies the following limit formula

$$\lim_{a_0 \to 0} \varphi_{\mathrm{b}}(c_{\mathrm{b}} a_0 - c_{\mathrm{b}}/\sqrt{N}) Z_{\mathrm{b}}^{(N)}(X) = \frac{e^{-\pi i N/12}}{\nu(c_0)} \chi_{4_1}^{(N)}(0) \tag{6.33}$$

Similarly let X be represented by the diagram



that is, the H–triangulation for the 5_2 knot. We denote T_0, T_1, T_2, T_3 the central, left, right, and top tetrahedra respectively and we choose the orientation so that the central tetrahedron T_0 is negative then all other tetrahedra are positive. The edge representing the knot 5_2 connects the last two edges of T_0, so that the weight on the knot is given by $2\pi a_0$. In the limit $a_0 \to 0$, all edges, except for the knot, become balanced under the conditions

$$a_1 = c_2 = a_3, \quad b_3 = c_1 + b_2,$$

which in particular imply (6.30). The partition function has the following expression

$$Z_{\mathrm{b}}^{(N)}(X) = \Theta \frac{e^{-\pi i N/12}}{\varphi_{\mathrm{b}}(c_{\mathrm{b}} a - c_{\mathrm{b}}\sqrt{N})} \chi_{5_2}^{(N)}(c_{\mathrm{b}}(a_1 - a_3)) \tag{6.34}$$

For some constant phase factor Θ.

6.4.4 Asymptotics of $\chi_{4_1}^{(N)}(0)$

In this section we want to study the asymptotic behavior of the invariant of the figure–8 knot

$$\begin{aligned} \chi_{4_1}^{(N)}(0) &= \int_{\mathbb{A}_N} \mathrm{D}_{\mathrm{b}}(-x,-k)\overline{\mathrm{D}_{\mathrm{b}}(x,-k)}\mathrm{d}(x,k) \\ &= \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k \in \mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi \mathrm{b}}, -k\right) \overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi \mathrm{b}}, -k\right)} \mathrm{d}x \end{aligned}$$

when $\mathrm{b} \to 0$. The analysis uses techniques similar to the one presented in [AK1] for $N = 1$, however higher level gives new informations that we will show here. The integration in the complex plane is a contour integral where $d > 0$ so that the integral is absolutely convergent. By means of the asymptotic formula for the quantum dilogarithm (3.22) we have that

$$\chi_{4_1}^{(N)}(0) = \frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \operatorname{Exp}\left[\frac{\mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})}{2\pi i \mathrm{b}^2 N}\right] \times \phi_{-x}(k)\overline{\phi_x(k)}(1+\mathcal{O}(\mathrm{b}^2))\mathrm{d}x$$

We want to apply the steepest descent method to this integral to get an asymptotic formula for $\mathrm{b} \to 0$. First we show the computation for the exact integral,

$$\frac{1}{2\pi \mathrm{b}\sqrt{N}} \sum_{k\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}-id} \operatorname{Exp}\left[\frac{\mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})}{2\pi i \mathrm{b}^2 N}\right] \phi_{-x}(k)\overline{\phi_x(k)}\mathrm{d}x \tag{6.35}$$

and then we will argue that the former one can be approximated by the latter when $\mathrm{b} \to 0$.
Let $h(x) := \mathrm{Li}_2(-e^{-\sqrt{N}x}) - \mathrm{Li}_2(-e^{\sqrt{N}x})$. Its critical points are solutions to

$$\begin{cases} h'(x) = 0 \\ h''(x) \neq 0 \end{cases}$$

which are $\mathcal{S} = \left\{\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}} : k \in \mathbb{Z}\right\}$. We compute the value of $\operatorname{Im} h$ at its critical points to be

$$\operatorname{Im} h\left(\pm\frac{2}{3}\frac{\pi i}{\sqrt{N}} + \frac{2\pi i k}{\sqrt{N}}\right) = \pm 4\Lambda(\frac{\pi}{6}) \tag{6.36}$$

where Λ is the Lobachevsky function

$$\Lambda(\alpha) = -\int_0^\alpha \log|2\sin\varphi|\,\mathrm{d}\varphi \tag{6.37}$$

and we refer the reader to [Kir] for the expressions that relate Lobachevsky function to the classical dilogarithm.

