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The Supply of Hours Worked and Endogenous Growth Cycles

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Abstract

We show that declining hours of work per worker in conjunction with a growing work force may give rise to growth cycles. This is accomplished in an overlapping generations model where individuals are endowed with Boppart-Krusell preferences (Boppart and Krusell (2020)), i. e., the wage elasticity of their supply of hours worked is negative. On the supply side, economic growth is due to the expansion of consumption-good varieties through endogenous research. We show that a sufficiently negative equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties opens up the possibility of growth cycles where the economy fluctuates between two regimes, one with and the other without an active research sector. We identify period-2 and period-3 cycles, conclude with Li and Yorke (1975) that cycles of any periodicity exists, and generalize our findings to period-n cycles. We show that the possibility of cycles occurs under empirically plausible conditions. Throughout, we emphasize that the economics of cycles is linked to the intergenerational trade of shares and their pricing in the asset market.

JEL-Codes: E32, J22, O33, O41.
Keywords: Endogenous Cycles, Technological Change, Endogenous Labor Supply, OLG-Model.

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1 Introduction

At least since 1870 hours worked per worker have substantially declined in many of today's industrialized countries. Estimates of Huberman (2004) and Huberman and Minns (2007) suggest that a male full-time production worker in the U.K. had a weekly workload of 56.9 hours in the year 1870. In 2000 this number comes down to 42 hours of work, an absolute decline of roughly 35%. According to these authors a similar tendency can be found in Australia, Belgium, Canada, Denmark, France, Germany, Ireland, Italy, the Netherlands, Spain, Sweden, Switzerland, and the US. At the same time the total workforce of these countries has been increasing.

Boppart and Krusell (2020) and Irmen (2018) interpret these stylized facts as properties of country-specific balanced growth paths driven by exogenous technical change. In contrast to this view, the present paper shows that declining hours of work per worker may cause growth cycles. Over time the evolution is unbalanced, yet, consistent with the data, i. e., income and wages increase and individual hours of work decline.

We obtain these features in an overlapping generations model with exogenous population growth where the individual supply of hours worked and technological change are both endogenous. The household side has two-period lived overlapping generations where individuals are endowed with a periodic log-log utility function that belongs to the Boppart-Krusell class (Boppart and Krusell (2020)). Hence, the wage elasticity of the individual supply of hours worked is negative. In addition, the individual labor supply of hours worked falls at a constant rate if the real wage increases at a constant rate (Irmen (2018)). In the spirit of Grossman and Helpman (1991) and Jones (1995), the production side has endogenous technological change through an expansion of the set of consumption-good varieties. New varieties are invented by research firms. Along cycles, the economy switches between a regime with an operating research sector and a regime without such a sector.

A key finding of our analysis concerns the economic mechanism of the market for assets and its repercussions on the evolution of the economy over time. In this market, ownership shares of the firms that produce a variety of the consumption good are traded. The demand side has the current young with their need to save for old age. In the regime without an operating research sector, the supply of shares stems solely from the current old. They own the shares of all firms and sell them to finance their consumption demand. In the regime with an operating research sector, there is an additional source of share supply. New firms emit ownership shares to finance the purchase of a blueprint that grants them the right to market a new consumption-good variety invented by a research firm.

In the regime without an active research sector the amount of existing consumption-good varieties is large relative to the cohort size of the current young. Accordingly, the supply of shares is large relative to its demand and the equilibrium share price will be low. Therefore, the financial resources that new firms may raise to pay for a blueprint are too small for a research firm to break even. As a result, the research sector remains inactive.

In the regime with an active research sector the amount of existing consumption-good varieties is small relative to the cohort size of the current young. Therefore, in spite of the additional supply of shares through the primary offering of new firms, the equilibrium price of shares is sufficiently high so that research firms enter the market and break even.

The supply of hours worked plays a key role for the transition between the two regimes. The transition into the regime with an active research sector is driven by the extensive margin of the supply of hours worked. Without an active research sector wages, the individual supply of hours worked, individual savings, and the amount of traded shares remains constant. However, through population growth the cohort size of the young increases over time. This boosts the total demand for shares. Therefore, the equilibrium share price increases over time and, eventually, reaches a level at which an additional share supply of new firms can be absorbed and research firms to break even.

The transition into the regime without an active research sector is driven by the effect of newly invented consumption-good varieties on the intensive margin of the supply of hours worked and its repercussions for the asset market. On its demand side, new varieties increase the wage which leads to a reduction in the individual supply of hours worked. Therefore, individual incomes and savings increase by less than wages. This attenuates the increase in the total demand for shares. On the supply side, new varieties affect the cost of research firms through higher wages and an intertemporal knowledge spillover. Since the latter increases the productivity of research labor the break-even price of a blueprint increases by less than wages. Then, what matters is the relative strength of the effects of new varieties on the reduction in the supply of hours worked and on the cost reduction through the knowledge spillover. If the former channel dominates the latter, then, in spite of the cost reduction through the knowledge spillover, the demand for shares becomes so weak that the asset market equilibrium cannot support a share price at which research firms can break even.

To understand the role of an endogenous supply of hours worked for the occurrence of cycles, observe that the economy has a globally stable steady state with monotone convergence if the wage elasticity of the individual supply of hours worked is zero. If this elasticity is negative, then the occurrence of cycles hinges in the *equilibrium elasticity of the individual supply of hours worked to an expansion in the set of consumption-good varieties*. If the latter is slightly negative, then the steady state remains stable, possibly with oscillatory convergence. If it becomes sufficiently negative then the steady state is unstable. In this case, the global dynamics gives rise to period-2 and period-3 cycles along which the economy fluctuates between both regimes. We fully identify these cycles and derive the conditions under which they exist. Then, we conclude with Li and Yorke (1975) that our dynamical system gives rise to cycles of any periodicity.

In our discussion section, we first generalize our analysis to period-n cycles with the property of spending one period in the regime with an active research sector followed by $n-1$ periods in the regime without such a sector. Again, we provide a comprehensive characterization of these cycles and derive the condition for their existence. We also establish a remarkable property of these cycles: over a full period-n cycle the average growth rates of new varieties and of share prices are the same as those obtained if the economy were in steady state. Moreover, we show that these cycles are unstable.

Second, we provide some evidence on how the possibility of growth cycles may be linked to existing empirical evidence. Finally, we study the evolution of GDP in both absolute and per-capita terms. Here, we show that an active research sector in a given period unequivocally increases the level of GDP in the following period. However, the level of per-capita GDP may fall.

This paper contributes to at least two strands of the literature. First, it contributes to the growth literature ignited by Boppart and Krusell (2020) that incorporates the secular decline in hours worked per worker into the neoclassical growth model of Ramsey (1928), Cass (1965), and Koopmans (1965). Irmen (2018) applies it to an overlapping generations setting. In both papers technical change is exogenous. Irmen (2020) studies the economic consequences of automation in a model with endogenous technical change where households are endowed with Boppart-Krusell preferences. These contributions have in common that the steady state is either a globally stable or a saddle-path stable balanced growth path. In contrast, the present paper establishes, that Boppart-Krusell preferences may be the source of growth cycles. Indeed, if the individual supply of hours worked does not respond to movements in the real wage, then the steady state of our model is globally stable. It is the negative wage elasticity of the individual supply of hours worked that opens the possibility of growth cycles involving fluctuations between a regime with and without an active research sector.

Second, our analysis contributes to the literature on endogenous growth cycles. Unlike the present paper, this literature maintains the assumption of an exogenous labor supply. Hence, the mechanics behind cycles that we identify are new to this literature.

A paper closely related to our study is Matsuyama (1999) who studies a variant of the lab-equipment model of Rivera-Batiz and Romer (1991). Matsuyama argues that economies may grow through cycles moving back and forth between a regime with capital accumulation alone and a regime that has capital accumulation and innovation. Moreover, since the accumulation of capital is subject to diminishing returns, the economy may be trapped in a regime without long-run growth.[1]

Our rationale for growth cycles substantially differs from Matsuyama's in at least two ways. First, there is no capital accumulation in our setup. Rather, the economy "accumulates" workers through population growth. As this accumulation is not subject to diminishing returns the economy either converges to a steady state with growth of per-capita variables or keeps on fluctuating between the regime with and without an active research sector.

Second, we maintain the assumption that new blueprints are sold in conjunction with a perpet-

[1] Other studies emphasizing the presence of physical capital and research as a source of growth cycles include Matsuyama (2001) with cyclical deterministic growth and Bental and Peled (1996), Wälde (2002), and Wälde (2005) with cyclical stochstic growth.

ual patent. Hence, a key mechanism for the emergence of cycles in Matsuyama (1999), namely, the temporary monopoly power of new products, is mute in our model. In contrast, we emphasize in line with the empirical evidence that individuals reduce their supply of hours worked in response to higher wages. This is the key driver of instability of the unique steady state.

Other contributions where innovation cycles arise because of the temporary nature of the monopoly power enjoyed by innovators include Deneckere and Judd (1992), Gale (1996), and Shleifer (1986). The growth cycles studied in Evans, Honkapohja, and Romer (1998) rely on sunspots, expectational indeterminacy, and multiply equilibria. Unlike the latter studies, the cycles identified in the present paper result from the instability of a unique steady state. This instability is driven by the strength of the negative wage elasticity of the individual supply of hours worked to higher wages.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 defines the intertemporal general equilibrium, proves its existence, and derives the dynamical system. Section 4 studies the structural and the comparative-static properties of the steady state. Section 5 deals with the transitional dynamics. Here, we derive the main results of our analysis. Section 5.1 has the local stability analysis of the steady state. Section 5.2 deals with the global dynamics and, in particular, identifies and interprets period-2 and period-3 cycles. The focus of Section 6 is on period-n cycles, on the empirical plausibility of growth cycles, and on the evolution of GDP in absolute and per-capita terms. Section 7 concludes. All proofs are contained in Section 8, Appendix A.

2 The Model

The economy has a household sector, a production sector, and a research sector in an infinite sequence of periods $t = 1, 2, ..., \infty$.

The household sector comprises overlapping generations of individuals who live for two periods, youth and old age. Their labor supply when young is endogenous. The individual lifetime utility features a Boppart-Krusell generalized log-log utility function that is compatible with a steady-state path along which per-capita income grows and the individual labor supply declines at a constant rate (Boppart and Krusell (2020)).

The production sector has monopolistically competitive firms producing differentiated varieties of a consumption good with labor as the sole contemporaneous input. The research sector invents new varieties of the consumption good using labor and existing technological knowledge as inputs.

In all periods, there are markets for the following objects of exchange. First, there are markets for a continuum of differentiated varieties of the consumption good. These varieties are supplied by the production sector and demanded by the household sector. Second, there is a labor market where the current young supply hours of work that the firms of the production and the research sector demand. Third, there is a market for the blueprints of newly invented varieties of the consumption good. These blueprints are supplied by the inventing research firms and demanded by new firms that enter the production sector. Finally, there is an asset market where bonds (in zero net supply) and ownership shares in the firms of the production sector are traded. At the beginning of each period, all assets are owned by the current old. To finance their consumption, the old sell their ownership shares to the young. In addition, there are primary stock offerings by the new firms of the production sector. They need to finance their purchase of a blueprint for a newly invented variety of the consumption good. The demand for both types of shares corresponds to the savings of the current young.

2.1 Household

The population at t consists of L_t young and L_{t-1} old individuals. Except for their age, individuals are identical. The number of young individuals between two adjacent periods grows at the exogenous rate $g_L > 0$. For short, we shall refer to g_L as the growth rate of the labor force.

When young, individuals supply labor, earn wage income, save, and enjoy leisure as well as the consumption goods. When old, they retire and consume their wealth.

For cohort t, denote consumption when young and old by c_t^y and c_{t+1}^o, and leisure time enjoyed when young by l_t. We normalize the maximum per-period time endowment supplied to the labor market to unity. Then, $1 - l_t = h_t$, where $h_t \in [0, 1]$ is the individual supply of hours worked when young. Individuals of all cohorts assess bundles (c_t^y, l_t, c_{t+1}^o) according to the lifetime utility function

$$U(c_t^y, l_t, c_{t+1}^o) = \ln c_t^y + \ln\left(1 - \phi(1 - l_t)(c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o; \tag{2.1}$$

here, the parameter $\phi > 0$ determines the strength of the disutility of labor, and $\beta \in (0, 1)$ is the discount factor. As shown in Irmen (2018), $\nu \in (0, 1)$ implies that consumption when young and leisure are complements in the sense that $\partial^2 U / \partial c^y \partial l > 0$.

Consumption when young and old, c_t^y and c_{t+1}^o, represent bundles of differentiated consumption goods, i. e.,

$$c_t^y = A_t^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_t} (x_t^y(j))^\alpha dj \right]^{\frac{1}{\alpha}} \quad \text{and} \quad c_{t+1}^o = A_{t+1}^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_{t+1}} (x_{t+1}^o(j))^\alpha dj \right]^{\frac{1}{\alpha}}, \tag{2.2}$$

where $\sigma > 1$ and $\alpha \in (0, 1)$. Here, $x_t^y(j)$ and $x_{t+1}^o(j)$ denote the respective quantity of consumption good j consumed when young and old. The "number" of available consumption goods at any time t is given by A_t. As $\sigma > 1$ there is a "taste for variety". The parameter $\alpha \in (0, 1)$ determines the elasticity of substitution between any pair of existing consumption good varieties. As α increases consumption goods become better substitutes.

