Travaux mathématiques, Volume 23 (2013), 173–214,

Model completeness of valued PAC fields

by Wulf-Dieter Geyer and Moshe Jarden[1]

Abstract

We present a theorem of Kollár on the density property of valued PAC fields and a theorem of Abraham Robinson on the model completeness of the theory of algebraically closed non-trivial valued fields. Then we prove that the theory T of non-trivial valued fields in an appropriate first order language has a model completion $\tilde{T}$. The models of $\tilde{T}$ are non-trivial valued fields (K, v) that are ω-imperfect, ω-free, and PAC.

MSC (2010): 12E30.

[1]Research supported by the Minkowski Center for Geometry at Tel Aviv University, established by the Minerva Foundation.

Contents

Introduction

A field K is said to be **PAC** (**pseudo algebraically closed**) if every absolutely irreducible variety V defined over K (i.e. a geometrically integral K-scheme) has a K-rational point. Here and throughout the paper we use $\tilde{K}$ to denote a fixed algebraic closure of K.

The notion of PAC fields has been introduced in [Ax68] (although not by this name) in connection with the decidability of the elementary theory of finite fields. Each countable Hilbertian field K has an abundance of separable algebraic extensions of K which are PAC. Indeed, for each positive integer e and for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, the fixed field $K_s(\boldsymbol{\sigma})$ is PAC [FrJ08, Thm. 18.6.1]. Here K_s is the separable algebraic closure of K, $\mathrm{Gal}(K) = \mathrm{Gal}(K_s/K)$ is the absolute Galois group of K, "almost all" is meant in the sense of the Haar measure of $\mathrm{Gal}(K)^e$ with respect to its Krull topology, and $K_s(\boldsymbol{\sigma})$ is the fixed field in K_s of the coordinates of $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_e)$.

Chapter 11 of [FrJ08] gives an extensive treatment of PAC fields. In particular, it points out that if K is PAC, then $V(K)$ is Zariski dense in $V(\tilde{K})$ for each absolutely irreducible variety V defined over K [FrJ08, p. 192, Prop. 11.1.1] and asks whether $V(K)$ is even v-dense in $V(\tilde{K})$ for each valuation v of $\tilde{K}$ [FrJ05, Problem 11.5.4]. If this happens, we say that K has the **density property**.

The latter problem goes back to [GeJ75, Problem 1], where the following theorem is proved: Let K be a countable Hilbertian field and e a positive integer. Then for every valuation v of $\tilde{K}$, for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$, and for every absolutely irreducible variety V defined over K, the set $V(K_s(\boldsymbol{\sigma}))$ is v-dense in $V(\tilde{K})$ [GeJ75, Thm. 6.2]. Note that the order of the quantifiers "for every valuation v" and "for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$" can not be exchanged without a substantial argument, because $\tilde{K}$ has in general uncountably many valuations. That argument is supplied in [FrJ76], where the "stability of PAC fields" is proved [FrJ76, Thm. 3.4]. As a result, it is proved that $K_s(\boldsymbol{\sigma})$ has the density property for almost all $\boldsymbol{\sigma} \in \mathrm{Gal}(K)^e$.

For a general PAC field K and an arbitrary valuation v of $\tilde{K}$, Prestel proved that K is v-dense in $\tilde{K}$ [FrJ08, p. 204, Prop. 11.5.3]. The proof is based on the observation that if $f \in K[X]$ is a nonconstant separable polynomial and $c \in K^\times$, then $f(X_1)f(X_2) - c^2$ is an absolutely irreducible polynomial. Thus, there exist $x_1, x_2 \in K$ with $f(x_1)f(x_2) = c^2$, so $v(f(x_1)) \geq v(c)$ or $v(f(x_2)) \geq v(c)$.

János Kollár refined Prestel's trick and proved that every PAC field has the density property [Kol07, Thm. 2]. Using the stability property of PAC fields with an extra condition (which Kollár proves), he reduces the theorem to proving that if K is a PAC field and $f \in K[X, Y]$ is an absolutely irreducible polynomial which is Galois in Y, then one can approximate each zero $(0, \tilde{b}) \in \tilde{K}^2$ of f by a zero $(a, b) \in K^2$. The main point is to find $c \in K^\times$ with $v(c)$ large such that the algebraic set defined by $f(X_1, Y_1) = 0$, $f(X_2, Y_2) = 0$, and $X_1X_2 = c^2$ is an absolutely irreducible variety defined over K. This is done by using a lemma of Enriques-Severi-Zariski followed by smoothness arguments.

The first goal of this note is to give a self contained presentation of Kollár's proof. Then we present Abraham Robinson's proof that the theory of non-trivial algebraically closed valued field is model complete. Finally, we apply the density property of PAC fields and Robinson's result to prove that the elementary theory of PAC valued fields, in an appropriate first order language, is itself model complete. Moreover, it admits elimination of quantifiers.

Acknowledgements: The authors are indebted to Dan Haran for a useful discussion on the subject and to Aharon Razon for a critical reading of the manuscript.

1 Convention

We follow [Wei62, Section I.1] and choose a **universal extension** $\mho$ of K that contains the algebraic closure $\tilde{K}$ of K. Thus, $\mho$ is an algebraically closed field containing $\tilde{K}$ with $\text{trans.deg}(\mho/K) = \infty$. For each non-negative integer n and every field $K \subseteq L \subseteq \mho$ we follow the classical algebraic geometry and consider $\mathbb{A}^n(L)$ as the set of all **points** $\mathbf{a} = (a_1, \ldots, a_n)$ with coordinates $a_1, \ldots, a_n \in L$. Likewise we consider $\mathbb{P}^n(L)$ as the set of all points $\mathbf{a} = (a_0{:}a_1{:}\cdots{:}a_n)$ which are, as usual, equivalence classes of $(n+1)$-tuples $(a_0, a_1, \ldots, a_n)$ of elements of $\mho$ modulo multiplication by a non-zero element of $\mho$ such that there exists $0 \le i \le n$ with $a_i \ne 0$ and $\frac{a_j}{a_i} \in L$ for $j = 0, \ldots, n$. In this case $K\big(\frac{a_0}{a_i}, \frac{a_1}{a_i}, \ldots, \frac{a_n}{a_i}\big)$ is the **residue field** of $\mathbf{a}$. The elements $a_0, a_1, \ldots, a_n$ are **homogeneous coordinates** of $\mathbf{a}$.

Next we consider the **affine n-dimensional space**

$$\mathbb{A}^n_K = \text{Spec}(K[X_1, \ldots, X_n])$$

over K and the **projective n-dimensional space**

$$\mathbb{P}^n_K = \text{Proj}(K[X_0, \ldots, X_n])$$

over K. We say that V is an **absolutely irreducible variety in $\mathbb{A}^n_K$ (resp. $\mathbb{P}^n_K$) defined over** K, if V is a Zariski-closed subscheme of $\mathbb{A}^n_K$ (resp. $\mathbb{P}^n_K$) such that the scheme $V \times_K \mho$ obtained from V by extending the field of scalars from K to $\mho$ is integral. Equivalently, $V \times_K \tilde{K}$ is an integral scheme.