We only remark that $4\Lambda(\frac{\pi}{6}) = \mathrm{Vol}(4_1)$, where by $\mathrm{Vol}(4_1)$ we mean the hyperbolic volume of knot complement $S^3 \setminus (4_1)$.
Fix $\mathbb{C} \ni x_0 = -\frac{2}{3}\frac{\pi i}{\sqrt{N}}$, which is accessible from the original contour without passing through other critical points, and consider the contour

$$\mathcal{C} = \{z \in \mathbb{C} : \mathrm{Re}(h(z)) = \mathrm{Re}(h(x_0)),\ \mathrm{Im}(h(z)) \leq \mathrm{Im}(h(x_0))\}$$

which is asymptotic to $\mathrm{Re}(z)+\mathrm{Im}(z)=0$ for $\mathrm{Re}(z)\to\infty$ and to $\mathrm{Re}(z)-\mathrm{Im}(z)=0$ for $\mathrm{Re}(z)\to-\infty$. Moreover

$$\lim_{\mathrm{Re}(z)\to\pm\infty}\mathrm{Im}(h(z))=\lim_{\mathrm{Re}(z)\to\pm\infty}\pm\mathrm{Re}(z)\,\mathrm{Im}(z)=-\infty \tag{6.38}$$

All together we have found a contour $\mathcal{C}$ along which the integral (6.35) can be computed with the steepest descent method (see [Won]), giving as the following approximation for $\mathrm{b}\to 0$

$$e^{\frac{h(x_0)}{2\pi i\mathrm{b}^2 N}}\frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1+\mathcal{O}(\mathrm{b}^2) \tag{6.39}$$

where

$$g_{4_1}(x):=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\phi_{-x}(k)\overline{\phi}_x(k).$$

We now go back to $\chi_{4_1}^{(N)}(0)$, and we write it as the following integral

$$\chi_{4_1}^{(N)}(0)=\frac{1}{2\pi\mathrm{b}\sqrt{N}}\sum_{k\in\mathbb{Z}/N\mathbb{Z}}\int_{\mathbb{R}-id}f_{\mathrm{b}}(x,k)\mathrm{d}(x,k) \tag{6.40}$$

where

$$f_{\mathrm{b}}(x,k)=\mathrm{D}_{\mathrm{b}}\left(\frac{-x}{2\pi\mathrm{b}},-k\right)\overline{\mathrm{D}_{\mathrm{b}}\left(\frac{x}{2\pi\mathrm{b}},-k\right)}. \tag{6.41}$$

Then consider the contour

$$\mathcal{C}_{\mathrm{b}}=\{z\in\mathbb{C}:\arg f_{\mathrm{b}}(z)=\arg f_{\mathrm{b}}(z_{\mathrm{b}}),\ |f_{\mathrm{b}}(z)|=|f_{\mathrm{b}}(z_{\mathrm{b}})|\} \tag{6.42}$$

where z_{b} is defined as the solution to

$$\frac{\partial}{\partial x}\log f_{\mathrm{b}}(x)=0 \tag{6.43}$$

which minimize the absolute value of f_{b}. Using the asymptotic formula for f_{b} it is simple to show that the contours $\mathcal{C}_{\mathrm{b}}$ approximates $\mathcal{C}$ as $\mathrm{b}\to 0$ and that the points z_{b}'s will converge to x_0. So, in the limit $\mathrm{b}\to 0$, the integral (6.40) is approximated by the integral (6.35), for which we already have an asymptotic formula. We have proved the following

$$\chi_{4_1}^{(N)}(0)=e^{\frac{h(x_0)}{2\pi i\mathrm{b}^2 N}}\frac{g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)}{\sqrt{iN^{-1}h''(x_0)}}(1+\mathcal{O}(\mathrm{b}^2)), \tag{6.44}$$