The optimal behavior of an individual of cohort t results from a two-stage budgeting procedure. First, it allocates its spending across the differentiated consumption goods available when young and old age. Second, it determines its labor supply when young as well as the consumption profile over her life cycle.

More precisely, in the first stage, the individual chooses the quantities $x_t^y(j)$ for given prices $p_t(j)$, $j \in [0, A_t]$, and the quantities $x_{t+1}^o(j)$ for given (perfect foresight) prices $p_{t+1}(j)$, $j \in [0, A_{t+1}]$, so as to minimize the costs of attaining aggregate consumption levels c_t^y and c_{t+1}^o. Stated differently, the two cost-minimization problems solved by cohort t are

$$\min_{\{x_t^y(j)\}_{j=0}^{A_t}} \int_0^{A_t} p_t(j) x_t^y(j) dj \qquad \text{s. t.} \qquad A_t^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_t} (x_t^y(j))^\alpha dj \right]^{\frac{1}{\alpha}} = c_t^y,$$
$$\min_{\{x_{t+1}^o(j)\}_{j=0}^{A_{t+1}}} \int_0^{A_{t+1}} p_{t+1}(j) x_{t+1}^o(j) dj \qquad \text{s. t.} \qquad A_{t+1}^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_{t+1}} (x_{t+1}^o(j))^\alpha dj \right]^{\frac{1}{\alpha}} = c_{t+1}^o. \tag{2.3}$$

This delivers the following conditional demands for each differentiated consumption good

$$x_t^y(j) = \frac{p_t(j)^{-\frac{1}{1-\alpha}} c_t^y}{A_t^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_t} p_t(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{\frac{1}{\alpha}}} \quad \text{and} \quad x_{t+1}^o(j) = \frac{p_{t+1}(j)^{-\frac{1}{1-\alpha}} c_{t+1}^o}{A_{t+1}^{\sigma - \frac{1}{\alpha}} \left[\int_0^{A_{t+1}} p_{t+1}(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{\frac{1}{\alpha}}}. \tag{2.4}$$

Using the latter in the respective cost definitions of (2.3) gives two expenditure functions pertaining to period t, i. e., $P_t c_t^y$ for all young and $P_t c_t^o$ for all old individuals. Here,

$$P_t = A_t^{\frac{1}{\alpha} - \sigma} \left[\int_0^{A_t} p_t(j')^{-\frac{\alpha}{1-\alpha}} dj' \right]^{-\frac{1-\alpha}{\alpha}} \tag{2.5}$$

is the *ideal price index*, i. e., the minimum cost of one unit of the consumption aggregates, c_t^y and c_t^o.

In the second stage, each member of cohort t chooses a plan $(c_t^y, h_t, c_{t+1}^o, s_t)$ that maximizes her lifetime utility subject to two periodic budget constraints. We follow Irmen (2018) and denote by $\mathcal{P}$ the set of *permissible* bundles (c_t^y, h_t, c_{t+1}^o) out of which the individual chooses. Elements of this set satisfy the condition

$$1 - 2\nu - (1 - \nu)\phi h_t (c_t^y)^{\frac{\nu}{1-\nu}} > 0, \tag{2.6}$$

which assures that U is strictly concave.

Let $w_t > 0$ denote the wage per hour worked and $R_{t+1} > 0$ the (perfect foresight) interest factor paid per unit saved.[2] Then, cohort t solves

$$\max_{(c_t^y, h_t, c_{t+1}^o, s_t) \in \mathscr{P} \times \mathbb{R}} \quad \ln c_t^y + \ln\left(1 - \phi h_t (c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o$$
$$\text{s. t.} \quad P_t c_t^y + P_t s_t \leq w_t h_t \quad \text{and} \quad P_{t+1} c_{t+1}^o \leq R_{t+1} P_t s_t. \tag{2.7}$$

Before we state and discuss the solution to this maximization problem it proves useful to define

$$w_c \equiv \left(\frac{(1+\beta)(1-\nu)}{\left(\phi\left(1 + (1+\beta)(1-\nu)\right)\right)^{1-\nu} \left(1 - \nu(1+\beta)\right)^{\nu}} \right)^{\frac{1}{\nu}}, \tag{2.8}$$

and to make the following assumption:

Assumption 1 *It holds that*

$$0 < \nu < \bar{\nu}(\beta) \equiv \frac{3 + \beta - \sqrt{5 + \beta(2+\beta)}}{2(1+\beta)},$$

and, for all t,

$$\frac{w_t}{P_t} \geq w_c.$$

Assumption 1 serves two purposes. The first inequality assures that the optimal plan derived in the following proposition satisfies condition (2.6). The second inequality implies that the individual chooses a positive demand for leisure.

Proposition 2.1 (Optimal Plan of Cohort t)

Suppose Assumption 1 holds. Then, the optimal plan of cohort t involves the conditional demands (2.4) and

$$h_t = w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{-\nu}, \qquad P_t s_t = \frac{\beta}{(1+\beta)(1-\nu)} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu},$$

$$P_t c_t^y = \frac{1 - \nu(1+\beta)}{(1+\beta)(1-\nu)} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu}, \qquad P_{t+1} c_{t+1}^o = \frac{\beta}{(1+\beta)(1-\nu)} R_{t+1} w_c^{\nu} \left(\frac{w_t}{P_t}\right)^{1-\nu}.$$

Hence, the optimal plan of cohort t hinges critically on the ratio w_t/P_t which has an interpretation as the real wage in units of contemporary consumption. Assumption 1 assures a positive demand for leisure since w_t/P_t is sufficiently high. Moreover, $(-\nu)$ is the the wage elasticity of the individual labor supply. Accordingly, the individual supply of hours worked declines in

[2] For simplicity our notation does not distinguish between a wage rate paid in the production and a wage rate paid in the research sector. As individuals are identical they may supply homogeneous labor to either sector. Hence, in any constellation that has both sectors operating there can only be one wage. Moreover, observe that the individual supply of hours worked is assumed to be perfectly divisible across occupations.

the wage. Finally, Proposition 2.1 implies that consumption when young and leisure are also "demand complements," i. e, in response to a higher P_t or a lower w_t, both, c_t^y and l_t fall.

2.2 Production

At all t the production sector has A_t monopolistically competitive firms. Each firm possesses the blueprint and a perpetual patent for the exclusive production of one variety $j \in [0, A_t]$ that it acquired in the past. All firms produce their variety with the same linear production function,

$$x_t(j) = h_{x,t}(j), \tag{2.9}$$

where $h_x(j)$ is the amount of working hours hired by firm j to produce $x_t(j)$ units of consumption good j. Each firm's profit is

$$\pi_t(j) = p_t(j)x_t(j) - w_t h_{x,t}(j), \tag{2.10}$$

where $x_t(j)$ and $h_{x,t}(j)$ are linked via (2.9), and, in light of (2.4),

$$x_t(j) = L_{t-1}x_t^o(j) + L_t x_t^y(j). \tag{2.11}$$

Then, revenue maximization delivers the price set by all firms as

$$p_t = p_t(j) = \frac{w_t}{\alpha}. \tag{2.12}$$

The less substitutable the differentiated consumption goods are, the higher their price. Since all firms charge the same price, p_t, they all supply the same quantity, i. e., $x_t^y = x_t^y(j)$, $x_t^o = x_t^o(j)$, and $x_t = x_t(j)$. Then, (2.9), (2.10) and (2.12) imply that the profit of each firm at t is

$$\pi_t = \pi_t(j) = (1-\alpha)p_t x_t. \tag{2.13}$$

These profits are paid as dividends to old individuals who are the shareholders of the firms in the production sector. Finally, since $h_{x,t} = h_{x,t}(j)$ the total amount of hours worked demanded by the production sector, $H_{x,t}$, equals

$$H_{x,t} = A_t h_{x,t} = A_t x_t. \tag{2.14}$$

2.3 Research

The research sector has many small competitive research firms. They may either enter the market or remain inactive. If they enter the market then they hire labor, invent new varieties of the consumption good, and sell the respective blueprints to a newly created firm of the production sector. All research firms have access to the same technology for the invention of new varieties. Therefore, the analysis of the research sector can be done through the lens of a competitive representative firm in conjunction with a free-entry condition.

Following Jones (1995), the representative firm has access to a technology for the creation of new consumption-good varieties given by

$$\Delta A_t = \frac{H_{A,t}}{a} A_t^{\psi}, \quad 0 < \psi < 1. \tag{2.15}$$

Here, $\Delta A_t \equiv A_{t+1} - A_t$ denotes the additional varieties invented in period t, $H_{A,t} \geq 0$ is the total amount of working hours demanded in the research sector, and $a > 0$ determines the productivity of labor in research. Since $\psi > 0$ the productivity of hours worked in research increases in the number of varieties invented in the past, A_t. The assumption $\psi < 1$ constrains the extent of intertemporal knowledge spillovers.

Let v_t denote the revenue generated from selling a blueprint of a newly created variety. Then, the profit associated with an invention is

$$v_t - w_t \frac{a}{A_t^{\psi}}, \tag{2.16}$$

where a/A_t^{ψ} is the amount of hours worked in the research sector required to invent one new variety.

Since the total amount of hours worked demanded by research firms must be finite the profit of (2.16) cannot be strictly positive in equilibrium. This gives rise to the following equilibrium free-entry condition:

$$v_t \leq w_t \frac{a}{A_t^{\psi}}, \quad \text{with "=", if } \Delta A_t > 0. \tag{2.17}$$

Hence, if at t the revenue obtained from selling a blueprint of a new variety is too low, then the research sector will not be active and $\Delta A_t = 0$. However, if the research sector is active at t, then condition (2.17) must hold as equality since in equilibrium entering research firms must be just as well-off as non-entering ones. This distinction plays a key role in our analysis of the economy's transitional dynamics below.

3 Intertemporal General Equilibrium

This section states and interprets the definition of the intertemporal general equilibrium and establishes the associated dynamical system.

3.1 Definition

For all $j \in [0, A_t]$, a price system corresponds to a sequence

$$\{w_t, R_t, P_t, p_t(j), \pi_t(j), v_t\}_{t=1}^{\infty},$$

and an allocation is a sequence

$$\left\{c_t^y, l_t, c_t^o, s_t, x_t^y(j), x_t^o(j), x_t(j), H_{x,t}, H_{A,t}, A_t\right\}_{t=1}^{\infty}.$$

It comprises a plan $\left\{c_t^y, l_t, c_t^o, s_t, x_t^y(j), x_t^o(j)\right\}_{t=1}^{\infty}$ for all cohorts, consumption of the old at $t = 1$, c_t^o, and strategies for the production and the research sector $\{x_t(j), H_{x,t}, H_{A,t}, A_t\}_{t=1}^{\infty}$.

For an exogenous evolution of the labor force, $L_t = L_1(1+g_L)^{t-1}$, $g_L > 0$, with $L_1 > 0$ and a given initial stock of varieties of the consumption good, $A_1 > 0$, an intertemporal general equilibrium with perfect foresight corresponds to a price system and an allocation that satisfy the following conditions for all $t = 1, 2, \cdots, \infty$:

(E1) The plan of each cohort satisfies Proposition 2.1.

(E2) The mark-up in the production sector satisfies (2.12).

(E3) The evolution of A_t is governed by (2.15).

(E4) The research sector satisfies the free-entry condition (2.17).

(E5) The market for the consumption good clears, i. e.,

$$P_t c_t^y L_t + P_t c_t^o L_{t-1} = p_t A_t x_t.$$

(E6) The asset market clears and values all shares according to fundamentals.

(E7) There is full employment of labor, i. e.,

$$H_t = H_{x,t} + H_{A,t} = h_t L_t.$$

(E1) guarantees the optimal behaviour of the household sector under perfect foresight. (E2) assures profit-maximizing behaviour of the production sector. (E4) states that research firms' equilibrium profits are non-positive. According to (E5) the total demand of the young and old individuals for consumption goods are equal to the supply.

In the asset market at t, the ownership shares of the A_t existing varieties and of the ΔA_t new varieties are traded. Since these shares are perfect substitutes as stores of value, they must have the same price denoted by v_t. Moreover, (E6) requires a market valuation of shares according to fundamentals. This means that

$$v_t = \frac{\pi_{t+1}}{R_{t+1}} + \frac{\pi_{t+2}}{R_{t+1}R_{t+2}} + \cdots, \tag{3.1}$$

where π_{t+i}, $i = 1, 2, ...$, are the (perfect foresight) dividends paid to the shareholders in the future. The latter implies the following no-arbitrage condition for bonds and stocks

$$\frac{v_{t+1}}{v_t} + \frac{\pi_{t+1}}{v_t} = R_{t+1}. \tag{3.2}$$

Finally, (E7) calls for the total demand of the production and research sectors for hours worked to equal the aggregate supply of hours worked by the young.