If $V = \text{Spec}(K[X_1, \ldots, X_n]/I)$ (resp. $V = \text{Proj}(K[X_0, \ldots, X_n]/I)$) is an absolutely irreducible variety in $\mathbb{A}^n_K$ (resp. $\mathbb{P}^n_K$) defined over K, then for each field $K \subseteq L \subseteq \mho$, we consider $V(L)$ as the set of all zeros $\mathbf{a} \in \mathbb{A}^n(L)$ (resp. $\mathbf{a} \in \mathbb{P}^n(L)$) of I. Note that there is a natural bijective correspondence between $V(L)$ and $\text{Mor}_K(\text{Spec}(L), V)$. In particular, we may identify each point $\mathbf{a} \in V(K)$ with a unique scheme theoretic K-rational point of V (i.e. a point of V whose residue field is K).

A point $\mathbf{x} \in V(\mho)$ is a **generic point** of V over K if the field $F = K(\mathbf{x})$ is regular over K and trans.deg$(F/K) = \dim(V)$. In this case, F is **the function field of** V over K. Note that F is unique up to a K-isomorphism. Moreover, it is always possible to choose the homogeneous coordinates of $\mathbf{x}$ in F. Indeed, if in the projective case $\mathbf{x} = (x_0 : \cdots : x_n)$ and $x_i \neq 0$, then $F = K(x_i^{-1}x_0, \ldots, x_i^{-1}x_n)$.

With this notation, a reduced closed subscheme V of $\mathbb{A}^n_K$ is an absolutely irreducible variety in $\mathbb{A}^n_K$ defined over K if and only if the scheme $V \times_K \mho$ (alternatively $V \times_K \tilde{K}$) is irreducible and the ideal of $\mho[X_1, \ldots, X_n]$ (alternatively $\tilde{K}[X_1, \ldots, X_n]$) of all polynomials that vanish on $V(\mho)$ (alternatively $V(\tilde{K})$) is generated by polynomials with coefficients in K.

Note that if $p = \mathrm{char}(K) > 0$, $a \in K$ has no pth root in K, and we set $V = \mathrm{Spec}(K[X]/K[X](X^p - a))$, then $V(\tilde{K})$ consists of one point, namely $\sqrt[p]{a}$. In particular $V_{\tilde{K}}$ is irreducible. However, the polynomial $X - \sqrt[p]{a}$ vanishes on $V(\tilde{K})$ but does not belong to $\tilde{K}[X](X^p - a)$, so V is not an absolutely irreducible variety in $\mathbb{A}^1_K$ defined over K.

A reduced closed subscheme V of $\mathbb{P}^n_K$ is **an absolutely irreducible variety in $\mathbb{P}^n_K$ which is defined over** K if and only if each of the standard affine open subsets of V is an absolutely irreducible variety in $\mathbb{A}^n_K$ defined over K.

By [FrJ08, p. 175, Cor. 10.2.2(a)] or [GoW10, p. 136, Prop. 5.51], a reduced irreducible closed subscheme V of $\mathbb{A}^n_K$ or $\mathbb{P}^n_K$ is an absolutely irreducible variety which is defined over K if and only if the function field F of V is a regular extension of K, i.e. K is algebraically closed in K and F/K is separable.

Thus, our definition of an "absolutely irreducible variety in $\mathbb{A}^n_K$ (resp. $\mathbb{P}^n_K$) defined over K" is equivalent to the definition of the same notion in the classical language of algebraic geometry (see also [FrJ08, Sec. 10.2]).

For each absolutely irreducible variety V defined over K we write $\tilde{V}$ for the variety $V \times_K \tilde{K}$ obtained from V by extending the base field from K to $\tilde{K}$. We also say that $\tilde{V}$ is **defined over** K. If $\varphi\colon V \to W$ is a morphism of absolutely irreducible varieties defined over K and L is a field extension of K in $\mho$, we abuse notation and write $\varphi\colon V(L) \to W(L)$ also for the set theoretic map induced from the morphism φ. Finally we write $\tilde{\varphi}\colon \tilde{V} \to \tilde{W}$ for the morphism obtained from φ by extending the base field from K to $\tilde{K}$.

2 Conics in $\mathbb{P}^1_K \times \mathbb{P}^1_K$

We consider the direct product $\mathbb{P}^1_K \times \mathbb{P}^1_K$ of two copies of the projective line over an arbitrary field K, define a pencil of conics in that space, and then blow it up at two points.

2.1 The Conics $H_{\mathbf{a}}$

We consider two copies of $\mathbb{P}^1_K$, one with homogeneous coordinates $(X_0{:}X_1)$ and the other one with homogeneous coordinates $(Y_0{:}Y_1)$. For each $\mathbf{a} = (a_0{:}a_1) \in \mathbb{P}^1(\tilde{K})$ we associate the conic $\tilde{H}_{\mathbf{a}}$ in $\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}$ defined by the bi-homogeneous equation

$$a_1 X_0 Y_0 = a_0 X_1 Y_1. \tag{2.1}$$

If $\mathbf{a} \in \mathbb{P}^1(K)$, we denote by $H_{\mathbf{a}}$ the conic in $\mathbb{P}^1_K \times \mathbb{P}^1_K$ defined by (2.1). In this case we also have $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$ and $\tilde{H}_{\mathbf{a}} = H_{\mathbf{a}} \times_K \tilde{K}$, in accordance with our convention.

The scheme $\mathbb{P}^1 \times \mathbb{P}^1$ is covered by the open subsets

$$U^{ij} = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid x_i y_j \neq 0\}, \qquad i, j = 0, 1$$

which are isomorphic to the affine plane $\mathbb{A}^2$ with the affine coordinates $X_{i'} = \frac{X_{i'}}{X_i}$ and $Y_{j'} = \frac{Y_{j'}}{Y_j}$, where $\{i, i'\} = \{j, j'\} = \{0, 1\}$. Therefore, the conic $\tilde{H}_{\mathbf{a}}$ is covered by the open affine subsets $\tilde{H}^{ij}_{\mathbf{a}} = U^{ij} \cap \tilde{H}_{\mathbf{a}}$ for $i, j = 0, 1$. As subsets of $\mathbb{A}^2$ the latter subsets are defined by the following equations:

$$\begin{array}{llll} \tilde{H}^{00}_{\mathbf{a}}: & a_1 = a_0 X_1 Y_1, & \tilde{H}^{01}_{\mathbf{a}}: & a_1 Y_0 = a_0 X_1 \\ \tilde{H}^{10}_{\mathbf{a}}: & a_1 X_0 = a_0 Y_1, & \tilde{H}^{11}_{\mathbf{a}}: & a_1 X_0 Y_0 = a_0. \end{array} \tag{2.2}$$

Thus, $\tilde{H}^{ij}_{\mathbf{a}}$ is a line or a hyperbola. Hence,

(3.3) if $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$ is not in the set $B = \{(1{:}0), (0{:}1)\}$ of the base points of $\mathbb{P}^1$, i.e. if $a_0 a_1 \neq 0$, then $\tilde{H}_{\mathbf{a}}$ is a smooth absolutely irreducible curve defined over $K(\mathbf{a})$.

The origins of the affine planes U^{01} and U^{10} are the points

$$\mathbf{q}_1 = ((1{:}0), (0{:}1)), \qquad \text{respectively} \qquad \mathbf{q}_2 = ((0{:}1), (1{:}0)),$$

which are the only points $(\mathbf{x}, \mathbf{y}) \in \mathbb{P}^1 \times \mathbb{P}^1$ with $x_0 y_0 = 0 = x_1 y_1$. They satisfy:

(3.4) For each $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$ we have $\mathbf{q}_1, \mathbf{q}_2 \in \tilde{H}_{\mathbf{a}}$. Moreover, if $\mathbf{a} \notin B$, then the slope of the tangent of $\tilde{H}_{\mathbf{a}}$ at $\mathbf{q}_1$ (resp. $\mathbf{q}_2$) is $\frac{a_0}{a_1}$ (resp. $\frac{a_1}{a_0}$) if $a_1 \neq 0$ (resp. $a_0 \neq 0$).