As we remarked above $\operatorname{Im} h(x_0) = -\operatorname{Vol}(4_1)$.
Next we look at the number $g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right)$ which is a topological invariant of the knot in the formula above. We have that

$$\sqrt{N} g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) = \sum_{k=1}^{N} \phi_{\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k)\overline{\phi}_{-\frac{2}{3}\frac{\pi i}{\sqrt{N}}}(k)$$
$$= \left|\prod_{j=1}^{N-1}\left(1 - e^{-\frac{\pi i}{3N}} e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}}\right| \sum_{k=0}^{N-1}\prod_{j=1}^{k} \frac{1}{\left|1 - e^{\frac{1}{3}\frac{-\pi i}{N}} e^{\frac{2\pi i j}{N}}\right|^2}$$

The last expression allows us to make the following remark

$$g_{4_1}\left(-\frac{2}{3}\frac{\pi i}{\sqrt{N}}\right) = \gamma_N \mathcal{H}_N^0(\overline{\rho_{comp}}) \tag{6.45}$$

where $H_N^0(\overline{\rho_{comp}})$ is the Baseilhac–Benedetti invariant for the figure–eight knot found in [BB], computed at the conjugate of the complete hyperbolic structure (meaning that the holonomies of the structure are all complex conjugated) and γ_N is a global rescaling given by

$$\gamma_N = \left|\prod_{j=1}^{N-1}\left(1 - e^{-\frac{2\pi i j}{N}}\right)^{\frac{j}{N}}\right| \tag{6.46}$$

Remark 6.29. The very same steps of the previous asymptotic computation for $\chi_{4_1}^{(N)}(0)$ can be applied to $\chi_{5_2}^{(N)}(0)$ up to the point of having an expression

$$\chi_{5_2}^{(N)}(0) = e^{\frac{\phi(x_{5_2})}{2\pi i \mathrm{b}^2 N})} \frac{g_{5_1}(x_{5_2})}{\sqrt{iN^{-1}h''_{5_2}(x_{5_2})}}(1 + \mathcal{O}(\mathrm{b}^2)), \tag{6.47}$$

where x_{5_2} is the only critical point in the complex plane that contributes to the steepest descent and

$$g_{5_1}(x) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}\overline{\phi_{-x}}(j)\phi_x(j)\overline{\phi_{-x}}(j). \tag{6.48}$$

The fact that $\operatorname{Im}\phi(x_{5_2}) = -\operatorname{Vol}(5_2)$, can be seen directly, see for example [AK1]. However this situation is already too complicated to allow us to check relations with other theories. The obvious guess is to look for the Baseilhac–Benedetti invariant, but no explicitly computed examples, other then 4_1, are known to the authors.

The following conjecture was originally stated in [AK1] for $N = 1$. Here we restate it in the updated setting.

Conjecture 6.30 ([AK1]). *Let M be a closed oriented compact 3-manifold. For any hyperbolic knot $K \subset M$, there exist a two parameters* (b, N) *family of smooth functions $J^{(\mathrm{b},N)}_{M,K}(x,j)$ on $\mathbb{R} \times \mathbb{Z}/N\mathbb{Z}$ which has the following properties.*

1. *For any fully balanced shaped ideal triangulation X of the complement of K in M, there exist a gauge invariant real linear combination of dihedral angles λ, a (gauge non-invariant) real quadratic polynomial of dihedral angles ϕ such that*
$$Z^{(N)}_{\mathrm{b}}(X) = e^{ic_{\mathrm{b}}^2\phi} \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \int_{\mathbb{R}} J^{(\mathrm{b},N)}_{M,K}(x,j) e^{ic_{\mathrm{b}}x\lambda} \mathrm{d}x$$

2. *For any one vertex shaped H-triangulation Y of the pair (M, K) there exists a real quadratic polynomial of dihedral angles φ such that*
$$\lim_{\omega_Y \to \tau} \mathrm{D}_{\mathrm{b}} \left(c_{\mathrm{b}} \frac{\omega_Y(K) - \pi}{\pi\sqrt{N}}, 0 \right) Z^{(N)}_{\mathrm{b}}(Y) = e^{ic_{\mathrm{b}}^2\varphi - i\frac{\pi N}{12}} J^{(\mathrm{b},N)}_{M,K}(0,0),$$
where $\tau \colon \Delta_1(Y) \to \mathbb{R}$ takes the value 0 on the knot K and the value 2π on all other edges.