The equilibrium conditions (E1) - (E7) imply for all t that the market for ownership shares satisfies

$$s_t L_t = v_t \left(A_t + \Delta A_t\right). \tag{3.3}$$

In other words, the savings of the young is equal to the period-t market capitalisation of *all* consumption-good varieties available in $t+1$, i. e., those already produced at t and those invented at t. Condition (3.3) reflects two economic transactions. First, the old at t sell their A_t shares to the current young at a price v_t. Second, the firms of the production sector that purchase by auction at t one of the ΔA_t new blueprints at a price v_t need to finance this purchase. This is done through the issue of new shares with an aggregate value of $v_t \Delta A_t$. Clearly, if (2.17) holds as inequality then $\Delta A_t = 0$ and $A_{t+1} = A_t$, if it holds as equality then $A_{t+1} = A_t + \Delta A_t$.

Henceforth, we choose the consumption aggregate when young as the numéraire, i. e., $P_t = 1$ for all t. In view of (2.5) this choice implies for any symmetric configuration that

$$p_t = A_t^{\sigma-1}. \tag{3.4}$$

Due to the "taste for variety," i. e., $\sigma > 1$, the real price of each consumption good variety increases in the number of differentiated consumption goods. Moreover, with the latter in the mark-up formula (2.12) the equilibrium real wage is

$$w_t = \alpha A_t^{\sigma-1}. \tag{3.5}$$

Accordingly, from Proposition 2.1 a higher A_t reduces the supply of hours worked and increases savings. Henceforth, we refer to $-\nu(\sigma-1)$ as the *equilibrium elasticity of the individual supply of hours worked to changes in* A_t.

3.2 Dynamical System

Throughout our focus is on equilibria with a positive demand for leisure. From Proposition 2.1 this requires $w_t \geq w_c$. Since A_t cannot decline and the equilibrium real wage is given by (3.5) this inequality holds for all t if the following assumption is satisfied.

Assumption 2 *It holds that*

$$A_1 \geq \left(\frac{w_c}{\alpha}\right)^{\frac{1}{\sigma-1}}.$$

The transitional dynamics of the intertemporal general equilibrium may be studied through the evolution of the transformed variable

$$z_t \equiv \frac{L_t}{A_t^{\eta}}, \tag{3.6}$$

where

$$\eta \equiv 1 - \psi + \nu(\sigma - 1) > 0,$$

and

$$z_c \equiv \left(\frac{\alpha}{w_c}\right)^{\nu} \frac{a(1+\beta)(1-\nu)}{\beta} \tag{3.7}$$

denotes a critical value of z_t. The following proposition shows that, depending on how z_t relates to z_c, the economy is in one of two distinct regimes. If $z_t \leq z_c$ then the economy is said to be in Regime 0, and the research sector is inactive. If $z_t \geq z_c$ then the research sector is active, and the economy is said to be in Regime 1.

Proposition 3.1 (Existence, Uniqueness, and Dynamical System)

Suppose Assumption 2 holds. Then, a unique intertemporal general equilibrium exists. Moreover, the transitional dynamics of this equilibrium is given by a unique equilibrium sequence, $\{z_t\}_{t=1}^{\infty}$, generated by the piecewise defined difference equation $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ where

$$z_{t+1} = \Phi(z_t) \equiv \begin{cases} (1+g_L)z_t & \text{if} \quad z_t \leq z_c, \\ \\ (1+g_L)z_c \left(\frac{z_t}{z_c}\right)^{\psi-\nu(\sigma-1)} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{3.8}$$

Proposition 3.1 establishes the existence and the uniqueness of an equilibrium for any permissible initial value $z_1 > 0$. Moreover, it shows that the transitional dynamics may involve a passage through two regimes. In Regime 0, z_t is small, i. e., cohort t is small and/or the stock of existing varieties is large. Under these circumstances inequality (2.17) is strict in equilibrium, and the research sector remains inactive. In Regime 1, z_t is large, (2.17) holds as equality, and the research sector is active.[3]

Why are there two regimes? The explanation starts with the observation that the free-entry condition (2.17) in conjunction with (3.5) determines the equilibrium value of a blueprint of a newly invented variety if $\Delta A_t > 0$ as

$$v_t = a\alpha A_t^{\sigma-1-\psi}. \tag{3.9}$$

Hence, if the research sector is active then the equilibrium value of a new variety equals the total labor cost associated with its invention. The exponent $\sigma - 1 - \psi$ reflects the two channels through which A_t affects these costs. First, from (3.5) a higher A_t increases the wage per hour worked, i. e., to break even v_t must increase. Second, a higher A_t increases the productivity of labor in research. Hence, research firms break even at a lower v_t.

The question is then whether the current young are ready to buy some $\Delta A_t > 0$ primary offerings of the newly invented varieties in addition to the shares of existing varieties at the break-even price of equation (3.9). The equilibrium condition of the market for ownership shares stated in (3.3) has the answer. Here, given A_t, the demand increases in L_t. This supports a higher equilibrium price of shares. Indeed, with Proposition 2.1 one readily verifies that the current young are ready to buy some $\Delta A_t > 0$ newly emitted shares at a price equal to v_t of (3.9) if $L_t > z_c A_t^{\eta}$ or $z_t > z_c$. Hence, in Regime 1, cohort t is sufficiently large and/or the stock

[3] Observe that equations (3.6)-(3.8) include the case $\nu = 0$ for which the individual supply of hours worked is time-invariant and equal to $h = (1+\beta)/[\phi(2+\beta)]$. The parameter restriction $\phi \geq (1+\beta)/(2+\beta)$ ensuring that $h \leq 1$ will then replace Assumption 2. Details for this case are available from the authors upon request.

of existing varieties is sufficiently small.

In Regime 0 the equilibrium value of ownership shares obtains directly from the equilibrium condition of the market for ownership shares. In other words, v_t is the equilibrium value of shares that solves (3.3) for $\Delta A_t = 0$. The comparison between v_t and $a\alpha A_t^{\sigma-1-\psi}$, the break-even price of a new blueprint as stated in (3.9), reveals that (see equation (8.12) in the Proof of Proposition 3.1) $z_t < z_c \Rightarrow v_t < a\alpha A_t^{\sigma-1-\psi}$. Hence, in equilibrium inequality (2.17) is strict. The intuition is the following. If $v_t < a\alpha A_t^{\sigma-1-\psi}$ then the shares of existing varieties are cheaper than those of newly invented ones. Since both types of shares generate the same stream of returns investors will only buy the shares of existing varieties. Accordingly, there is no demand for newly emitted shares. In equilibrium, potential research firms anticipate this and, therefore, will not become active.

4 Steady-State Analysis

Define a steady-state equilibrium as a path along which all variables except leisure grow at constant rates. Let $g_{m,t} \equiv \Delta m_t / m_t$, $\Delta m_t \equiv m_{t+1} - m_t$, denote the growth rate of an arbitrary variable m_t, and m and g_m their respective steady-state values. The following subsections establish the existence of a steady-state equilibrium and study its structural properties.

4.1 Existence and Uniqueness

From (3.8) a steady-state equilibrium has $z_t = z$ for all t. The following proposition establishes the existence and the uniqueness of the steady state equilibrium.

Proposition 4.1 (Existence of a Unique Steady-State Equilibrium)

The dynamical system of Proposition 3.1 has a unique steady state. It is given by

$$z \equiv z_c (1 + g_L)^{\frac{1}{\eta}} > z_c.$$

Hence, the unique steady state is in Regime 1. A simple argument shows why the steady state has to be in this Regime. Suppose the economy starts in Regime 0. Then, the initial values are such that $z_1 < z_c$. The research sector is not competitive and remains inactive since $v_1 < a\alpha A_1^{\sigma-1-\psi}$. As long as the economy remains in Regime 0, the real wage, the individual supply of hours worked, and individual savings remain constant. However, the cohort size and, therefore, the demand for shares and the equilibrium share price grow exponentially at rate g_L. Then, there is a period $\tau_c \geq 2$ such that $v_1(1 + g_L)^{\tau_c - 1} \geq a\alpha A_1^{\sigma-1-\psi}$, or, equivalently, $z_1(1 + g_L)^{\tau_c - 1} \geq z_c$. The equilibrium at τ_c involves $v_{\tau_c} = a\alpha A_1^{\sigma-1-\psi}$ and $\Delta A_{\tau_c} \geq 0$. In other words, ongoing population growth implies in finite time that the demand for ownership shares becomes sufficiently large and the young are willing to by $A_1 + \Delta A_{\tau_c}$ shares at a price v_{τ_c}.

4.2 Structural Properties of the Steady State

The following proposition derives the steady-state growth rates of key variables.

Proposition 4.2 (Structural Properties of the Steady State)

Consider the steady state of Proposition 4.1. The steady-state growth factor of consumption-good varieties is

$$1 + g_A = (1 + g_L)^{\frac{1}{\eta}} > 1. \tag{4.1}$$

Moreover, it holds that

a) $1 + g_w = (1 + g_L)^{\frac{\sigma-1}{\eta}}$,

b) $1 + g_h = (1 + g_L)^{\frac{-\nu(\sigma-1)}{\eta}}$,

c) $1 + g_{c^y} = 1 + g_{c^o} = (1 + g_L)^{\frac{(1-\nu)(\sigma-1)}{\eta}}$,

d) $1 + g_H = (1 + g_L)^{\frac{1-\psi}{\eta}}$,

e) $1 + g_v = (1 + g_L)^{\frac{\sigma-1-\psi}{\eta}}$.

According to Proposition 4.2 the number of available consumption-good varieties increases in steady state at a rate approximately equal to g_L/η. The intuition is the following.

From the research technology (2.15) we have

$$g_A = \frac{\Delta A_t}{A_t} = \frac{H_{A,t}}{a} A_t^{\psi-1} \quad \Rightarrow \quad 1 + g_A = (1 + g_{H_A})^{\frac{1}{1-\psi}}. \tag{4.2}$$

Since increments in A_t improve the productivity of research at a decreasing rate, the amount of hours devoted to research, $H_{A,t}$, has to increase at a constant rate to support the steady-state growth rate g_A.

To detect the determinants of g_{H_A} consider the labor-market equilibrium which requires $g_{H_x} = g_{H_A} = g_H$. Hence, the fraction of the workforce allocated to research is constant over time. Accounting for the extensive and the intensive margin of the supply of hours worked, g_H satisfies

$$1 + g_H = (1 + g_L)(1 + g_h) = (1 + g_L)(1 + g_w)^{-\nu}, \tag{4.3}$$

where the last equality follows from Proposition 2.1. Combining (4.2) and (4.3) delivers

$$1 + g_A = \left[(1 + g_L)(1 + g_w)^{-\nu}\right]^{\frac{1}{1-\psi}}, \tag{4.4}$$

i. e., the research technology and the negative wage elasticity of the individual labor supply imply that faster wage growth induces a stronger decline in the amount of hours supplied by research workers and, hence, a smaller g_A.[4]

[4] Observe that (4.4)delivers the steady-state growth rate of the discrete-time version of Jones (1995) if $\nu = 0$. However, here, $\nu > 0$ which calls for an additional relationship to pin down g_A.

A second steady-state relationship between the growth rates g_A and g_w obtains from (3.5), i. e.,

$$1 + g_w = (1 + g_A)^{\sigma-1}. \tag{4.5}$$

It reflects the role of the taste for variety for the evolution of the real wage. As A_t increases so does the real price, p_t, charged by the monopolistically competitive firms. The constant mark-up rule (2.12) implies that p_t and w_t grow at the same rate. Solving (4.4) and (4.5) for the steady-state growth rates g_A and g_w gives the results stated in the proposition.

The steady-state growth rate of hours worked follows from Proposition 2.1 as $1 + g_h = (1 + g_w)^{-\nu}$. The corresponding growth rates of consumption when young and old coincide with the growth rate of the individual wage income, $w_t h_t$. The steady-state growth rate of aggregate hours worked satisfies $1 + g_H = (1 + g_L)(1 + g_h)$ and takes growth at the extensive and at the intensive margin into account. Finally, the growth rate of the steady-state value of ownership shares follows with (3.9). Accordingly, population growth speeds up this rate if $\sigma - 1 > \psi$.

5 Transitional Dynamics

This section develops the main results of the paper. Section 5.1 analyses the local stability of the steady state. Building on this, Section 5.2 deals with the global dynamics.

5.1 Local Stability of the Steady State

Proposition 3.1 and 4.1 allow to write the equilibrium difference equation $\Phi(z_t)$ for $z_t > z_c$ as

$$z_{t+1} = z^{\eta} z_t^{\psi-\nu(\sigma-1)}. \tag{5.1}$$

Then, the local stability properties of the steady state are as follows.