(3.5) The conic $\tilde{H}_{(1:0)}$ is defined by the equation $X_1 Y_1 = 0$. Let $L_1 = (1{:}0) \times \mathbb{P}^1_{\tilde{K}}$ be the line defined by $X_1 = 0$ and $L_2 = \mathbb{P}^1_{\tilde{K}} \times (1{:}0)$ be the line defined by $Y_1 = 0$. Then L_1 goes through $\mathbf{q}_1$, L_2 goes through $\mathbf{q}_2$, $\tilde{H}_{(1:0)} = L_1 \cup L_2$, and both lines go through $((1{:}0), (1{:}0))$ which is therefore a node of $\tilde{H}_{(1:0)}$ and actually its only singular point.

(3.6) The conic $\tilde{H}_{(0:1)}$ is defined by the equation $X_0Y_0 = 0$. Let $L_1' = (0{:}1) \times \mathbb{P}^1_{\tilde{K}}$ be the line defined by $X_0 = 0$, $L_2' = \mathbb{P}^1_{\tilde{K}} \times (0{:}1)$ be the line defined by $Y_0 = 0$. Then L_1' goes through $\mathbf{q}_2$, L_2' goes through $\mathbf{q}_1$, $\tilde{H}_{(0:1)} = L_1' \cup L_2'$, and both lines go through $((0{:}1),(0{:}1))$ which is therefore a node of $\tilde{H}_{(0:1)}$ and actually its only singular point.

Summing up the information about the tangents of the $\tilde{H}_{\mathbf{a}}$'s at $\mathbf{q}_1$ and $\mathbf{q}_2$, we have:

(3.7) Let $i \in \{1,2\}$. If $\mathbf{a}$ and $\mathbf{a}'$ are distinct points of $\mathbb{P}^1(\tilde{K})$, then the tangents of $\tilde{H}_{\mathbf{a}}$ and $\tilde{H}_{\mathbf{a}'}$ at $\mathbf{q}_i$ are distinct. Moreover, as $\mathbf{a}$ ranges over all points of $\mathbb{P}^1(\tilde{K})$, the tangents of $\tilde{H}_{\mathbf{a}}$ at $\mathbf{q}_1$ (resp. $\mathbf{q}_2$) form the full pencil of lines through $\mathbf{q}_1$ (resp. $\mathbf{q}_2$) in U^{01} (resp. in U^{10}).

(3.8) If $\mathbf{a} \neq \mathbf{a}'$, then $\mathbf{q}_1$ and $\mathbf{q}_2$ are the only points of intersection of $\tilde{H}_{\mathbf{a}}$ and $\tilde{H}_{\mathbf{a}'}$.

Indeed, suppose $\mathbf{q} \in \tilde{H}_{\mathbf{a}} \cap \tilde{H}'_{\mathbf{a}}$ with $\mathbf{q} = ((x_0{:}x_1),(y_0{:}y_1)) \neq \mathbf{q}_1, \mathbf{q}_2$ and $\mathbf{a} \neq \mathbf{a}'$. Then,

(3.8a) $a_1x_0y_0 = a_0x_1y_1$ and $a_1'x_0y_0 = a_0'x_1y_1$,

and

(3.8b) $x_0y_0 \neq 0$ or $x_1y_1 \neq 0$ (otherwise $x_0 = y_1 = 0$ and $\mathbf{q} = \mathbf{q}_2$ or $y_0 = x_1 = 0$ and $\mathbf{q} = \mathbf{q}_1$).

If $\mathbf{a}, \mathbf{a}' \notin B$, then $\frac{a_1}{a_0}x_0y_0 = \frac{a_1'}{a_0'}x_0y_0$ and $\frac{a_0}{a_1}x_1y_1 = \frac{a_0'}{a_1'}x_1y_1$, hence $\mathbf{a} = \mathbf{a}'$. If $\mathbf{a} = (1{:}0)$, then $x_1y_1 = 0$ and $a_1' \neq 0$, so $x_0y_0 = 0$ in contrast to (3.8b). If $\mathbf{a} = (0{:}1)$, then $x_0y_0 = 0$ and $a_0' \neq 0$, so $x_1y_1 = 0$, in contrast to (3.8b). Similarly, the assumption $\mathbf{a}' \in B$ contradicts (3.8b).

(3.9) $\mathbb{P}^1(\tilde{K}) \times \mathbb{P}^1(\tilde{K}) = \bigcup_{\mathbf{a} \in \mathbb{P}^1(\tilde{K})} \tilde{H}_{\mathbf{a}}(\tilde{K})$.

Indeed, if $((x_0{:}x_1),(y_0{:}y_1)) \neq \mathbf{q}_1, \mathbf{q}_2$, then $x_0y_0 \neq 0$ or $x_1y_1 \neq 0$. Hence, the equality $(x_1y_1)x_0y_0 = (x_0y_0)x_1y_1$ implies that $((x_0{:}x_1),(y_0{:}y_1)) \in \tilde{H}_{(x_0y_0:x_1y_1)}$. This together with (3.8) proves (3.9).

Lemma 2.1. *Let $\varphi\colon H' \to H$ be a birational surjective morphism of a curve H' onto a normal curve H over $\tilde{K}$. Then φ is an isomorphism.*

Proof. Let F be the common field of functions of H and H' over $\tilde{K}$. Consider $\mathbf{p}' \in H'(\tilde{K})$ and let $\mathbf{p} = \varphi(\mathbf{p}')$. Then $\mathcal{O}_{H,\mathbf{p}} \subseteq \mathcal{O}_{H',\mathbf{p}'} \subset F$. By assumption, $\mathcal{O}_{H,\mathbf{p}}$ is a discrete valuation ring and $\mathcal{O}_{H',\mathbf{p}'}$ a proper local ring of F. Hence, $\mathcal{O}_{H,\mathbf{p}} = \mathcal{O}_{H',\mathbf{p}'}$. Therefore, φ is an isomorphism. □

2.2 Blowing up

We blow up $\mathbb{P}^1_K \times \mathbb{P}^1_K$ at the set $\{\mathbf{q}_1, \mathbf{q}_2\}$ to obtain a surface S in $(\mathbb{P}^1_K \times \mathbb{P}^1_K) \times \mathbb{P}^1_K \times \mathbb{P}^1_K$ such that the projection $\sigma\colon S \to \mathbb{P}^1_K \times \mathbb{P}^1_K$ on the first factor is a birational projective K-morphism, (the second factor $\mathbb{P}^1_K$ comes from blowing up at $\mathbf{q}_1$, and the third factor $\mathbb{P}^1_K$ comes from blowing up at $\mathbf{q}_2$). The morphism σ has the following properties [Mum88, pp. 219-225]:

(3.10a) S is an absolutely irreducible surface defined over K. We set $\tilde{S} = S \times_K \tilde{K}$ but use σ to denote also the map $\tilde{S} \to \mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}$ obtained by extending the base field from K to $\tilde{K}$. Note that $\tilde{S}(\tilde{K})$ can be naturally identified with $S(\tilde{K})$.