3. *The hyperbolic volume of the complement of K in M is recovered as the following limit*
$$\lim_{\mathrm{b}\to 0} 2\pi \mathrm{b}^2 N \log |J^{(\mathrm{b},N)}_{M,K}(0,0)| = -\operatorname{Vol}(M \setminus K)$$

Remark 6.31. We have proved this extended conjecture for the knots $(S^3, 4_1)$ and $(S^3, 5_2)$, see formulas (6.45), (6.47) and (6.48). Moreover we gave a more explicit expansion, showing the appearance of an extra interesting therm g_K, and a precise relation between g_{4_1} and a known invariant of hyperbolic knots, defined by Baseilhac–Benedetti in [BB], see equation (6.45). We could have been more bold and extend the conjecture declaring the appearance of $g_{(M,K)}$ to be general, and it to be proportional to the Baseilhac–Benedetti invariant. However we feel that there are not enough evidences to state it as general conjecture.

Appendices

A Tempered Distributions

For standard references for the topics of this appendix see e.g. [Hör2, Hör1] and [RS1, RS2].

Definition A.1. The Schwartz space $\mathcal{S}(\mathbb{R}^n)$ is the space of all the functions $\phi \in C^\infty(\mathbb{R}^n, \mathbb{C})$ such that

$$||\phi||_{\alpha,\beta} := \sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indeces α, β.
The space of Tempered Distributions $\mathcal{S}'(\mathbb{R}^n)$ is the space of linear functionals on $\mathcal{S}(\mathbb{R}^n)$ which are continuous with respect to all these semi-norms.

Both these spaces are stable under the action of the Fourier transform $\mathcal{F}$ and we use the notation $\hat{u} = \mathcal{F}(u)$. Let Z_n be the zero section of $T^*(\mathbb{R}^n)$.

Definition A.2. For a temperate distribution $u \in \mathcal{S}'(\mathbb{R}^n)$, we define its *Wave Front Set* to be the following subset of the cotangent bundle of $\mathbb{R}^n$

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) - Z_{\mathbb{R}^n} |\ \xi \in \Sigma_x(u)\}$$

where

$$\Sigma_x(u) = \cap_{\phi \in C_x^\infty(\mathbb{R}^n)} \Sigma(\phi u).$$

Here

$$C_x^\infty(\mathbb{R}^n) = \{\phi \in C_0^\infty(\mathbb{R}^n) | \phi(x) \neq 0\}$$

and $\Sigma(v)$ are all $\eta \in \mathbb{R}^n - \{0\}$ having no conic neighborhood V such that

$$|\hat{v}(\xi)| \leq C_N(1 + |\xi|)^{-N},\ N \in \mathbb{Z}_{>0},\ \xi \in V.$$

Lemma A.3. *Suppose u is a bounded density on a C^∞ sub-manifold Y of $\mathbb{R}^n$, then $u \in \mathcal{S}'(\mathbb{R}^n)$ and*

$$\mathrm{WF}(u) = \{(x, \xi) \in T^*(\mathbb{R}^n) | x \in \operatorname{Supp} u,\ \xi \neq 0 \text{ and } \xi(T_x Y) = 0\}.$$

In particular if $\operatorname{Supp} u = Y$, then we see that $\mathrm{WF}(u)$ is the co-normal bundle of Y.