Proposition 5.1 (Local Stability Properties of the Steady State)

Consider the steady state of Proposition 4.1. It is locally stable if and only if

$$\left| \psi - \nu(\sigma - 1) \right| < 1.$$

Moreover, for z_t sufficiently close to z, the following holds:

1. *if $-\nu(\sigma-1) \in (-\psi, 0)$ then the convergence to the steady state is monotone;*
2. *if $-\nu(\sigma-1) = -\psi$ then the economy jumps instantaneously to its steady state;*
3. *if $-\nu(\sigma-1) \in (-(1+\psi), -\psi)$ then the convergence to the steady state is oscillating;*
4. *if $-\nu(\sigma-1) = -(1+\psi)$ then the economy embarks on a period-2 cycle;*
5. *if $-\nu(\sigma-1) < -(1+\psi)$ then the steady state is unstable.*

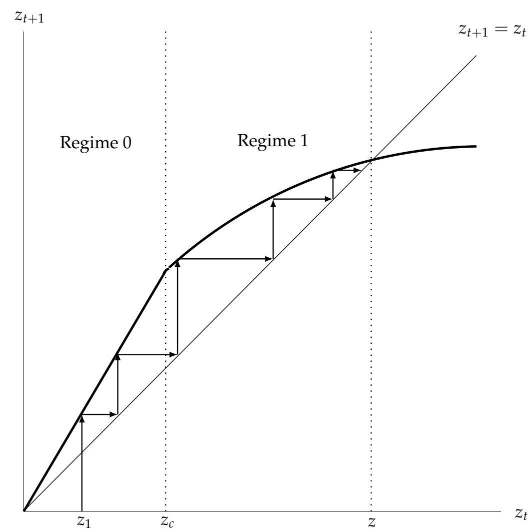

Figure 5.1: Case 1: Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) \in (-\psi, 1-\psi)$.

Hence, the local stability properties of the steady state hinge on how the intertemporal knowledge spillover represented by $\psi \in (0,1)$ relates to $-\nu(\sigma-1)$, the equilibrium elasticity of the individual supply of hours worked to changes in A_t. The latter vanishes for $\nu = 0$. In this case, the individual supply of hours worked does not respond to changes in the real wage, and the steady state is unequivocally locally stable with monotone convergence. Hence, any deviation from this transitional behavior results since $\nu \in (0,1)$. Figures 5.1 - 5.5 provide an illustration for the local (and global) dynamics obtained for the first three cases.

To develop an intuition for Proposition 5.1 consider the equilibrium condition of the market for ownership shares (3.3) at t and $t+1$ for an economy in Regime 1

$$\frac{s_{t+1}L_{t+1}}{s_t L_t} = \frac{v_{t+1}\left(A_{t+1} + \Delta A_{t+1}\right)}{v_t\left(A_t + \Delta A_t\right)}. \tag{5.2}$$

From Proposition 2.1 and (3.5) we have $s_{t+1}/s_t = (h_{t+1}/h_t)(w_{t+1}/w_t) = (1+g_{A,t})^{(1-\nu)(\sigma-1)}$. Hence, the growth factor of individual savings reflects an effect of an increasing amount of varieties on wages and an effect on the supply of hours worked. Similarly, with (2.17) and (3.9) we obtain $v_{t+1}/v_t = (w_{t+1}/w_t)(A_{t+1}/A_t)^{-\psi} = (1+g_{A,t})^{\sigma-1-\psi}$. Hence, the growth factor of the share price reflects an effect of an increasing amount of varieties on wages and on the productivity of labor in research through the knowledge spillover. As the effects through wages on s_{t+1}/s_t and v_{t+1}/v_t on the left and the right-hand side of (5.2) cancel, we write this equation as[5]

$$(1+g_{A,t})^{\psi-\nu(\sigma-1)}(1+g_L) = 1 + g_{A,t+1}, \tag{5.3}$$

[5] Equation (5.3) is equivalent to the second difference equation in the dynamical system (3.8).

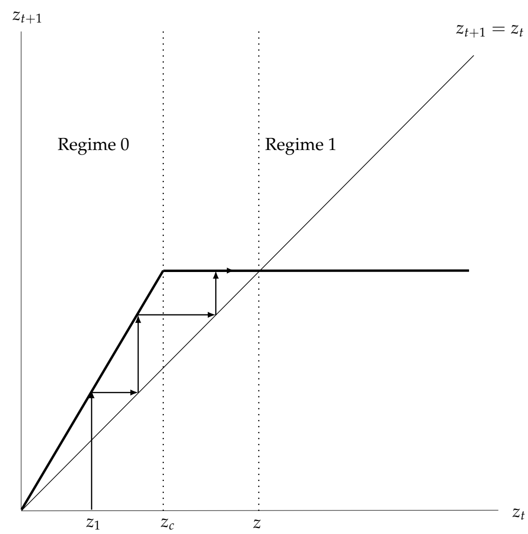

Figure 5.2: Case 2: Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) = -\psi$.

The steady state of this difference equation is given in Proposition 4.2. Its local stability properties hinge on the exponent $\psi - \nu(\sigma - 1)$ which captures the respective effect of $g_{A,t}$ on the demand and the supply side of the market for ownership shares. On the demand side, $-\nu(\sigma - 1) < 0$ means that a higher $g_{A,t}$ reduces the individual supply of hours worked, the individual wage income and savings at $t + 1$. Accordingly, for a given share price at $t + 1$ fewer primary offerings can be placed and ΔA_{t+1} falls. On the supply side, $\psi > 0$ means that a higher $g_{A,t}$ reduces the costs of creating a blueprint. Accordingly, for a given demand for shares, more primary offers can be placed and ΔA_{t+1} increases.

Hence, if $\psi > \nu(\sigma - 1)$ then a higher $g_{A,t}$ implies a higher $g_{A,t+1}$ and the steady state is locally stable with monotone convergence. If $\psi = \nu(\sigma - 1)$ then the two effects outweigh each other. If $\psi < \nu(\sigma - 1)$ then the relationship between $g_{A,t}$ and $g_{A,t+1}$ becomes negative. Then, either the convergence to the locally stable steady state is oscillatory, or the steady state becomes unstable with the possibility of a period-2 cycles within Regime 1.

5.2 Global Dynamics

The extensive and the intensive margin of the supply of hours worked determine the qualitative properties of the global dynamics. On the one hand, a growing labor force enlarges the demand for ownership shares. This force pushes the economy into Regime 1 in finite time irrespective of its initial position in Regime 0. On the other hand, the decline in the individual supply of hours worked to a higher wage determines the stability of the steady state. This section shows that the value of $-\nu(\sigma - 1)$ decides whether the economy remains eventually inside Regime 1 (Proposition 5.2) or keeps on fluctuating between both regimes (Proposition 5.3).

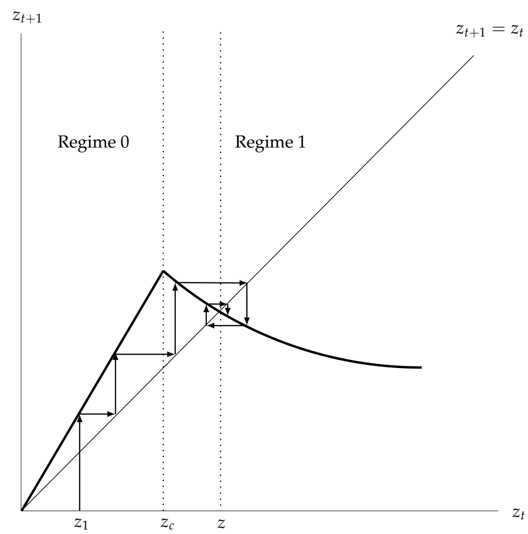

Figure 5.3: Case 3: Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) \in (-(1+\psi), -\psi)$.

Proposition 5.2 (Global Dynamics I)

Consider the dynamical system of Proposition 3.1. For any initial value $z_1 \in \mathbb{R}_{++}$, $z_1 \neq z$, the evolution of z_t for $t > 1$ satisfies the following:

1. *if $-\nu(\sigma - 1) \in (-\psi, 0)$, then $\lim_{t\to\infty} z_t = z$;*
2. *if $-\nu(\sigma - 1) = -\psi$, then there is $\tau < \infty$ such that $z_\tau = z$ for all $t > \tau$;*
3. *if $-\nu(\sigma - 1) \in (-(1+\psi), -\psi) < 0$, then $\lim_{t\to\infty} z_t = z$. Moreover,*

   (a) if $z_1 < z_c$ then the economy eventually settles down in Regime 1, then oscillates about and eventually converges to the steady state,

   (b) if $z_c(1+g_L)^{-\frac{1}{\psi-\nu(\sigma-1)}} \geq z_1 \geq z_c$ then the economy evolves in Regime 1 with oscillating convergence to the steady state,

   (c) if $z_1 > z_c(1+g_L)^{-\frac{1}{\psi-\nu(\sigma-1)}}$, then the economy immediately transits to Regime 0, and after a finite amount of periods moves back to Regime 1, where it oscillates about and eventually converges to the steady state z;

4. *if $-\nu(\sigma - 1) = -(1+\psi)$, then the economy eventually transits to Regime 1 and embarks immediately on a period-2 cycle inside Regime 1.*

Proposition 5.2 characterizes the global dynamics for all permissible parameter constellations that satisfy $-\nu(\sigma - 1) \geq -(1+\psi)$. The four cases have their respective counterparts in the Cases 1 - 4 of Proposition 5.1 (Figures 5.1 - 5.5 illustrate the Cases 1, 2, and 3(a)). As explained above, the economy transits into Regime 1 in finite time. Following this switch there is monotone convergence if $-\nu(\sigma-1) \in (-\psi, 0)$, immediate convergence if $-\nu(\sigma-1) = -\psi$, oscillatory convergence inside Regime 1 if $-\nu(\sigma-1) \in (-(1+\psi), -\psi)$, or an embarkment on a period-2 cycle inside Regime 1 if $-\nu(\sigma-1) = -(1+\psi)$.

Finally, the comparison between Case 3(b) and 3(c) reveals an interesting phenomenon. In both cases, the economy starts in Regime 1, however, with different initial values. The equilibrium in the market for ownership shares, $z_1/z_c = A_2/A_1$, reveals that A_2 is smaller in Case 3(b) than in Case 3(c). Therefore, Case 3(b) has the economy remaining inside Regime 1 with oscillatory convergence towards the steady state. In contrast, A_2 will be large in Case 3(c). As $-\nu(\sigma-1) < -\psi$ the induced reduction in the individual supply of hours worked and savings will be more pronounced than the increase in the productivity of research labor. As a consequence, inequality (2.17) will be strict in period 2. Hence, a switch from Regime 1 into Regime 0 is possible for $-(1+\psi) < -\nu(\sigma-1) < -\psi$ if the initial value z_1 is sufficiently high.[6] The focus of the following proposition is on constellations where $-\nu(\sigma-1) < -(1+\psi)$.

Proposition 5.3 (Global Dynamics II)

Consider the dynamical system of Proposition 3.1. If $-\nu(\sigma-1) < -(1+\psi)$ then for any initial value $z_1 \in \mathbb{R}_{++}$, $z_1 \neq z$, the economy enters a trapping region $[\Phi^2(z_c), \Phi(z_c)]$ in finite time. Inside this region the evolution fluctuates between Regime 0 and 1.

In particular, the following holds.

1. *If*

$$z_1 = z_c(1+g_L)^{\frac{1+\psi-\nu(\sigma-1)}{\eta}} \tag{5.4}$$

then the economy embarks on a period-2 cycle with fluctuations between Regime 0 and Regime 1.

2. *If $-\nu(\sigma-1) < -(2+\psi)$ and*

$$z_1 = z_c(1+g_L)^{\frac{1+2[\psi-\nu(\sigma-1)]}{\eta}}, \tag{5.5}$$

then the economy embarks on a period-3 cycle with two consecutive periods in Regime 0 followed by one period in Regime 1.

Hence, if $-\nu(\sigma-1)$ is sufficiently small, then the economy evolves eventually inside a predefined region. This trapping region is given by $\left[\Phi^2(z_c), \Phi(z_c)\right]$ and ranges over Regime 0 and 1 (see Figure 5.4 for an illustration). The economy will then fluctuate between both regimes.

As to the type of fluctuation, Claim 1 of Proposition 5.3 asserts the possibility of a period-2

[6] Formally, one readily verifies that $v_2 = s_2L_2/A_2 < a\alpha A_2^{\sigma-1-\psi}$ is equivalent to $z_2 < z_c$. In turn, from the dynamical system of (3.8), the latter holds if $z_1 > z_c(1+g_L)^{-\frac{1}{\psi-\nu(\sigma-1)}}$.

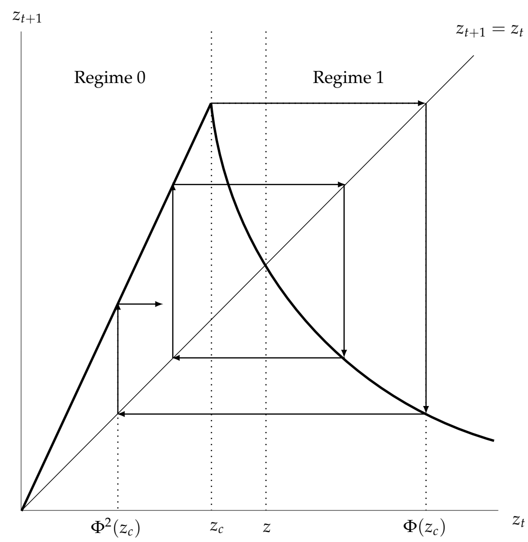

Figure 5.4: The Trapping Region and the Period-2 Cycle Arising if $-\nu(\sigma-1) < -(1+\psi)$.

cycle (see again Figure 5.4). One readily verifies for $t = 1, 3, 5, 9, ...$ that this cycle has

$$z_t = z_c(1+g_L)^{\frac{1+\psi-\nu(\sigma-1)}{\eta}} < z_c < z_{t+1} = z_c(1+g_L)^{\frac{2}{\eta}}. \tag{5.6}$$

The intuition is as follows. Consider an economy that embarks on a period-2 cycle in Regime 0, i. e., $z_1 < z_c$, and then transits into Regime 1, i. e., $z_2 > z_c$. This switch is due to labor force growth that strengthens the demand for ownership shares. Accordingly, the equilibrium share price increases which allows research firms to break even. An active research sector in period 2 means that $A_3 > A_2 = A_1$. This has two consequences for the market of ownership shares in period 3. First, $A_3 > A_2$ reduces the individual supply of hours worked, hence, individual savings, and the demand for shares. Second, $A_3 > A_2$ increases the productivity of research labor, thereby, reducing the break-even price of a new blueprint. As $-\nu(\sigma-1) < -(1+\psi)$ the former effect dominates the latter. Therefore, in period 3 inequality (2.17) will be strict and the economy back in Regime 0.