(3.10b) The restriction of σ to $S \smallsetminus \sigma^{-1}(\{\mathbf{q}_1, \mathbf{q}_2\})$ is an isomorphism onto $\mathbb{P}^1_K \times \mathbb{P}^1_K \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\}$. In particular, over each point $(\mathbf{a}, \mathbf{a}') \in \mathbb{P}^1(\tilde{K}) \times \mathbb{P}^1(\tilde{K})$ not in $\{\mathbf{q}_1, \mathbf{q}_2\}$ there lies a unique point of $S(\tilde{K})$ which we also denote by $(\mathbf{a}, \mathbf{a}')$.

(3.10c) For $i = 1, 2$, the fiber $\sigma^{-1}(\mathbf{q}_i)$ is of dimension 1, indeed the fiber is isomorphic to $\mathbb{P}^1_K$.

(3.10d) For each $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$ let $H'_{\mathbf{a}}$ be the Zariski-closure in $\tilde{S}$ of $\sigma^{-1}(\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\})$. Then σ maps $H'_{\mathbf{a}}$ isomorphically onto $\tilde{H}_{\mathbf{a}}$.

Proof of (3.10d). Since σ is a morphism, it is Zariski-continuous. Hence, using a bar to denote the Zariski-closure, we have

$$\sigma(H'_{\mathbf{a}}) = \sigma\big(\overline{\sigma^{-1}(\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\})}\big) \subseteq \overline{\sigma(\sigma^{-1}(\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\}))} = \overline{\tilde{H}_{\mathbf{a}} - \{\mathbf{q}_1, \mathbf{q}_2\}} = \tilde{H}_{\mathbf{a}}.$$

Since σ is projective, it is closed [Liu06, p. 108]. Hence, $\sigma(H'_{\mathbf{a}})$ is a Zariski-closed subset of $\tilde{H}_{\mathbf{a}}$. Since $\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\} \subseteq \sigma(H'_{\mathbf{a}})$, we get that $\sigma(H'_{\mathbf{a}}) = \tilde{H}_{\mathbf{a}}$. Further, since by (3.10b) σ maps $\sigma^{-1}(\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\})$ isomorphically onto $\tilde{H}_{\mathbf{a}} \smallsetminus \{\mathbf{q}_1, \mathbf{q}_2\}$, σ maps $H'_{\mathbf{a}}$ birationally onto $\tilde{H}_{\mathbf{a}}$.

If $\mathbf{a} \neq (0{:}1), (1{:}0)$, then, by (3.3), $\tilde{H}_{\mathbf{a}}$ is smooth, hence normal. By Lemma 2.1, σ maps $H'_{\mathbf{a}}$ isomorphically onto $\tilde{H}_{\mathbf{a}}$.

If $\mathbf{a} = (1{:}0)$, then by (3.5), $\tilde{H}_{\mathbf{a}}$ is defined in $\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}$ by the equation $X_1 Y_1 = 0$. Thus, $\tilde{H}_{\mathbf{a}} = L_1 \cup L_2$, where L_1 is the line defined by $X_1 = 0$ and L_2 is the line defined by $Y_1 = 0$. For $i = 1, 2$ consider the closed set $L'_i = \sigma^{-1}(L_i) \cap H'_{\mathbf{a}}$. Since L_i is smooth, we have as in the previous case that σ maps L'_i isomorphically onto L_i. Next note that the point $L_1 \cap L_2 = ((1{:}0), (1{:}0))$ (see (3.5)) is different from $\mathbf{q}_1$ and from $\mathbf{q}_2$. Since σ is an isomorphism beyond $\sigma^{-1}(\{\mathbf{q}_1, \mathbf{q}_2\})$, σ maps $H'_{\mathbf{a}} = L'_1 \cup L'_2$ isomorphically onto $\tilde{H}_{\mathbf{a}} = L_1 \cup L_2$.

The case where $\mathbf{a} = (0{:}1)$ is symmetric to the case where $\mathbf{a} = (1{:}0)$. □

(3.11) $S(\tilde{K}) = \bigcup_{\mathbf{a} \in \mathbb{P}^1(\tilde{K})} H'_{\mathbf{a}}(\tilde{K})$.

Proof of (3.11). By the second part of (3.7), the lines $\sigma^{-1}(\mathbf{q}_1)$ and $\sigma^{-1}(\mathbf{q}_2)$ are contained in the right hand side of (3.11). Taking the inverse images of (3.9) under σ and using (3.10d), we have that $S(\tilde K) = \bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde K)} H'_{\mathbf{a}}(\tilde K)$. It remains to prove that the union is disjoint. To this end let $i \in \{1,2\}$ and note that since $\mathbf{q}_i$ is a simple point of $\mathbb{P}^1(K)\times\mathbb{P}^1(K)$, $\sigma^{-1}(\mathbf{q}_i)$ is isomorphic to the projectivised tangent cone of $\mathbb{P}^1_K \times \mathbb{P}^1_K$ at that point [Mum88, p. 225 (V.)]. Let $i \in \{1,2\}$ and let $\mathbf{a},\mathbf{a}'$ be distinct points of $\mathbb{P}^1(\tilde K)$. Then, $H'_{\mathbf{a}}(\tilde K)\cap\sigma^{-1}(\mathbf{q}_i)(\tilde K)$ and $H'_{\mathbf{a}'}(\tilde K)\cap\sigma^{-1}(\mathbf{q}_i)(\tilde K)$ are points of $S(\tilde K)$ that correspond under that isomorphism to the tangents of $\tilde H_{\mathbf{a}}$ and $\tilde H_{\mathbf{a}'}$, respectively, at $\mathbf{q}_i$. By (3.7), those tangents are distinct. Hence, $H'_{\mathbf{a}}(\tilde K) \cap H'_{\mathbf{a}'}(\tilde K) \cap \sigma^{-1}(\{\mathbf{q}_1,\mathbf{q}_2\})(\tilde K) = \emptyset$. Since, by (3.8), $\tilde H_{\mathbf{a}}(\tilde K) \cap \tilde H_{\mathbf{a}'}(\tilde K) = \{\mathbf{q}_1,\mathbf{q}_2\}$, it follows from (3.10b) that $H'_{\mathbf{a}}(\tilde K) \cap H'_{\mathbf{a}'}(\tilde K) = \emptyset$, as claimed. □

(3.12) For each $\mathbf{a} \in \mathbb{P}^1(\tilde K)$, the projection π of $\tilde S$ on the second factor $\mathbb{P}^1_{\tilde K}$ of $(\mathbb{P}^1_{\tilde K} \times \mathbb{P}^1_{\tilde K}) \times \mathbb{P}^1_{\tilde K} \times \mathbb{P}^1_{\tilde K}$ is an epimorphism that maps each $H'_{\mathbf{a}}$ onto $\mathbf{a}$.

Proof of (3.12). The affine (Y_0,X_1)-plane $A = U^{01}$ is a Zariski-open neighborhood of $\mathbf{q}_1$ in $\mathbb{P}^1_{\tilde K} \times \mathbb{P}^1_{\tilde K}$. By (3.2), the intersection $\tilde H^{01}_{\mathbf{a}} = \tilde H_{\mathbf{a}} \cap A$ is defined by $a_1Y_0 = a_0X_1$. The blow up of A at $\mathbf{q}_1$ is the subset A' of $A \times \mathbb{P}^1_{\tilde K}$ defined by the equation $Z_1Y_0 = Z_0X_1$, where $(Z_0{:}Z_1)$ are the homogeneous coordinates of $\mathbb{P}^1_{\tilde K}$. Let $\pi_1\colon A\times \mathbb{P}^1_{\tilde K} \to \mathbb{P}^1_{\tilde K}$ be the projection on the second factor. Then, $\pi_1^{-1}(\mathbf{a}) \cap A' = \tilde H^{01}_{\mathbf{a}}$. Since the blow up of $\mathbb{P}^1_K \times \mathbb{P}^1_K$ is done in two stages, first in $\mathbf{q}_1$ and then in the inverse image of $\mathbf{q}_2$ (which we identify with $\mathbf{q}_2$) and since $\mathbf{q}_2 \notin A(\tilde K)$, we have $\pi(\sigma^{-1}(\tilde H^{01}_{\mathbf{a}})) = \mathbf{a}$. By the Zariski-continuity of π, we have $\pi(H'_{\mathbf{a}}) = \mathbf{a}$, as claimed. □

Since, by (3.11), the $H'_{\mathbf{a}}$ are disjoint,

(3.13) $\pi^{-1}(\mathbf{a}) = H'_{\mathbf{a}}$.