Definition A.4. Let u and v be temperate distributions on $\mathbb{R}^n$. Then we define

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) = \{(x, \xi_1 + \xi_2) \in T^*(\mathbb{R}^n) | (x, \xi_1) \in \mathrm{WF}(u),\ (x, \xi_2) \in \mathrm{WF}(v)\}.$$

Theorem A.5. *Let u and v be temperate distributions on $\mathbb{R}^n$. If*

$$\mathrm{WF}(u) \oplus \mathrm{WF}(v) \cap Z_n = \emptyset,$$

then the product of u and v exists and $uv \in \mathcal{S}'(\mathbb{R}^n)$.

Definition A.6. We denote by $\mathcal{S}(\mathbb{R}^n)_m$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\beta \partial^\alpha (\phi)(x)| < \infty$$

for all multi-indices α and β such that if $\alpha_i = 0$ then $\beta_i = 0$ for $n - m < i \leq n$. We define $\mathcal{S}'(\mathbb{R}^n)_m$ to be the continuous dual of $\mathcal{S}(\mathbb{R}^n)_m$ with respect to these semi-norms.

We observe that if $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-m}$ is the projection onto the first $n-m$ coordinates, then $\pi^*(\mathcal{S}(\mathbb{R}^{n-m})) \subset \mathcal{S}(\mathbb{R}^n)_m$. This means we have a well defined push forward map

$$\pi_* : \mathcal{S}'(\mathbb{R}^n)_m \longrightarrow \mathcal{S}'(\mathbb{R}^{n-m}).$$

Proposition A.7. *Suppose Y is a linear subspace in $\mathbb{R}^n$, u a density on Y with exponential decay in all directions in Y. Suppose $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a projection for some $m < n$. Then $u \in \mathcal{S}'(\mathbb{R}^n)_m$ and $\pi_*(u)$ is a density on $\pi(Y)$ with exponential decay in all directions of the subspace $\pi(Y) \subset \mathbb{R}^m$.*

Tempered distributions can be thought of as functions of growth at most polynomial, thanks to the following

Theorem A.8. *Let $T \in \mathcal{S}'(\mathbb{R}^n)$, then $T = \partial^\beta g$ for some polynomially bounded continuous function g and some multi-index β. That is, for $f \in \mathcal{S}(\mathbb{R}^n)$,*

$$T(f) = \int_{\mathbb{R}^n} (-1)^{|\beta|} g(x)(\partial^\beta f)(x)\mathrm{d}x$$

In particular it is possible to show that $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}(\mathbb{R}^n) \ni f \mapsto T_f \in \mathcal{S}'(\mathbb{R}^n)$ with $T_f(g) = \int_{\mathbb{R}^n} f(x)g(x)\mathrm{d}x$.

Denoting by $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$ the space of continuous linear maps from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^m)$, we remark that we have an isomorphism

$$\tilde{\cdot} : \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m)) \to \mathcal{S}'(\mathbb{R}^{n \sqcup m}) \tag{A.1}$$

determined by the formula

$$\varphi(f)(g) = \tilde{\varphi}(f \otimes g) \tag{A.2}$$

for all $\varphi \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^m))$, $f \in \mathcal{S}(\mathbb{R}^n)$, and $g \in \mathcal{S}(\mathbb{R}^m)$. This is the content of the Nuclear theorem, see e.g. [RS2]. Since we can not freely multiply distributions we end up with a categroid instead of a category. The partially defined composition in this categroid is defined as follows. Let n, m, l be three finite sets and $A \in \mathcal{S}'(\mathbb{R}^{n \sqcup m})$ and $B \in \mathcal{S}'(\mathbb{R}^{m \sqcup l})$. We have pull back maps

$$\pi^*_{n,m} : \mathcal{S}'(\mathbb{R}^{n \sqcup m}) \to \mathcal{S}'(\mathbb{R}^{n \sqcup m \sqcup l}) \text{ and } \pi^*_{m,l} : \mathcal{S}'(\mathbb{R}^{m \sqcup l}) \to \mathcal{S}'(\mathbb{R}^{n \sqcup m \sqcup l}).$$