Observe that a period-2 cycle requires $z_3 = z_1$ so that

$$1 + g_{A,2} = (1+g_L)^{\frac{2}{\eta}} \tag{5.7}$$

Hence, the average growth rate of A_t on the period-2 cycle is equal to the steady-state growth rate stated in Proposition 4.2. Moreover, one readily verifies that the share prices increase at different rates along a period-2 cycle whereby $(v_3/v_2) > (v_2/v_1) > 1$. However, the average growth rate of v_t on such a cycle coincides with the steady-state growth rate, g_v, stated in Proposition 4.2.

According to Claim 2 a period-3 cycle may arise. Figure 5.6 provides an illustration. It satisfies

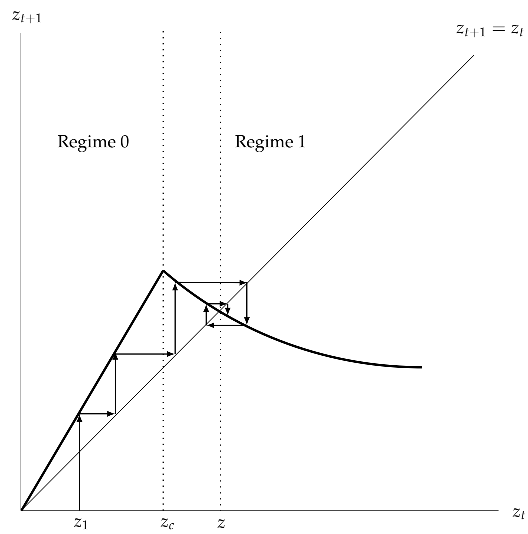

Figure 5.5: Case 3: Local and Global Dynamics of a Stable Steady-State Equilibrium for $-\nu(\sigma - 1) \in (-(1+\psi), -\psi)$.

$z_1 < z_2 < z_c < z_3$ and, for all $k = 1, 2, 3, \dots$,

$$\Phi^{3k}(z_1) = z_1 = z_c(1+g_L)^{\frac{1+2[\psi - \nu(\sigma-1)]}{\eta}},$$
$$\Phi^{3k}(z_2) = z_2 = z_c(1+g_L)^{\frac{2+\psi-\nu(\sigma-1)}{\eta}},$$
$$\Phi^{3k}(z_3) = z_3 = z_c(1+g_L)^{\frac{3}{\eta}}.$$

Hence, on such a cycle the economy remains for two periods in Regime 0 before it switches into Regime 1. Intuitively, the distance between z_1 and z_c is so large that it needs the labor force growth of two cohorts to support an equilibrium share price at which research firms can break even. The economic mechanisms of the two regime switches mimic those of the period-2 cycle and are elaborated further in the discussion of period-n cycles below. Here, we rather stress the following corollary that builds on Claim 2 of Proposition 5.3 and Theorem 1 of Li and Yorke (1975).

Corollary 5.1 (Chaos)

Suppose $-\nu(\sigma - 1) < -(2+\psi)$. Then,

1. *for every $k = 1, 2, \dots$ there is a periodic point in $\mathbb{R}_+$ having period k.*
2. *there is an uncountable set $S \subset \mathbb{R}_+$, which satisfies the following conditions:*
   (a) *For every $p, q \in S$, with $p \neq q$,*

$$\limsup_{n\to\infty} \mid \Phi^n(p) - \Phi^n(q) \mid \; > 0$$

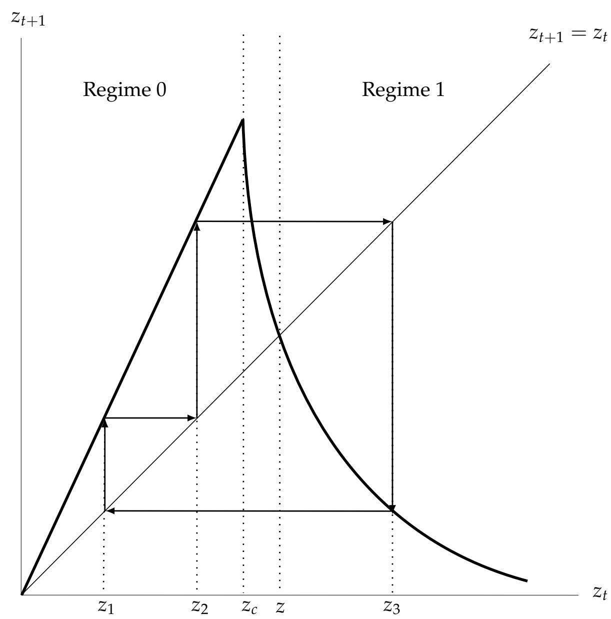

Figure 5.6: The Period-3 Cycle Arising under $-\nu(\sigma - 1) < -(2 + \psi)$.

and

$$\liminf_{n\to\infty} \mid \Phi^n(p) - \Phi^n(q) \mid = 0.$$

(b) For every $p \in S$ and periodic point $q \in \mathbb{R}_+$,

$$\limsup_{n\to\infty} \mid \Phi^n(p) - \Phi^n(q) \mid > 0.$$

Claim 1 of Corollary 5.1 says that, under the stated parameter constellation, starting values for cycles of any periodicity can be found. While this extends the findings of Proposition 5.3, the mechanics behind switching regimes or staying in one regime remain qualitatively the same as those identified in the context of period-2 and period-3 cycles.

Claim 2 of Corollary 5.1 reminds us of the fact that "period three implies chaos," i. e., small changes in initial conditions may have a substantial affect on the predicted evolution of the economy over time.

6 Discussion

In this section, we first generalize Proposition 5.3 beyond period-2 and period-3 cycles. Then, we study the empirical plausibility of cycles in Section 6.2. Finally, in Section 6.3 we derive the evolution of GDP in absolute and per-capita terms.

6.1 Period-n Cycles

According to Corollary 5.1 the dynamical system of Proposition 3.1 is consistent with cycles of any periodicity if $-\nu(\sigma-1) < -(2+\psi)$. Here, we derive a condition under which period-n cycles with a particular structure can be identified. Then, we discuss some economic implications of such cycles.

Proposition 6.1 (Existence of Period-n Cycles)

Let $n \in \{2, 3, \dots\}$ and suppose $-\nu(\sigma-1) < -(n-1+\psi)$. Then, there exists a period-n cycle with $n-1$ consecutive periods in Regime 0 followed by one period in Regime 1.

Moreover, suppose $z_1 < z_2 < \cdots < z_{n-1} < z_c < z_n$. Then for $t = 1, 2, \dots, n$,

$$z_t = z_c(1+g_L)^{\frac{t+(n-t)[\psi-\nu(\sigma-1)]}{\eta}}. \tag{6.1}$$

Hence, if the equilibrium elasticity of the individual supply of hours worked to changes in A_t is sufficiently small, then period-n cycles can be pinned down that have $n-1$ periods in Regime 0 followed by one period in Regime 1.

The following heuristic may be helpful to develop an intuition for this finding. Suppose all parameters except σ are fixed. Then, condition $-\nu(\sigma-1) < -(n-1+\psi)$ can be satisfied for a larger n if σ is larger. What is the effect of increasing σ on the qualitative properties of the dynamical system of Proposition 3.1? First, observe that changing σ does not affect z_c. Hence, the rightward boundary of the trapping region does not change either as $\Phi(z_c) = (1+g_L)z_c$. However, the leftward boundary declines as $\Phi^2(z_c) = (1+g_L)^{1+\psi-\nu(\sigma-1)}z_c$. Hence, a higher σ allows for a smaller initial value, z_1, and, therefore, for more periods in Regime 0.

Economically, this means for a given value of A_1 that L_1 can be smaller for a higher σ. Then, through population growth it takes more periods before a sufficiently large cohort is alive willing to purchase $A_1 + \Delta A_n$ shares at a price $v_n = a\alpha A_1^{\sigma-1-\psi}$.

To elicit some further economic implications of these cycles denote by $\overline{g}_{A,n}$ and $\overline{g}_{v,n}$ the respective average growth rate of A_t and v_t over a period-n cycle, i. e.,

$$\overline{g}_{A,n} \equiv [\Pi_{t=1}^{n}(1+g_{A,t})]^{\frac{1}{n}} - 1 \quad \text{and} \quad \overline{g}_{v,n} \equiv [\Pi_{t=1}^{n}(1+g_{v,t})]^{\frac{1}{n}} - 1.$$

Then, the following holds.

Corollary 6.1 (Average Growth Rates of A_t and v_t)

Consider some period-n cycle of Proposition 6.1. Then, the average growth rates of A_t and v_t over the period-n cycle coincide with their respective steady-state growth rate, i. e.,

$$\overline{g}_{A,n} = g_A \quad \textit{and} \quad \overline{g}_{v,n} = g_v.$$

Corollary 6.1 establishes two remarkable results. First, along a period-n cycle with no growth of A_t for $n-1$ periods, the average growth rate over the entire cycle, $\overline{g}_{A,n}$, is equal to the steady-state growth rate, g_A, of Proposition 4.2. Intuitively, the mechanism behind this finding is the following. On a period-n cycle the only positive growth rate is $g_{A,n}$ which is given by $g_{A,n} = z_n/z_c - 1$. Hence, it is greater the greater is z_n. From Proposition 6.1 we have $z_n = z_c(1+g_L)^{n/\eta}$ with $dz_n/dn > 0$. Hence, $g_{A,n}$ increases in n. More precisely, combining the expressions for $g_{A,n}$ and z_n gives

$$g_{A,n} = \frac{z_c(1+g_L)^{\frac{n}{\eta}}}{z_c} - 1 = (1+g_L)^{\frac{n}{\eta}} - 1.$$

Since $g_{A,t} = 0$ for $t < n$ the average growth rate of A_t along a period-n cycle is indeed equal to the steady-state growth rate as $\overline{g}_{A,n} = (1+g_L)^{1/\eta} - 1 = g_A$.

Second, the average growth rate of shares over a period-n cycle, $\overline{g}_{v,n}$, also coincides with its steady-state growth rate, g_v, of Proposition 4.2. This result obtains even though the growth rates of v_t for the first $n-2$ periods differ from those of period $n-1$ and n when the economy moves, respectively, from Regime 0 into Regime 1 and back.

Finally, we turn to the stability properties of period-n cycles.

Corollary 6.2 (Stability of Period-*n* Cycles)

The period-n cycles of Proposition 6.1 are unstable.

6.2 Empirical Plausibility of Cycles

The aim of this section is to investigate the empirical plausibility of cycles. To accomplish this, we first calibrate the steady state of our model. Then, we check whether the condition for instability of Proposition 5.1, i. e., $-\nu(\sigma-1) < -(1+\psi)$, and the condition for period-3 cycles of Proposition 5.3, i. e., $-\nu(\sigma-1) < -(2+\psi)$, are satisfied. The calibration exercise is conducted for given values of g_h, g_c, g_L, β, and γ_A where $g_c = g_{c^y} = g_{c^o}$. We approximate these values with real world data and use them to determine ν, σ, and ψ.

Boppart and Krusell (2020) estimate the average annual growth rate of hours worked per worker to equal -0.57%. Then, over 30 years we have $1+g_h = 0.9943^{30}$. As to g_c, we stipulate an average annual growth rate of per capita consumption of 1.9% so that $1+g_c = (1.019)^{30}$.[7] Then, Claim b) and c) of Proposition 4.2 imply

$$\nu = \frac{\ln 0.9943}{\ln 1.019}\left(\frac{\ln 0.9943}{\ln 1.019} - 1\right)^{-1} = 0.233.$$

Moreover, Claim c) of Proposition 4.2 and equation (9.1) deliver the expression for the steady-

[7] According to the Penn World Table 9.1, this is the average annual growth rate of per capita consumption in the United Kingdom over the period from 1950 to 2017.

$1+g_L$	ψ	$\psi - \nu(\sigma-1)$
1.0058^{30}	0.9342	-4.6871
1.006^{30}	0.7387	-4.8826
1.0065^{30}	0.2500	-5.3712

Table 6.1: Calibration Results.

state allocation of aggregate hours worked in research as

$$\gamma_A = \frac{\beta}{(1+\beta)(1-\nu)}\left[1-(1+g_c)^{\frac{-1}{(1-\nu)(\sigma-1)}}\right].$$

The annual discount factor is often estimated to be around 0.96 implying $\beta = 0.294$. For γ_A we use the fraction of researchers as a percentage of the labor force in the United Kingdom in 2017. According to the OECD Main Science and Technology Indicators database, this figure is 0.87%. Accordingly, given the values for g_c, ν, β and γ_A, the calibrated value of σ is 25.13.