3 Irreducible Curves

Let $\tilde K$ be a fixed algebraic closure of a field K, $\tilde D$ a smooth projective irreducible curve over $\tilde K$, and $\tilde\varphi\colon \tilde D \to \mathbb{P}^1_{\tilde K}$ a surjective morphism. Given a morphism $\alpha\colon X \to Y$ of schemes and a point $y \in Y$, we say that α is **smooth over** y if α is smooth at each $x \in X$ with $\alpha(x) = y$. With this terminology, we assume

(4.1) $\tilde\varphi$ is smooth over (0:1).

As in Section 2 we consider two copies of $\mathbb{P}^1_{\tilde K}$ with respective homogeneous coordinates $(X_0{:}X_1)$ and $(Y_0{:}Y_1)$. For each $\mathbf{a} = (a_0{:}a_1) \in \mathbb{P}^1(\tilde K)$ we consider the conic $\tilde H_{\mathbf{a}}$ defined in $\mathbb{P}^1_{\tilde K} \times \mathbb{P}^1_{\tilde K}$ by the equation $a_1X_0Y_0 = a_0X_1Y_1$. Let $\tilde\delta\colon \tilde D \times \tilde D \to \mathbb{P}^1_{\tilde K} \times \mathbb{P}^1_{\tilde K}$ be the product $\tilde\varphi \times \tilde\varphi$ and consider the inverse image $\tilde I_{\mathbf{a}} = \tilde\delta^{-1}(\tilde H_{\mathbf{a}})$. By [AlK70, p. 129, Prop. 1.7(d)], (4.2a) $\tilde\delta\colon \tilde D\times\tilde D\to\mathbb{P}^1_{\tilde K}\times\mathbb{P}^1_{\tilde K}$ is smooth over $((0{:}1),(0{:}1))$.

Since $\tilde D$ is a smooth projective curve and $\tilde\varphi\colon \tilde D\to\mathbb{P}^1_{\tilde K}$ is a surjective morphism, $\tilde\varphi$ is finite (see [Sha77, p. 122, Thm. 11] or [Har77, p. 137, Prop. 6.8]). Since the property of being finite is stable under composition and base change [GoW, p. 325, Prop. 12.11(3)],

(4.2b) $\tilde\delta$ is a finite morphism, hence proper [GoW10, p. 325, Prop. 12.12].

Since $\dim(\tilde H_{\mathbf a})=1$, it follows from (4.2b) that $\dim(\tilde I_{\mathbf a})=1$. By (3.9) of Section 2, $\tilde D(\tilde K)\times\tilde D(\tilde K)=\bigcup_{\mathbf a\in\mathbb{P}^1(\tilde K)}\tilde I_{\mathbf a}(\tilde K)$.

Lemma 3.1. *For all* $\mathbf a\in\mathbb{P}^1(\tilde K)\smallsetminus\{(0{:}1),(1{:}0)\}$, $\tilde I_{\mathbf a}$ *is a connected scheme.*

Proof. First we note that

(4.3) each of the conics $\tilde H_{\mathbf a}$, with $\mathbf a=(a_0{:}a_1)\in\mathbb{P}^1(\tilde K)\smallsetminus\{(0{:}1),(1{:}0)\}$, considered as an irreducible divisor of $\mathbb{P}^1_{\tilde K}\times\mathbb{P}^1_{\tilde K}$, is very ample.

To this end we consider the Segre embedding $s\colon\mathbb{P}^1_{\tilde K}\times\mathbb{P}^1_{\tilde K}\to\mathbb{P}^3_{\tilde K}$ given by $s((x_0{:}x_1),(y_0{:}y_1))=(z_0{:}z_1{:}z_2{:}z_3)$, where $z_0=x_0y_0$, $z_1=x_0y_1$, $z_2=x_1y_0$, and $z_3=x_1y_1$. Then s is a closed immersion onto a closed subsurface P of $\mathbb{P}^3_{\tilde K}$ [GoW10, p. 112, Prop. 4.39]. Hence, s induces an isomorphism of $\mathcal{O}_P(1)$ represented by the divisor $P_{\mathbf a}$ of $\mathbb{P}^3_{\tilde K}$ defined by the linear equation $a_1Z_0=a_0Z_3$ onto the invertible sheaf $\mathcal{L}_{\mathbf a}$ of $\mathbb{P}^1_{\tilde K}\times\mathbb{P}^1_{\tilde K}$ corresponding to $\tilde H_{\mathbf a}$. By definition, $\tilde H_{\mathbf a}$ is very ample [Har77, p. 120, Def. and p. 307].

By (4.3) and by definition [Har77, p. 307], $\tilde H_{\mathbf a}$ is an effective ample divisor. By (4.2b) and [Har77, p. 232, Exer. 5.7(d)], $\tilde I_{\mathbf a}=\tilde\delta^{-1}(\tilde H_{\mathbf a})$ is an effective ample divisor of $\tilde D\times\tilde D$. By assumption, $\tilde D\times\tilde D$ is an integral smooth (hence normal) projective variety. It follows from a Lemma of Enriques-Severi-Zariski [Har77, p. 244, Cor. 7.9] that $\tilde I_{\mathbf a}$ is connected. □

Remark 3.2 (Singular points)**.** Let $\kappa\colon X\to Y$ be a morphism of finite type between schemes of finite type over the algebraically closed field $\tilde K$. We set

$$\mathrm{Sing}(\kappa)=\{x\in X\mid \kappa \text{ is not smooth at } x\}.$$

By [Gro64, 6.8.7] or [Liu06, p. 224, Cor. 2.12], $\mathrm{Sing}(\kappa)$ is a closed subset of X. If $\lambda\colon Y\to Z$ is another morphism of finite type between schemes of finite type over $\tilde K$, then

$$\mathrm{Sing}(\lambda\circ\kappa)\subseteq\mathrm{Sing}(\kappa)\cup\kappa^{-1}(\mathrm{Sing}(\lambda)), \tag{3.4}$$

because composition of smooth morphisms is smooth [Liu06, p. 143, Prop. 3.38]. If $Y=\mathrm{Spec}(\tilde K)$, then $\mathrm{Sing}(X)=\mathrm{Sing}(\kappa)$ is the set of singular points of X. Applying the equivalent definition of smoothness given by [Liu06, p. 142, Definition 3.35], we find that

$$\mathrm{Sing}(\kappa) = \bigcup_{y \in Y} \mathrm{Sing}(\kappa^{-1}(y)) \tag{3.5}$$

if κ is flat.