By what we summarised above, the product

$$\pi^*_{n,m}(A)\pi^*_{m,l}(B) \in \mathcal{S}'(\mathbb{R}^{n \sqcup m \sqcup l})$$

is well defined provided the wave front sets of $\pi^*_{n,m}(A)$ and $\pi^*_{m,l}(B)$ satisfy the condition

$$(\mathrm{WF}(\pi^*_{n,m}(A)) \oplus \mathrm{WF}(\pi^*_{m,l}(B))) \cap Z_{n \sqcup m \sqcup l} = \varnothing \tag{A.3}$$

If we now further assume that $\pi^*_{n,m}(A)\pi^*_{m,l}(B)$ continuously extends to $\mathcal{S}(\mathbb{R}^{n \sqcup m \sqcup l})_m$, then we obtain a well defined element

$$(\pi_{n,l})_*(\pi^*_{n,m}(A)\pi^*_{m,l}(B)) \in \mathcal{S}'(\mathbb{R}^{n \sqcup l}).$$

A.1 Bra-Ket Notation

We often use the Bra-Ket notation to make computations with distributions. For $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ a density and $x \in \mathbb{R}^n$ we will write

$$\langle x|\varphi\rangle := \varphi(x),$$

with distributional meaning

$$\varphi(f) = \int_{\mathbb{R}^n} \langle x|\varphi\rangle f(x)\mathrm{d}x = \int_{\mathbb{R}^n} \varphi(x) f(x)\mathrm{d}x.$$

In particular if $\varphi \in \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, then

$$\langle x|\varphi\rangle = \varphi(x) = \delta_x(\varphi)$$

The integral kernel of the operator T, if it exists, is a distribution k_{T} such that

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} k_{\mathsf{T}}(x,y)\psi(y)\mathrm{d}y \tag{A.4}$$

Working with Schwartz functions, the nuclear theorem expressed by formula (A.2) guarantees that the kernel k_T exists and that it is a tempered distribution. We will usually write the kernel from equation (A.4), in Bra-Ket notation as follows

$$\mathsf{T}(\psi)(x) = \int_{\mathbb{R}^n} \langle x|\mathsf{T}|y\rangle\psi(y)\mathrm{d}y \tag{A.5}$$

and the nuclear theorem morphism (A.2) can be read as

$$\langle x|\mathsf{T}|y\rangle = \langle x,\, y|\widetilde{T}\rangle. \tag{A.6}$$

A.2 $\mathrm{L}^2(\mathbb{A}_N)$ and $\mathcal{S}(\mathbb{A}_N)$

$\mathbb{A}_N \equiv \mathbb{R} \times (\mathbb{Z}/N\mathbb{Z})$ has the structure of a locally compact abelian group, with the normalized Haar measure $\mathrm{d}(x,n)$ defined by

$$\int_{\mathbb{A}_N} f(x,n)\mathrm{d}(x,n) := \frac{1}{\sqrt{N}} \sum_{n\in\mathbb{Z}/N\mathbb{Z}} \int_{\mathbb{R}} f(x,n)\mathrm{d}x$$

where $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ is an integrable function. By definition $\mathrm{L}^2(\mathbb{A}_N)$ is the space of functions $\mathsf{f}\colon \mathbb{A}_N \longrightarrow \mathbb{C}$ such that

$$\int_{\mathbb{A}_N} |\mathsf{f}(a)|^2 \,\mathrm{d}a \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \int_{\mathbb{R}} |\mathsf{f}(x,n)|^2 \,\mathrm{d}x < \infty \tag{A.7}$$

with standard inner product

$$\langle \mathsf{f}, \mathsf{g} \rangle \equiv \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \int_{\mathbb{R}} \mathsf{f}(x,n)\overline{\mathsf{g}(x,n)}\mathrm{d}(x,n) \tag{A.8}$$

Finite square integrable sequences are just a finite dimensional vector space

$$\mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{C}^N,$$

with a preferred basis given by mod N Kronecker delta functions

$$\delta_j(n) \equiv \begin{cases} 1 & \text{if } j = n \bmod N \\ 0 & \text{otherwise} \end{cases} \tag{A.9}$$