Finally, Claim b) of Proposition 4.2 links ψ and g_L through

$$1+g_h = (1+g_L)^{\frac{-\nu(\sigma-1)}{1-\psi+\nu(\sigma-1)}}.$$

Table 6.1 presents the calibration results. Column 2 shows the calibrated values of ψ using the numerical values stated above for differing values of g_L. As required, for average annual population growth rates around 0.6% the calibrated values for ψ lie in the intreval $(0,1)$.[8] Column 3 reveals that $\psi - \nu(\sigma-1) < -2$ holds. Hence, the corresponding steady states are unstable, and the condition for period-3 cycles is satisfied.

6.3 The Evolution of GDP

How does *GDP* in absolute and per-capita terms evolve over time? To address this question, denote the economy's total value added at t by GDP_t. This value is equal to the sum of the added values in the production and the research sector, i. e., $GDP_t = A_t p_t x_t + v_t \Delta A_t$. Then, expressed in terms of the state variable z_t we have

$$GDP_t = \begin{cases} aA_t^{\sigma-\psi}\xi\left(\frac{z_t}{z_c}\right), & \text{if } z_t \le z_c, \\ \\ aA_t^{\sigma-\psi}\left[(\xi-(1-\alpha))\frac{z_t}{z_c}+(1-\alpha)\right], & \text{if } z_t \ge z_c, \end{cases} \tag{6.2}$$

[8] Observe that average annual population growth rates of roughly 0.6% is in line with the empirical evidence. For instance, for the U.K. and Sweden over the time span 1870 to 2010 this rate is equal to 0.59% and 0.58%, respectively (see UN Population Division (2019)).

where $\xi \equiv (1+\beta)(1-\nu)/\beta > 1$. Moreover, let gdp_t denote per-capita GDP at t so that

$$gdp_t \equiv \frac{1+g_L}{2+g_L} \times \frac{GDP_t}{L_t}.$$

To describe the evolution of GDP and gdp over two consecutive periods, four cases must be distinguished. The first and simplest is the one with the economy staying in Regime 0 for two consecutive periods, t and $t+1$. In these periods, the economy differs only with respect to its population size since $L_{t+1} = (1+g_L)L_t$. Hence, $GDP_{t+1} = (1+g_L)GDP_t$ and $gdp_{t+1} = gdp_t$.

The second case has the economy in Regime 1 for two consecutive periods. Then, in addition to population growth, we have $\Delta A_t > 0$ and $\Delta A_{t+1} >$ as the research sector is active in t and $t+1$. In light of (6.2), one finds that[9]

$$GDP_{t+1} > GDP_t \quad \text{since} \quad (1+g_{A,t})^{\sigma-\psi}\left[\frac{(\xi-(1-\alpha))\frac{z_{t+1}}{z_c}+(1-\alpha)}{(\xi-(1-\alpha))\frac{z_t}{z_c}+(1-\alpha)}\right] > 1.$$

For per-capita GDP the result is not unequivocal as

$$gdp_{t+1} \gtreqless gdp_t \quad \Leftrightarrow \quad (1+g_{A,t})^{\sigma-\psi}\left[\frac{(\xi-(1-\alpha))\frac{z_{t+1}}{z_c}+(1-\alpha)}{(\xi-(1-\alpha))\frac{z_t}{z_c}+(1-\alpha)}\right] \gtreqless 1+g_L.$$

Hence, if the economy stays for two consecutive periods in Regime 1 then its GDP will increase. This is due to the growth of A and to population growth. However, the growth of A is not sufficient to guarantee that per-capita GDP increases, too.

Third, consider the case with an economy starting in Regime 0 at t and switching into Regime 1 in period $t+1$. Then, with (6.2) it holds that

$$GDP_{t+1} \gtreqless GDP_t \quad \Leftrightarrow \quad \frac{z_{t+1}}{z_t} - \frac{1-\alpha}{\xi}\left[\frac{z_{t+1}}{z_t} - \frac{z_c}{z_t}\right] \gtreqless 1.$$

This inequality cannot be signed in general, not even if we impose more structure and assume that period t and $t+1$ are included in a period-n cycle of Proposition 6.1. Hence, in spite of population growth, the value added created in the research sector in period $t+1$ may be offset by a reduction in the value added of the production sector at $t+1$. However, in per-capita terms, one finds unequivocally that

$$gdp_{t+1} < gdp_t \quad \text{since} \quad -\frac{1-\alpha}{\xi}\left[\frac{z_{t+1}}{z_t} - \frac{z_c}{z_t}\right] < 0,$$

where we use the fact that $z_{t+1}/z_t > 1$ and $z_c/z_t < 1$. Hence, the per-capita value added in the research sector of period $t+1$ is strictly smaller than the reduction in the per-capita value added in the production sector.

[9] The argument is as follows. The stated inequality clearly holds if $z_{t+1} \geq z_t$ since $\sigma > \psi$ and $\xi > 1$. If $z_{t+1} < z_t$ then it holds since the lowest value the term in brackets can take on is z_{t+1}/z_t. With $1+g_{A,t} = z_t/z_c$ and $z_{t+1}/z_t = (1+g_L)(z_t/z_c)^{\psi-\nu(\sigma-1)-1}$ the inequality follows since $(\sigma-1)(1-\nu) > 0$.

Finally, consider the case where the economy is in Regime 1 at t and switches to Regime 0 in period $t+1$. Then, it holds that[10]

$$GDP_{t+1} > GDP_t \quad \text{since} \quad (1+g_L)\left(\frac{z_t}{z_c}\right)^{\sigma(1-\nu)+\nu} > \frac{z_t}{z_c}.$$

Again, population growth and growth of A drive this result. In fact, here the growth of A is sufficiently large so that per-capita GDP grows, too, i. e.,

$$gdp_{t+1} > gdp_t \quad \text{since} \quad \left(\frac{z_t}{z_c}\right)^{\sigma(1-\nu)+\nu} > \frac{z_t}{z_c}.$$

The following proposition summarizes the findings of this section.

Proposition 6.2 (Evolution of GDP and gdp)

Consider the evolution of GDP and gdp as defined above over two consecutive periods, t and $t+1$. Then, the following holds.

1. *If the economy starts in Regime* 0 *and*
   - *stays there, then $GDP_{t+1} > GDP_t$ and $gdp_{t+1} = gdp_t$;*
   - *switches into Regime* 1*, then $GDP_{t+1} \gtreqless GDP_t$ and $gdp_{t+1} < gdp_t$.*

2. *If the economy starts in Regime* 1 *and*
   - *stays there, then $GDP_{t+1} > GDP_t$ and $gdp_{t+1} \gtreqless gdp_t$;*
   - *switches into Regime* 0*, then $GDP_{t+1} > GDP_t$ and $gdp_{t+1} > gdp_t$.*

The intuition is then as follows. Population growth drives GDP growth but not growth of per-capita GDP. Opening a research sector in $t+1$ may reduce GDP and will reduce gdp. Moreover, an active research sector at t unequivocally induces a higher GDP at $t+1$. However, it necessarily boosts gdp at $t+1$ only if research sector is shut down at $t+1$.

7 Concluding Remarks

In the literature of endogenous growth cycles, the labor-leisure choice of households plays no role. This omission is hard to justify on empirical grounds. More importantly, we show that it leaves our understanding of endogenous growth cycles incomplete. The main point of this paper is that a negative and sufficiently strong equilibrium elasticity of the individual supply of hours worked to an expansion of the set of consumption-good varieties may be a cause of instability. Moreover, it opens up the possibility of growth cycles. These findings require a negative wage elasticity of the individual supply of hours worked that is consistent with the

[10] The argument is as follows. From (6.2) $GDP_{t+1} > GDP_t$ holds if and only if $A_{t+1}^{\sigma-\psi}\xi\,(z_{t+1}/z_c) > A_t^{\sigma-\psi}\left[(\xi-(1-\alpha))\,(z_t/z_c)+(1-\alpha)\right]$. Then, using Proposition 3.1, $\sigma(1-\nu)+\nu > 1$, and $z_t/z_c > 1$ delivers the result.

empirical evidence for many of today's industrialized countries.

Since the household sector of our model features two-period lived overlapping generations the evolutions that we describe and explain apply to periods with a length of roughly 30 years. An interesting question is then whether our analytical framework can be adapted to capture fluctuations in the short run that the empirical literature detects (see, e. g., Wälde and Woitek (2004)). One route to accomplish this would be to stipulate a constant savings rate and an exogenous labor supply with a negative and constant wage elasticity. As an alternative, one may attempt to derive such functions endogenously from a representative Ramsey household equipped with Boppart-Krusell preferences.

Finally, at the technical front, one may wonder whether our model harbors additional properties that are beyond the mathematical scope of the present paper. Contributions like Mukherji (2005), Gardini, Sushko, and Naimzada (2008), and Deng and Khan (2018) elicit such properties for the model of Matsuyama (1999). Studies of this kind applied to our model would certainly shed light on the role of the linear segment of our dynamical system for the global dynamics. At this stage, we leave these questions for future research.

8 Appendix A: Proofs

8.1 Proof of Proposition 2.1

Consolidating the two periodic budget constraints, the problem stated in (2.7) gives rise to the following Lagrangian

$$\mathscr{L} = \ln c_t^y + \ln\left(1 - \phi h_t (c_t^y)^{\frac{\nu}{1-\nu}}\right) + \beta \ln c_{t+1}^o + \lambda_t \left[w_t h_t - P_t c_t^y - \frac{P_{t+1} c_{t+1}^o}{R_{t+1}} \right].$$

Then, standard arguments following the algorithm developed in the proof of Proposition 1 in Irmen (2018) complete the proof. ■

8.2 Proof of Proposition 3.1

First, we derive algebraically the difference equation (3.8) in Section 8.2.1. Second, we prove the existence of a unique intertemporal general equilibrium in Section 8.2.2.

8.2.1 The Equilibrium Difference Equation

In Regime 0, inequality (2.17) is strict, and, accordingly, $H_{A,t} = \Delta A_t = 0$. Then, with s_t of Proposition 2.1 and (3.5), the asset market clearing condition (3.3) delivers

$$v_t = \frac{a \alpha L_t A_t^{(\sigma-1)(1-\nu)-1}}{z_c}. \tag{8.1}$$

Using the latter and (3.5) in inequality (2.17), gives

$$z_t \equiv \frac{L_t}{A_t^{\eta}} < z_c.$$

If this inequality holds, then the economy is in Regime 0, i. e., A_t is constant and L_t grows exogenously at rate g_L. Therefore,

$$z_{t+1} = (1+g_L)z_t \quad \text{if} \quad z_t < z_c. \tag{8.2}$$

In Regime 1, (2.17) holds as equality, and, accordingly, $H_{A,t} \geq 0$ and $\Delta A_t \geq 0$. Then, with s_t of Proposition 2.1, (3.5), and (3.9) the asset market equilibrium condition (3.3) becomes

$$\frac{A_t^{(1-\nu)(\sigma-1)} L_t}{z_c} = A_t^{\sigma-1-\psi} A_{t+1}.$$

Solving for A_{t+1}/A_t delivers

$$\frac{z_t}{z_c} = \frac{A_{t+1}}{A_t},$$

hence $z_t \geq z_c$. From the definition of z_t given in (3.6), we have

$$\frac{z_{t+1}}{z_t} = \frac{L_{t+1}A_{t+1}^{-\eta}}{L_t A_t^{-\eta}} \quad \text{or} \quad \frac{A_{t+1}}{A_t} = (1+g_L)^{\frac{1}{\eta}} \left(\frac{z_{t+1}}{z_t}\right)^{\frac{-1}{\eta}}.$$

Combining the latter two equations and solving for z_{t+1} delivers

$$z_{t+1} = (1+g_L)z_c \left(\frac{z_t}{z_c}\right)^{\psi-\nu(\sigma-1)} \quad \text{if} \quad z_t \geq z_c. \tag{8.3}$$

Hence, for $z_t < z_c$ the evolution of z_t is governed by (8.2), for $z_t \geq z_c$ the evolution of z_t is given by (8.3). It is straightforward to show from (8.2) that $\lim_{z_t \uparrow z_c} z_{t+1} = z_c$ and from (8.3) that $\lim_{z_t \downarrow z_c} z_{t+1} = z_c$. Hence, the piecewise defined difference equation stated in the proposition is continuous.

8.2.2 Existence and Uniqueness of the Intertemporal General Equilibrium

The proof has two steps. Step 1 shows that all elements of the equilibrium price system, $\{w_t, R_t, p_t, \pi_t, v_t\}_{t=1}^{\infty}$, and of the equilibrium allocation, $\left\{c_t^y, l_t, c_t^o, s_t, x_t^y, x_t^o, x_t, H_{x,t}, H_{A,t}, A_t\right\}_{t=1}^{\infty}$, can be expressed as a function of z_t. Step 2 proves that in equilibrium the shares are valued according to fundamentals.