Lemma 3.3. *For almost all* $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$, $\tilde{I}_{\mathbf{a}}$ *is a smooth scheme.*

Proof. As in Section 2, we consider the points $\mathbf{q}_1 = ((1{:}0),(0{:}1))$ and $\mathbf{q}_2 = ((0{:}1),(1{:}0))$ of $\mathbb{P}^1(\tilde{K}) \times \mathbb{P}^1(\tilde{K})$. As in Subsection 2.2, let S be the closed $\tilde{K}$-subsurface of $(\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}) \times \mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}$ obtained by blowing up $\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}}$ at the set $\{\mathbf{q}_1, \mathbf{q}_2\}$, let $\sigma\colon S \to (\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}})$ be the projection on the first factor, and $\pi\colon S \to \mathbb{P}^1_{\tilde{K}}$ the projection on the second factor. Both morphisms are projective, hence proper [Liu06, p. 108, Thm. 3.30]. Now we break up the rest of the proof into several parts.

Part A: *A commutative diagram.* Let $T = (\tilde{D} \times \tilde{D}) \times_{(\mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}})} S$ be the fibred product of $\tilde{\delta}$ and σ. Let $\tau\colon T \to S$ be the projection on the second factor and let $\pi_T = \pi \circ \tau$. Since $\tilde{\delta}$ is proper (by (4.2b)), so is τ [Liu06, p. 104, Cor. 3.16(c)]. Since also π is proper,

(4.6) $\pi_T = \pi \circ \tau$ is also proper [Liu06, p. 104, Cor. 3.16(b)].

By (3.11) of Section 2, $S(\tilde{K}) = \bigcup_{\mathbf{a} \in \mathbb{P}^1(\tilde{K})} H'_{\mathbf{a}}(\tilde{K})$, where $H'_{\mathbf{a}}$ is a curve on S that σ maps isomorphically onto $\tilde{H}_{\mathbf{a}}$. For each $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$ let $I'_{\mathbf{a}} = \tau^{-1}(H'_{\mathbf{a}}) = \tilde{I}_{\mathbf{a}} \times_{\tilde{H}_{\mathbf{a}}} H'_{\mathbf{a}}$. Then, $T(\tilde{K}) = \bigcup_{\mathbf{a} \in \mathbb{P}^1(\tilde{K})} I'_{\mathbf{a}}(\tilde{K})$. Moreover, by (3.13) of Section 2, $\pi^{-1}(\mathbf{a}) = H'_{\mathbf{a}}$, so with $\pi_T = \pi \circ \tau$, we have

(4.7) $\pi_T^{-1}(\mathbf{a}) = \tau^{-1}(\pi^{-1}(\mathbf{a})) = \tau^{-1}(H'_{\mathbf{a}}) = I'_{\mathbf{a}}$ for each $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$.

This gives a commutative diagram

$$\begin{array}{ccccc}
\bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} \tilde{I}_{\mathbf{a}} = \tilde{D} \times \tilde{D} & \xleftarrow{\sigma_T} & T = \bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} I'_{\mathbf{a}} & \xrightarrow{\pi_T} & \mathbb{P}^1_{\tilde{K}} \\
\downarrow{\scriptstyle \tilde{\delta}} & & \downarrow{\scriptstyle \tau} & & \| \\
\bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} \tilde{H}_{\mathbf{a}} = \mathbb{P}^1_{\tilde{K}} \times \mathbb{P}^1_{\tilde{K}} & \xleftarrow{\sigma} & S = \bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} H'_{\mathbf{a}} & \xrightarrow{\pi} & \mathbb{P}^1_{\tilde{K}}
\end{array}, \tag{3.8}$$

where the left square is cartesian. By (4.2b), $\tilde{\delta}$ is finite. Since finiteness of morphisms is preserved under base change [GoW10, p. 325, Prop. 12.11(2)],

(4.9) τ is a finite morphism.

Since $\dim(\mathbb{P}^1_{\tilde{K}}) = 1$ and $\mathbb{P}^1_{\tilde{K}}$ is smooth, $\mathbb{P}^1_{\tilde{K}}$ is a Dedekind scheme. In addition π is not a constant map. Since σ is birational, S is integral, so by [Liu06, p. 137, Cor. 3.10],

(4.10) π is flat.

Similarly, since σ_T is birational, T is integral, so by [Liu06, p. 137, Cor. 3.10],

(4.11) $\pi_T = \pi \circ \tau$ is flat.

Part B: *Finiteness of* $\tau^{-1}(\mathrm{Sing}(\pi))$. By (4.10) and (4.5), and by (3.13) of Section 2,

$$\text{(3.12)} \qquad \mathrm{Sing}(\pi) = \bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} \mathrm{Sing}(\pi^{-1}(\mathbf{a})) = \bigcup_{\mathbf{a}\in\mathbb{P}^1(\tilde{K})} \mathrm{Sing}(H'_{\mathbf{a}}).$$

Let $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$. By (3.10d) of Section 2, $H'_{\mathbf{a}} \cong \tilde{H}_{\mathbf{a}}$. By (3.3) of Section 2, $\tilde{H}_{\mathbf{a}}$ is smooth if $\mathbf{a} \neq (1{:}0), (0{:}1)$, so $\mathrm{Sing}(H'_{\mathbf{a}})$ is empty. By (3.5) and (3.6) of Section 2, each of the conics $H_{(1:0)}$ and $H_{(0:1)}$ has a unique singular point. It follows from (4.12) that $\mathrm{Sing}(\pi)$ is finite. By (4.9), the set $\tau^{-1}(\mathrm{Sing}(\pi))$ is finite.

Part C: *Finiteness of* $\pi_T(\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau))$. Let $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$. By (3.10d) of Section 2, the morphism σ maps $H'_{\mathbf{a}}$ isomorphically onto $\tilde{H}_{\mathbf{a}}$. Since the diagram

$$\begin{array}{ccc} \tilde{\delta}^{-1}(\tilde{H}_{\mathbf{a}}) = \tilde{I}_{\mathbf{a}} & \xleftarrow{\sigma_{T,\mathbf{a}}} & I'_{\mathbf{a}} = \tau^{-1}(H'_{\mathbf{a}}) \\ \downarrow{\scriptstyle \tilde{\delta}_{\mathbf{a}}} & & \downarrow{\scriptstyle \tau_{\mathbf{a}}} \\ \tilde{H}_{\mathbf{a}} & \xleftarrow{\sigma_{\mathbf{a}}} & H'_{\mathbf{a}} \end{array},$$

where the arrows are the corresponding restrictions of the arrows of the left square of Diagram (4.8), is cartesian,

(4.13) $\sigma_{T,\mathbf{a}}$ maps $I'_{\mathbf{a}}$ isomorphically onto $\tilde{I}_{\mathbf{a}}$.