There is a natural isomorphism

$$\mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \simeq \mathrm{L}^2(\mathbb{A}_N) \tag{A.10}$$

defined by

$$f \otimes \delta_j\,(a) = f(x)\delta_j(n), \text{ for } a = (x,n) \in \mathbb{A}_N \tag{A.11}$$

with inverse

$$\mathbb{A}_N \ni \mathsf{f} \mapsto \sum_{j=0}^{N-1} \mathsf{f}(\cdot, j) \otimes \delta_j \in \mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{Z}/N\mathbb{Z}) \tag{A.12}$$

Everything just said holds true substituting L^2 with $\mathcal{S}$, with the isomorphism $\mathcal{S}(\mathbb{A}_N) \simeq \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^N$ and further also, the space of tempered distributions on $\mathbb{A}_N$, defined as linear continuous functionals over $\mathcal{S}(\mathbb{A}_N)$, are simply $\mathcal{S}'(\mathbb{R}) \otimes \mathbb{C}^N$. All the Bra-Ket notation extends trivially to $\mathcal{S}(\mathbb{A}_N)$, including the nuclear theorem (A.6), substituting all the integrals over $\mathbb{R}$ with integrals over $\mathbb{A}_N$.
We use a bracket notation for Fourier coefficients and Gaussian exponentials in $\mathbb{A}_N$, following the notation introduced in [AK3]

$$\langle (x,n),\, (y,m) \rangle \equiv e^{2\pi i x y} e^{-2\pi i n m/N} \qquad \langle (x,n) \rangle \equiv e^{\pi i x^2} e^{-\pi i n(n+N)/N} \tag{A.13}$$

For (x,n) and (y,m) in $\mathbb{A}_N$. The Fourier transform then takes the form

$$\mathcal{F}(f)(x,n) = \int_{\mathbb{A}_N} f(y,m)\langle (x,n),(y,m) \rangle \mathrm{d}(y,m).$$

For any operator A of order N, we can define the operator $\mathrm{L}_N(A)$ via the spectral theorem, such that it formally satisfies

$$A = e^{2\pi i\, \mathrm{L}_N(A)/N}.$$

We can define, for any function $f : \mathbb{A}_N \longrightarrow \mathbb{C}$ the operator function $f\!\!/(\mathsf{x}, A) \equiv f(\mathsf{x}, \mathrm{L}_N(A))$ for any commuting pair of operators x and A, where the former is self adjoint and the latter is of order N. We have, for x and A as above, that

$$f\!\!/(\mathsf{x}, A) = \int_{\mathbb{A}_N} \tilde{f}(y, m) e^{2\pi i y \mathsf{x}} A^{-m} \mathrm{d}(y, m) \tag{A.14}$$

where we use the following notation for the inverse Fourier transforms

$$\tilde{f}(x, n) = \int_{\mathbb{A}_N} f(y, m) \overline{\langle (y, m); (x, n) \rangle} \mathrm{d}(y, m). \tag{A.15}$$

B Categroids

We need a notion which is slightly more general than categories to define the Teichmüller TQFT functor.

Definition B.1. [AK1]
A *Categroid* $\mathcal{C}$ consist of a family of objects $\mathrm{Obj}(\mathcal{C})$ and for any pair of objects A, B from $\mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Mor}_{\mathcal{C}}(A, B)$ such that the following holds

A For any three objects A, B, C there is a subset $K^{\mathcal{C}}_{A,B,C} \subset \mathrm{Mor}_{\mathcal{C}}(A, B) \times \mathrm{Mor}_{\mathcal{C}}(B, C)$, called the composable morphisms and a *composition* map

$$\circ : K^{\mathcal{C}}_{A,B,C} \to \mathrm{Mor}_{\mathcal{C}}(A, C).$$

such that composition of composable morphisms is associative.

B For any object A we have an identity morphism $1_A \in \mathrm{Mor}_{\mathcal{C}}(A, A)$ which is composable with any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(A, B)$ or $g \in \mathrm{Mor}_{\mathcal{C}}(B, A)$ and we have the equations

$$1_A \circ f = f, \text{ and } g \circ 1_A = g.$$

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