Step 1

Given L_t, A_t can be pinned downed by z_t using (3.6), i. e.,

$$A_t = \left(\frac{L_t}{z_t}\right)^{\frac{1}{\eta}}. \tag{8.4}$$

The price of consumption goods and the wage rate are solely determined by A_t according to (3.4) and (3.5) respectively. Then using (8.4), we have

$$p_t = \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1}{\eta}} \quad \text{and} \quad w_t = \alpha \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1}{\eta}}. \tag{8.5}$$

The optimal plan of cohort t is given in Proposition 2.1, i. e., h_t, c_t^y, and s_t hinge on w_t. Then, using (8.5), we have

$$\begin{aligned} h_t &= \left(\frac{w_c}{\alpha}\right)^{\nu} \left(\frac{L_t}{z_t}\right)^{\frac{-\nu(\sigma-1)}{\eta}}, \quad s_t = \alpha\zeta \left(\frac{L_t}{z_t}\right)^{\frac{(1-\nu)(\sigma-1)}{\eta}}, \\ c_t^y &= \frac{1-\nu(1+\beta)}{(1+\beta)(1-\nu)} w_c^{\nu} \alpha^{1-\nu} \left(\frac{L_t}{z_t}\right)^{\frac{(1-\nu)(\sigma-1)}{\eta}}. \end{aligned} \tag{8.6}$$

Using (8.7) the aggregate supply of hours worked becomes

$$H_t = L_t h_t = \frac{a(1+\beta)(1-\nu)}{\beta} \frac{z_t}{z_c} \left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}}. \tag{8.7}$$

Using (8.3) in (2.15) gives aggregate hours worked in research as

$$H_{A,t} = \begin{cases} 0 & \text{if} \quad z_t \leq z_c, \\ a\left(\frac{z_t}{z_c} - 1\right)\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{8.8}$$

Accordingly, aggregate hours worked in the consumption-good sector obtain with (8.8) and (8.7) as

$$H_{x,t} = \begin{cases} \frac{a(1+\beta)(1-\nu)}{\beta} \frac{z_t}{z_c} \left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \leq z_c, \\ a\left[\frac{1-\nu(1+\beta)}{\beta} \frac{z_t}{z_c} + 1\right]\left(\frac{L_t}{z_t}\right)^{\frac{1-\psi}{\eta}} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{8.9}$$

The supply of each consumption goods is given by $H_{x,t}/A_t$. With (8.4) and (8.9) this gives

$$x_t = \begin{cases} \frac{a(1+\beta)(1-\nu)}{\beta} \frac{z_t}{z_c} \left(\frac{L_t}{z_t}\right)^{\frac{-\psi}{\eta}} & \text{if} \quad z_t \leq z_c, \\ a\left[\frac{1-\nu(1+\beta)}{\beta} \frac{z_t}{z_c} + 1\right]\left(\frac{L_t}{z_t}\right)^{\frac{-\psi}{\eta}} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{8.10}$$

Substituting (8.5) and (8.10) into (2.13) delivers the profit of each firm in the production sector as

$$\pi_t = \begin{cases} (1-\alpha)\frac{a(1+\beta)(1-\nu)}{\beta} \frac{z_t}{z_c} \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \leq z_c, \\ (1-\alpha)a\left[\frac{1-\nu(1+\beta)}{\beta} \frac{z_t}{z_c} + 1\right]\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{8.11}$$

The value of the shares of these firms in Regime 0 and Regime 1 result from the substitution of

(8.4) into (8.1) and (3.9), respectively, as

$$v_t = \begin{cases} a\alpha \frac{z_t}{z_c} \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ a\alpha \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-1-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{8.12}$$

Consumption of the old can be obtained using (8.5), (8.7) and (8.10) in (E5). This gives

$$c_t^o L_{t-1} = \left(\frac{L_t}{z_t}\right)^{\frac{\sigma}{\eta}} x_t - (1-\nu(1+\beta))\frac{a\alpha}{\beta}\frac{z_t}{z_c}\left(\frac{L_t}{z_t}\right)^{\frac{\sigma-\psi}{\eta}}.$$

With x_t of (8.10), we have

$$c_t^o = \begin{cases} \frac{(1+\beta)(1-\nu)-\alpha(1-\nu(1+\beta))}{\beta}\frac{z_t}{z_c} a \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-\psi}{\eta}} & \text{if} \quad z_t \le z_c, \\ \left[\frac{(1+\beta)(1-\nu)-\alpha(1-\nu(1+\beta))}{\beta}\frac{z_t}{z_c} - \left(\frac{z_t}{z_c}-1\right)\right] a \left(\frac{L_t}{z_t}\right)^{\frac{\sigma-\psi}{\eta}} & \text{if} \quad z_t \ge z_c. \end{cases} \tag{8.13}$$

One readily verifies that (2.4) in conjunction with (8.4) and, respectively, c_t^y of (8.7) and c_t^o of (8.13) delivers x_t^y and x_t^o as functions of z_t.

It remains to be shown that R_{t+1} can be expressed as a function of z_t. The following lemma accomplishes this.

Lemma 8.1 *(Perfect Foresight Interest Factor Along the Transition)*

Denote by R_{t+1} the perfect foresight interest factor at $t+1$. Then, the following holds.

1. *If $z_t < z_c/(1+g_L)$ then the economy is in Regime 0, stays there, and*

$$R_{t+1} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta} + \frac{1}{\alpha}(1+g_L).$$

2. *If $z_c/(1+g_L) \le z_t \le z_c$ then the economy is in Regime 0, transits to Regime 1, and*

$$R_{t+1} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta} + \frac{1}{\alpha}\frac{z_c}{z_t}.$$

3. *If $z_c \le z_t \le z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}}$ then the economy is in Regime 1, stays there, and*

$$R_{t+1} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta}\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\left(\frac{z_t}{z_c}\right)^{\sigma-1-\psi}.$$

4. *If $z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}} \le z_t$ then the economy is in Regime 1, transits to Regime 0, and*

$$R_{t+1} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta}\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}(1+g_L)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)}.$$

Proof of Lemma 8.1

The no-arbitrage condition (3.2) ensures that the rate of return on bonds and on shares are equal. Accordingly, the change in the value of shares, v_{t+1}/v_t, and the dividend yield, π_{t+1}/v_t, can be used to obtain an expression for the interest factor, R_{t+1}. Moreover, the economy will either stay in the same regime or move to another regime. Hence, there are four possible cases. For each of them, we first derive v_{t+1}/v_t and π_{t+1}/v_t and then use these expressions to derive the corresponding value of R_{t+1}.

Under (E5) and (E1), with (3.5), the profit of (2.13) can be expressed as

$$\begin{aligned} \pi_{x,t+1} = \quad & (1-\alpha) A_{t+1}^{(1-\nu)(\sigma-1)-1} \frac{w_c^\nu \alpha^{1-\nu}}{(1+\beta)(1-\nu)} L_t \\ \times \quad & \left[(1-\nu(1+\beta))(1+g_L) + \beta R_{t+1} \left(\frac{A_{t+1}}{A_t} \right)^{-(1-\nu)(\sigma-1)} \right]. \end{aligned} \tag{8.14}$$

Suppose the economy is in Regime 0 at time t. Then, $A_{t+1} = A_t$. With (8.11), (8.12) and the result above, the dividend yield is

$$\frac{\pi_{x,t+1}}{v_t} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\beta} + (1-\alpha) R_{t+1}. \tag{8.15}$$

Suppose the economy is in Regime 0 at time t and stays in Regime 0 at time $t+1$. Then, equation (8.12) implies that the change in the value of shares is

$$\frac{v_{t+1}}{v_t} = \frac{L_{t+1}}{L_t} = (1+g_L). \tag{8.16}$$

Suppose the economy is in Regime 0 at time t and in Regime 1 at time $t+1$. Then, equation (8.12) implies that the change of the value of the shares is

$$\frac{v_{t+1}}{v_t} = \frac{z_c}{z_t}. \tag{8.17}$$

For $z_t \leq z_c$, (8.16) and (8.17) together give

$$\frac{v_{t+1}}{v_t} = \begin{cases} 1+g_L & \text{if } z_t(1+g_L) < z_c, \\ \frac{z_c}{z_t} & \text{if } z_t(1+g_L) \geq z_c. \end{cases} \tag{8.18}$$

Then, substituting (8.15) and (8.18) into (3.2) and solving for R_{t+1} delivers

$$R_{t+1} = \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta} + \begin{cases} \frac{1+g_L}{\alpha} & \text{if } z_t(1+g_L) < z_c, \\ \frac{1}{\alpha}\frac{z_c}{z_t} & \text{if } z_t(1+g_L) \geq z_c. \end{cases} \tag{8.19}$$

Suppose the economy is in Regime 1 at time t. Then, dividing (8.14) by (3.9) gives the dividend yield as

$$\frac{\pi_{x,t+1}}{v_t} = \frac{1-\alpha}{\beta}\frac{1}{z_c}L_t\frac{A_t^{\psi-(\sigma-1)}}{A_{t+1}}\left[(1-\nu(1+\beta))A_{t+1}^{(1-\nu)(\sigma-1)}(1+g_L)+\beta R_{t+1}A_t^{(1-\nu)(\sigma-1)}\right].$$

Note that $A_{t+1}/A_t = z_t/z_c$. Hence,

$$\frac{\pi_{x,t+1}}{v_t} = \frac{(1-\alpha)(1-\nu(1+\beta))}{\beta}(1+g_L)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} + (1-\alpha)R_{t+1}. \tag{8.20}$$

Suppose the economy is in Regime 1 at time t and stays there at time $t+1$. Then, (3.9) gives the change in the value of shares as

$$\frac{v_{t+1}}{v_t} = \left(\frac{A_{t+1}}{A_t}\right)^{\sigma-1-\psi} = \left(\frac{z_t}{z_c}\right)^{\sigma-1-\psi}. \tag{8.21}$$

Suppose the economy is in Regime 1 at time t and transits to Regime 0 at time $t+1$. Then, (3.9) and (8.12) give the change in the value of the shares

$$\frac{v_{t+1}}{v_t} = (1+g_L)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)}. \tag{8.22}$$

For $z_t \geq z_c$, (8.21) and (8.22) together yield

$$\frac{v_{t+1}}{v_t} = \begin{cases} (1+g_L)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} & \text{if } z_t \geq z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}}, \\ \left(\frac{z_t}{z_c}\right)^{\sigma-1-\psi} & \text{if } z_t \leq z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}}. \end{cases} \tag{8.23}$$

Then, substituting (8.20) and (8.23) into (3.2) and solving for R_{t+1} delivers

$$\begin{aligned} R_{t+1} &= \frac{(1-\alpha)(1-\nu(1+\beta))}{\alpha\beta}(1+g_L)\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} \\ &+ \begin{cases} \frac{1+g_L}{\alpha}\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} & \text{if } z_t \geq z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}}, \\ \frac{1}{\alpha}\left(\frac{z_t}{z_c}\right)^{\sigma-1-\psi} & \text{if } z_t \leq z_c(1+g_L)^{\frac{1}{\psi-\nu(\sigma-1)}}. \end{cases} \end{aligned} \tag{8.24}$$

■

Finally, we use Lemma 8.1 to prove that in equilibrium condition (3.1) holds so that shares are indeed valued according to fundamentals. Lemma 8.2 accomplishes this.

Lemma 8.2 *(Equilibrium Share Valuation According to Fundamentals)*

The intertemporal general equilibrium satisfies condition (3.1).

Proof of Lemma 8.2

The no arbitrage condition (3.2) implies (3.1) if

$$\lim_{j\to\infty}\frac{v_{t+j}}{\Pi_{i=1}^{j}R_{t+j}} = 0. \tag{8.25}$$

Since

$$v_{t+j} = v_t \left(\frac{v_{t+1}}{v_t}\right)\left(\frac{v_{t+2}}{v_{t+1}}\right)\dots\left(\frac{v_{t+j}}{v_{t+j-1}}\right) = v_t \Pi_{i=1}^{j} \frac{v_{t+i}}{v_{t+i-1}},$$

condition (8.25) can be written as

$$v_t \lim_{j\to\infty} \Pi_{i=1}^{j} \left(\frac{\frac{v_{t+i}}{v_{t+i-1}}}{R_{t+j}}\right) = 0. \tag{8.26}$$

Equations (8.15) and (8.20) deliver the equilibrium ratio of $\pi_{x,t+1}/v_t$ for $z_t \leq z_c$ and $z_t \geq z_c$, respectively. Using this infomation in the no-arbitrage condition (3.2) delivers the equilibrium value of R_{t+1} as a function of v_{t+1}/v_t as

$$R_{t+1} = \begin{cases} \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta} + \frac{1}{\alpha}\frac{v_{t+1}}{v_t} & \text{if} \quad z_t \leq z_c, \\ \frac{(1-\alpha)(1-\nu(1+\beta))(1+g_L)}{\alpha\beta}\left(\frac{z_t}{z_c}\right)^{(1-\nu)(\sigma-1)} + \frac{1}{\alpha}\frac{v_{t+1}}{v_t} & \text{if} \quad z_t \geq z_c. \end{cases} \tag{8.27}$$

In both cases, the first summands are positive. Moreover, $\alpha \in (0,1)$. Therefore, $R_{t+1} > v_{t+1}/v_t$ for all t. As a consequence, the left-hand side of (8.26) has the product of infinitely many factors each being strictly smaller than unity. Hence, the limit of these factors vanishes and equation (8.25) holds. ■

8.3 Proof of Proposition 4.1

A steady-state of the equilibrium difference equation $z_{t+1} = \Phi(z_t)$ is a value $z > 0$ such that $z = \Phi(z)$. Since $g_L > 0$ there is no such solution for $z_t \leq z_c$. For $z_t > z_c$, the only solution delivers the expression for z stated in the proposition. Since $g_L > 0$, $z > z_c$. ■

8.4 Proof of Proposition 4.2

The steady-state growth rate of A_t is derived in the main text. The growth rate of w_t follows from (3.5). The growth rates of h_t, c_t^y, and c_t^o follow with Proposition 2.1. Since $H_t = L_t h_t$, the growth rate of H_t equals to $(1+g_L)(1+g_h)$. Finally, g_v follows with (3.9). ■

8.5 Proof of Proposition 5.1

According to Proposition 1.9 of Galor (2007), z of the dynamical system (3.8) is locally stable iff

$$\left| \left.\frac{dz_{t+1}}{dz_t}\right|_{z_t=z} \right| = \left| \psi - \nu(\sigma-1) \right| < 1.$$

Then, the qualitative findings concerning the local stability are immediate. ■

8.6 Proof of Proposition 5.2

We first state and prove two lemmata before using them in the proof of the proposition.