By (3.6) of Section 2, $\tilde{H}_{(0:1)} = ((0{:}1) \times \mathbb{P}^1_{\tilde{K}}) \cup (\mathbb{P}^1_{\tilde{K}} \times (0{:}1))$. Therefore, by (4.13), $I'_{(0:1)}$ is isomorphic to $I_{(0:1)} = (\tilde{\varphi}^{-1}((0{:}1)) \times \tilde{D}) \cup (\tilde{D} \times \tilde{\varphi}^{-1}((0{:}1))$. Since $\tilde{D}$ is smooth, the latter scheme is smooth except for nodes lying over the intersection point of the two components of $\tilde{H}_{(0:1)}$, namely over $((0{:}1), (0{:}1))$. Therefore, using Convention (3.10b) of Section 2,

$$\text{(3.14)} \qquad \tau(\mathrm{Sing}(I'_{(0:1)})) = \{((0{:}1), (0{:}1))\}.$$

By (3.13) of Section 2, $\pi^{-1}((0{:}1)) = H'_{(0:1)}$. Hence,

$$\text{(3.15)} \qquad \pi_T^{-1}((0{:}1)) = \tau^{-1}(\pi^{-1}((0{:}1))) = \tau^{-1}(H'_{(0:1)}) = I'_{(0:1)}.$$

Therefore, by (4.11) and (4.5),

$$\mathrm{Sing}(I'_{(0:1)}) = \mathrm{Sing}(\pi_T^{-1}((0{:}1))) = \mathrm{Sing}(\pi_T) \cap I'_{(0:1)}.$$

It follows by (4.14) that

$$\tau(\mathrm{Sing}(\pi_T) \cap I'_{(0:1)}) = \tau(\mathrm{Sing}(I'_{(0:1)})) = \{((0{:}1),(0{:}1))\}. \tag{3.16}$$

On the other hand, by (4.2a), $\tilde{\delta}$ is smooth over $((0{:}1),(0{:}1))$, so by [Liu06, p. 143, Prop. 3.38], τ is also smooth over the point $((0{:}1),(0{:}1))$ of S (Convention (3.10b) of Section 2). In other words,

$$\tau^{-1}(((0{:}1),(0{:}1))) \cap \mathrm{Sing}(\tau) = \emptyset. \tag{3.17}$$

It follows from (4.16) and (4.17) that

$$\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau) \cap I'_{(0:1)} = \emptyset. \tag{3.18}$$

By (4.6), π_T is proper. By Remark 3.2, $\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau)$ is closed in T. Hence, the set $\pi_T(\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau))$ is closed in $\mathbb{P}^1_{\tilde{K}}$. Therefore, $\pi_T(\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau))$ is either $\mathbb{P}^1_{\tilde{K}}$ or a finite set. In the former case, each point in $I'_{(0:1)} = \pi_T^{-1}((0{:}1))$ (see (4.15)) lies in $\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau)$, which contradicts (4.18). Therefore,

(4.19) the set $\pi_T(\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau))$ is finite.

Part D: *Finiteness of* $\pi_T(\mathrm{Sing}(\pi_T))$. Now note by (4.4) that $\mathrm{Sing}(\pi_T) = \mathrm{Sing}(\pi \circ \tau) \subseteq \mathrm{Sing}(\tau) \cup \tau^{-1}(\mathrm{Sing}(\pi))$. Hence,

$$\pi_T(\mathrm{Sing}(\pi_T)) \subseteq \pi_T(\mathrm{Sing}(\pi_T) \cap \mathrm{Sing}(\tau)) \cup \pi_T(\tau^{-1}(\mathrm{Sing}(\pi))).$$

It follows from (4.19) and the finiteness of $\tau^{-1}(\mathrm{Sing}(\pi))$ (Part B) that $\pi_T(\mathrm{Sing}(\pi_T))$ is finite.

Part E: *End of proof.* We consider $\mathbf{a} \in \mathbb{P}^1(\tilde{K}) \smallsetminus \pi_T(\mathrm{Sing}(\pi_T))$. By (4.7), $\pi_T^{-1}(\mathbf{a}) = I'_{\mathbf{a}}$. Hence, $I'_{\mathbf{a}}$ is smooth. By (4.13), $I'_{\mathbf{a}}$ is isomorphic to $\tilde{I}_{\mathbf{a}}$. Hence, $\tilde{I}_{\mathbf{a}}$ is smooth. It follows from Part D that $\tilde{I}_{\mathbf{a}}$ is smooth for almost all $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$, as claimed. □

Since normal (in particular, smooth) connected $\tilde{K}$-schemes of finite type are irreducible (e.g. [GoW10, p. 168, Exer. 6.20]), a combination of Lemma 3.1 and Lemma 3.3 yields the following result:

Corollary 3.4. *For almost all* $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$, $\tilde{I}_{\mathbf{a}}$ *is an irreducible smooth curve.*

4 The Open Mapping Theorem

As in Section 2, we fix an algebraic closure $\tilde{K}$ of a field K. We also fix a valuation v of $\tilde{K}$ and prove an open map theorem for varieties over $\tilde{K}$ in the v-topology.

Our proof is based on a theorem about the continuity of roots for not necessarily separable polynomials. A convenient reference is [Jar91, Prop. 12.2].

Lemma 4.1. *Let $p(X) = \prod_{i=1}^n (X - a_i)$ be a monic polynomial with coefficients in $\tilde{K}$. Then, for each $c \in \tilde{K}^\times$ there exists $c' \in \tilde{K}^\times$ such that if $q \in \tilde{K}[X]$ is a monic polynomial of degree n and $v(q - p) > v(c')$, then $q(X)$ may be presented as a product $q(X) = \prod_{i=1}^n (X - b_i)$ such that $v(b_i - a_i) > v(c)$ for $i = 1, \dots, n$.*

Here and throughout we write $v\big(\sum_{i=0}^n c_i X^i\big) > v(b)$ for $c_0, \dots, c_n \in \tilde{K}$ and $b \in \tilde{K}^\times$ as an abbreviation for "$v(c_i) > v(b)$ for $i = 0, \dots, n$."

Lemma 4.2. *Let V be a vector space of finite dimension d over an infinite field K_0. Then V has an infinite subset V_0 such that every subset of V_0 of cardinality d is a basis of V.*

Proof. By assumption, V has a subset of cardinality d which is a basis of V. Inductively suppose that U is a finite subset of V with $e \geq d$ elements such that every subset of U of cardinality d is a basis of V. We denote the collection of all subsets of U of cardinality $d-1$ by $\mathcal{U}$. By assumption, for each $U_0 \in \mathcal{U}$ the dimension of the subspace $\sum_{u \in U_0} K_0 u$ of V is $d-1$, so that subspace is properly contained in V. It follows that also $\bigcup_{U_0 \in \mathcal{U}} \sum_{u \in U_0} K_0 u$ is a proper subset of V. We choose an element $v \in V \smallsetminus \bigcup_{U_0 \in \mathcal{U}} \sum_{u \in U_0} K_0 u$. Then, for each $U_0 \in \mathcal{U}$ we have $\dim(K_0 v + \sum_{u \in U_0} K_0 u) = d$, so $\{v\} \cup U_0$ is a basis of V. Consequently, every subset of $U \cup \{v\}$ of cardinality d is a basis of V. This completes the induction and the proof of the lemma. □

Remark 4.3 (The v-topology on $\mathrm{Max}(A)$)**.** Let B be a finitely generated integral domain over $\tilde{K}$. We denote the set of all maximal ideals of B by $\mathrm{Max}(B)$. For each $\mathfrak{q} \in \mathrm{Max}(B)$ we identify $B/\mathfrak{q}$ with $\tilde{K}$ and let $x(\mathfrak{q})$ be the residue of $x \in B$ at $\mathfrak{q}$. The v-topology of $\tilde{K}$ induces a v**-topology** on $\mathrm{Max}(B)$. A basic open neighborhood of a point $\mathfrak{q}_0 \in \mathrm{Max}(B)$ is a set

$$\mathcal{V} = \bigcap_{i=1}^{r} \{\mathfrak{q} \in \mathrm{Max}(B) \mid v(y_i(\mathfrak{q}) - y_i(\mathfrak{q}_0)) > v(c_0)\}, \tag{4.1}$$

where $y_1, \dots, y_r \in B$ and $c_0 \in \tilde{K}^\times$. Thus, each $x \in B$ may be viewed as a v**-continuous** (i.e. continuous in the v-topology) map from $\mathrm{Max}(B)$ to $\tilde{K}$.