Lemma 8.3

Suppose $z_t \geq z_c$. As long as the economy stays in Regime 1 *for s periods, the dynamics of z_t is given by*

$$\frac{z_{t+s}}{z} = \left(\frac{z_t}{z}\right)^{[\psi-\nu(\sigma-1)]^s}. \tag{8.28}$$

Moreover, if $\mid \psi - \nu(\sigma-1) \mid < 1$, then

$$\lim_{s\to\infty} \frac{z_{t+s}}{z} = 1. \tag{8.29}$$

Proof of Lemma 8.3

From Proposition 4.1, the population growth factor satisfies $1 + g_L = (z/z_c)^\eta$. If $z_t > z_c$, then with (3.8) we have

$$\frac{z_{t+1}}{z} \quad = \quad \left(\frac{z_t}{z}\right)^{\psi-\nu(\sigma-1)}. \tag{8.30}$$

If $z_{t+s} \geq z_c$ for all $s = 0, 1, 2, \ldots$, then by successive iteration we obtain (8.28). If $\mid \psi - \nu(\sigma - 1) \mid < 1$, then $\lim_{s\to\infty}[\psi - \nu(\sigma-1)]^s = 0$, i. e., (8.29) holds. ■

Lemma 8.4

Suppose $\psi - \nu(\sigma-1) < 0$. Then, for any $z_1 \in \mathbb{R}_{++}$, there exist $\tau \geq 1$, such that $z_\tau \in [z_c, \Phi(z_c)]$.

Proof of Lemma 8.4

The piecewise defined difference equation (3.8) is strictly increasing on the subdomain $z_t \leq z_c$ and strictly decreasing on the subdomain $z_t \geq z_c$. Therefore, its maximum is $\Phi(z_c) = (1 + g_L)z_c$. If $z_1 \leq z_c$, then $z_\tau \in [z_c, \Phi(z_c)]$ where $\tau = 1 + \lceil \ln(z_c/z_1)/\ln(1+g_L) \rceil$. If $z_1 > \Phi(z_c)$, then $z_2 = \Phi(z_c)(z_1/z_c)^{\psi-\nu(\sigma-1)} < \Phi(z_c)$. Then, either $z_2 < z_c$ or $z_2 \in [z_c, \Phi(z_c)]$. In the first case, $z_\tau \in [z_c, \Phi(z_c)]$ where $\tau = 2 + \lceil \ln(z_c/z_2)/\ln(1+g_L) \rceil$. ■

Proof of Case 1: Suppose $-\nu(\sigma-1) \in (-\psi, 0)$. Then, the difference equation (3.8) is increasing. If $z_1 < z_c$, then the economy transits to Regime 1 in finite time and stays there. If $z_c < z_1 < z$, then the economy stays in Regime 1. With Lemma 8.3, the economy converges to z.

Proof of Case 2: Suppose $-\nu(\sigma-1) = -\psi$. Then, for $z_t \geq z_c$ the difference equation (3.8) is a constant, i. e., there is no transitional dynamics in Regime 1. If $z_1 < z_c$, then the economy leaves Regime 0 in finite time and jumps immediately to z. If $z_1 \geq z_c$, then the economy jumps immediately to z in $t = 2$.

Proof of Case 3: Suppose $-\nu(\sigma-1) \in ((-(1+\psi), -\psi)$. Then, $z_c < \Phi^2(z_c) < z$ where $\Phi^2(z_c) = (1+g_L)^{1+\psi-\nu(\sigma-1)}z_c$. With Lemma 8.4, we have that $z_{\tau+1} \in [\Phi^2(z_c), \Phi(z_c)] \subset [z_c, \Phi(z_c)]$. Hence, for all $t \geq \tau$, $z_t \in [z_c, \Phi(z_c)]$. Finally, Lemma 8.3 implies $\lim_{t\to\infty} z_t = z$.

Proof of Case 4: Suppose $-\nu(\sigma-1) = -(1+\psi)$. At τ defined in Lemma 8.3 the economy arrives in Regime 1. Then, it embarks immediately on a period-2 cycle inside Regime 1. More specifically, $z_{\tau+2k} = z_\tau$ and $z_{\tau+2k-1} = (1+g_L)(z_\tau/z_c)^{-1}$ for all $k = 1, 2, 3, \ldots$. ■

8.7 Proof of Proposition 5.3

First, we prove that $-\nu(\sigma-1) < -(1+\psi)$ implies for any initial value $z_1 \in \mathbb{R}_{++}$ that the economy enters the indicated trapping region in finite time. The argument is a follows.

Some straightforward algebra reveals that $\Phi^2(z_c) < z_c < \Phi(z_c)$. Since $\Phi'(z_t) > 0$ for all $z_t \in [\Phi^2(z_c), z_c]$ and $\Phi'(z_t) < 0$ for all $z_t \in [z_c, \Phi(z_c)]$, $\Phi(z_c) \geq \Phi(z_t)$ and $\Phi(z_t) \geq \Phi^2(z_c)$ for all $z_t \in [\Phi^2(z_c), \Phi(z_c)]$. Therefore, $[\Phi^2(z_c), \Phi(z_c)]$ is a trapping region in the sense that $z_t \in [\Phi^2(z_c), \Phi(z_c)]$ implies $z_{t+\tau} \in [\Phi^2(z_c), \Phi(z_c)]$ for all $\tau > 0$.

Second, we prove that inside the trapping region the evolution fluctuates between Regime 0 and 1. The argument is as follows.

If $z_t \in (\Phi^{-1}(z_c), \Phi(z_c)]$, then $z_{t+1} < z_c$. Moreover, since $\psi - \nu(\sigma-1) < (-1)$, if $z_t \in [z_c, \Phi^{-1}(z_c)]$ and $z_t \neq z$, then $z_{t+\tau} \in (\Phi^{-1}(z_c), \Phi(z_c)]$ where τ is finite. Therefore, the economy will move from Regime 1 to Regime 0 after a finite number of periods. Together with Lemma 8.4, it follows that inside the trapping region, $[\Phi^2(z_c), \Phi(z_c)]$, the evolution of z_t fluctuates between Regime 0 and 1.

Third, we prove Claim 1. In order to show the existence of a period-2 cycle, we need to show that $\Phi^2(z_1) - z_1 = 0$ has a solution. Suppose $z_1 < z_c < z_2$. Then $z_2 = \Phi(z_1) = (1+g_L)z_1$ and $z_3 = \Phi^2(z_1) = (1+g_L)z_c((1+g_L)z_1/z_c)^{\psi-\nu(\sigma-1)}$. One readily verifies that $z_1 = (1+g_L)^{(1+\psi-\nu(\sigma-1))/\eta}z_c$ solves $\Phi^2(z_1) - z_1 = 0$. Furthermore, $z_2 = (1+g_L)^{2/\eta}z_c$. Finally, the condition $\psi - \nu(\sigma-1) < (-1)$ ensures that $z_1 < z_c < z_2$.

Finally, we prove Claim 2. In order to establish the existence of a period-3 cycle, we need to show that $\Phi^3(z_1) - z_1 = 0$ has a solution. Suppose $z_1 < z_2 < z_c < z_3$. Then $z_2 = \Phi(z_1) = (1+g_L)z_1$, $z_3 = \Phi^2(z_1) = (1+g_L)^2 z_1$ and $z_4 = \Phi^3(z_1) = (1+g_L)z_c((1+g_L)^2 z_1/z_c)^{\psi-\nu(\sigma-1)}$. One readily verifies that $z_1 = (1+g_L)^{(1+2(\psi-\nu(\sigma-1)))/\eta}z_c$ solves $\Phi^3(z_1) - z_1 = 0$. Furthermore, $z_2 = (1+g_L)^{(2+\psi-\nu(\sigma-1))/\eta}z_c$ and $z_3 = (1+g_L)^{3/\eta}z_c$. Finally, the condition $\psi - \nu(\sigma-1) < (-2)$ ensures that $z_1 < z_2 < z_c < z_3$. ■

8.8 Proof of Corollary 5.1

Follows directly from Theorem 1 in Li and Yorke (1975) in conjunction with Claim 2 of Proposition 5.3. ■

8.9 Proof of Proposition 6.1

To establish the existence of a period-n cycle, we need to show that $\Phi^n(z_1) - z_1 = 0$ has a solution. Suppose $z_1 < z_2 < \cdots < z_{n-1} < z_c < z_n$. Then, the dynamical system in Proposition 3.1

gives, for all $t = 2, 3, \dots, n$,

$$\Phi^{t-1}(z_1) = (1+g_L)^{t-1} z_1 \quad \text{and} \quad \Phi^n(z_1) = (1+g_L) z_c \left[\frac{(1+g_L)^{n-1} z_1}{z_c} \right]^{\psi - \nu(\sigma-1)}.$$

Then, solving $\Phi^n(z_1) - z_1 = 0$ for z_1 gives

$$z_1 = z_c (1+g_L)^{\frac{1+(n-1)[\psi-\nu(\sigma-1)]}{\eta}}.$$

Moreover, $n-1$ iterative substitutions of the latter equation into the equilibrium difference equation for Regime 0 gives for $t = 1, 2, \dots, n$, the expression for z_t of (6.1). To ensure that $z_1 < z_2 < \cdots < z_{n-1} < z_c$, it is sufficient to show $z_{n-1} < z_c$. Equation (6.1) implies that

$$z_{n-1} = z_c (1+g_L)^{\frac{n-1+\psi-\nu(\sigma-1)}{\eta}}.$$

Simple manipulations show that the condition $-\nu(\sigma-1) < -(n-1+\psi)$ ensures $z_{n-1} < z_c$.■

8.10 Proof of Corollary 6.1

The period-n cycle begins without an active research sector for $n-1$ periods, i. e., $g_{A,t} = 0$ for all $t = 1, 2, \dots, n-1$. Then, in period n, the growth rate satisfies $1 + g_{A,n} = z_n / z_c$. With Proposition 6.1, we obtain $1 + g_{A,n} = (1+g_L)^{\frac{n}{\eta}}$. Therefore, the average growth factor of A_t over a period-n cycle satisfies $(1 + \overline{g}_{A,n}) = (1+g_L)^{\frac{1}{\eta}}$. Accordingly, $\overline{g}_{A,n}$ is equal to the steady-state growth rate, g_A, given in Proposition 4.2.

On readily verifies that

$$1 + \overline{g}_{v,n} = \left[\left(\frac{v_2}{v_1} \right) \cdot \left(\frac{v_3}{v_2} \right) \cdot \dots \cdot \left(\frac{v_{n-1}}{v_{n-2}} \right) \cdot \left(\frac{v_n}{v_{n-1}} \right) \cdot \left(\frac{v_n}{v_{n-1}} \right) \right]^{\frac{1}{n}},$$

where the first $n-2$ factors are equal to $(1+g_L)$, $v_n / v_{n-1} = (1+g_L)^{1-n/\eta}$, and $v_{n+1}/v_n = (1+g_L)^{1+n(1-\nu)(\sigma-1)/\eta}$. Then, the result is immediate. ■

8.11 Proof of Corollary 6.2

Following, e. g., Elaydi (2005), p. 39, the period-n cycles of Proposition 6.1 are asymptotically stable if $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| < 1$, and unstable if $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| > 1$. ■
For all $t = 1, 2, \dots, n-1$, $\Phi'(z_t) = (1+g_L)$ and $\Phi'(z_n) = [\psi - \nu(\sigma-1)](1+g_L)^{-(n-1)}$. Therefore, $|\Phi'(z_1)\Phi'(z_2), \dots, \Phi'(z_n)| = |\psi - \nu(\sigma-1)|$. Since $-\nu(\sigma-1) < -(n-1+\psi)$ and $n \geq 2$, $|\psi - \nu(\sigma-1)| > 1$. Hence, the period-$n$ cycles of Proposition 6.1 are unstable. ■

8.12 Proof of Proposition 6.2

Given in the main text. ■

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