Let A be an integral domain that contains $\tilde{K}$ with quotient field E. Suppose that $F = \mathrm{Quot}(B)$ is an extension of E and that B is an integral extension of A. If $\mathfrak{q} \in \mathrm{Max}(B)$, then $\mathfrak{p} = \mathfrak{q} \cap A \in \mathrm{Max}(A)$ and we identify $x(\mathfrak{p})$ with $x(\mathfrak{q})$ for each $x \in A$. Then, the canonical morphism $\varphi\colon \mathrm{Max}(B) \to \mathrm{Max}(A)$ defined by $\varphi(\mathfrak{q}) = \mathfrak{q} \cap A$ is v-continuous.

Indeed, consider a basic open neighborhood

$$\mathcal{U} = \bigcap_{i=1}^{m} \{\mathfrak{p} \in \mathrm{Max}(A) \mid \ v(x_i(\mathfrak{p}) - x_i(\mathfrak{p}_0)) > v(a_0)\}$$

of a point $\mathfrak{p}_0 \in \mathrm{Max}(A)$ with $x_1, \ldots, x_m \in A$ and $a_0 \in \tilde{K}^\times$. By the going up theorem, $\varphi(\mathrm{Max}(B)) = \mathrm{Max}(A)$ [Lan93, p. 339, Prop. 1.10]. Hence, with $\mathfrak{q}_0 \in \varphi^{-1}(\mathfrak{p}_0)$,

$$\varphi^{-1}(\mathcal{U}) = \bigcap_{i=1}^{m} \{\mathfrak{q} \in \mathrm{Max}(B) \mid \ v(x_i(\mathfrak{q}) - x_i(\mathfrak{q}_0)) > v(a_0)\},$$

is v-open in $\mathrm{Max}(B)$.

We use Lemma 4.1 to prove that the map φ of Remark 4.3 is v-open if F/E is finite and separable and A is integrally closed.

Lemma 4.4. *Let A be an integrally closed domain which contains $\tilde{K}$, $E = \mathrm{Quot}(A)$, F a finite separable extension of E, and B a subdomain of F which contains A and is integral over A. Then the canonical morphism $\varphi\colon \mathrm{Max}(B) \to \mathrm{Max}(A)$ defined by $\varphi(\mathfrak{q}) = \mathfrak{q} \cap A$ is a v-open map.*

Proof. Let $\hat{F}$ be the Galois closure of F/E, $\hat{B}$ the integral closure of B in $\hat{F}$, and $\psi\colon \mathrm{Max}(\hat{B}) \to \mathrm{Max}(B)$ the canonical map. Then, $\hat{B}$ is also the integral closure of A in $\hat{F}$ and $\hat{\varphi} = \varphi \circ \psi$ is the canonical map of $\mathrm{Max}(\hat{B})$ to $\mathrm{Max}(A)$. Let $\mathcal{V}$ be a v-open subset of $\mathrm{Max}(B)$. By Remark 4.3, ψ is v-continuous. Hence, $\hat{\mathcal{V}} = \psi^{-1}(\mathcal{V})$ is a v-open subset of $\mathrm{Max}(\hat{B})$. If $\hat{\varphi}(\hat{\mathcal{V}})$ is v-open in $\mathrm{Max}(A)$, then so is $\varphi(\mathcal{V}) = \hat{\varphi}(\psi^{-1}(\mathcal{V}))$. Therefore, replacing F by $\hat{F}$, we may assume that F/E is Galois of degree n with $G = \mathrm{Gal}(F/E)$ and B is the integral closure of A in F.

Consider a point $\mathfrak{q} \in \mathrm{Max}(B)$ and let $\mathfrak{p} = \varphi(\mathfrak{q})$. It suffices to prove that for every basic v-open neighborhood $\mathcal{V} = \bigcap_{i=1}^{r} \{\mathfrak{q}' \in \mathrm{Max}(B) \mid \ v(y_i(\mathfrak{q}') - y_i(\mathfrak{q})) > v(c)\}$ of $\mathfrak{q}$ in $\mathrm{Max}(B)$ with $y_1, \ldots, y_r \in B$, and $c \in \tilde{K}^\times$, the point $\mathfrak{p}$ of $\mathrm{Max}(A)$ has a v-open neighborhood in $\varphi(\mathcal{V})$. We break the proof of this statement into several parts.

Part A: *Many bases of a vector space.* We consider the vector space $V = \sum_{i=1}^{r} \tilde{K} y_i$ spanned by $y_1, \ldots, y_r$ over $\tilde{K}$ and let $d = \dim(V)$. By Lemma 4.2, there exists an infinite subset Z'' of V such that every subset of Z'' of cardinality d is a basis of V. We choose a finite subset Z' of Z'' of cardinality greater than $(d-1)n$. Since $\tilde{K} \subseteq A$, the vector space V is contained in B, hence $Z' \subseteq B$. In particular, every $z \in Z'$ is integral over A.

Let $\mathcal{Z}$ be the collection of all subsets of Z' of cardinality d. For every $1 \le i \le r$ and $Z \in \mathcal{Z}$ there exists a presentation

$$y_i = \sum_{z \in Z} a_{i,Z,z} z \text{ with } a_{i,Z,z} \in \tilde{K}. \tag{4.2}$$

We set

$$\alpha = \min(v(a_{i,Z,z})_{i=1,\dots,r\,,Z\in\mathcal{Z},\,z\in Z}). \tag{4.3}$$

PART B: *Continuity of roots.* Since A is integrally closed, Part A implies that for each $z \in Z'$

(5.4) $f_z(X) = \prod_{\sigma\in G}(X - \sigma z)$ is a monic polynomial of degree n with coefficients in A.

Then,

$$f_z(\mathfrak{p})(X) = f_z(\mathfrak{q})(X) = \prod_{\sigma\in G}(X - (\sigma z)(\mathfrak{q})) = \prod_{\sigma\in G}(X - z(\sigma^{-1}\mathfrak{q})). \tag{4.5}$$

Let $\mathfrak{p}' \in \mathrm{Max}(A)$ and choose $\mathfrak{q}' \in \mathrm{Max}(B)$ over $\mathfrak{p}'$. As in (5.5), $f_z(\mathfrak{p}')(X) = \prod_{\sigma\in G}(X - z(\sigma^{-1}\mathfrak{q}'))$. By Lemma 4.1, there exists $c' \in \tilde{K}^\times$ and there exists $\sigma_z \in G$ such that

(5.6) if $v(f_z(\mathfrak{p}') - f_z(\mathfrak{p})) > v(c')$, then $v(z(\sigma_z^{-1}\mathfrak{q}') - z(\mathfrak{q})) > v(c) - \alpha$.

Let $s\colon Z' \to G$ be the map defined by $s(z) = \sigma_z$. Then $(d-1)n < |Z'| = \sum_{\sigma\in G}|s^{-1}(\sigma)|$. Hence, there exists $\sigma \in G$ such that $|s^{-1}(\sigma)| \geq d$. Choose a subset Z of $s^{-1}(\sigma)$ of cardinality d, in particular $Z \in \mathcal{Z}$. By (5.6), if $v(f_z(\mathfrak{p}') - f_z(\mathfrak{p})) > v(c')$ for each $z \in Z$, then

$$v(z(\sigma^{-1}\mathfrak{q}') - z(\mathfrak{q})) > v(c) - \alpha \text{ for all } z \in Z. \tag{4.7}$$

PART C: *Conclusion of the proof.* We prove that the open neighborhood

$$\mathcal{U} = \{\mathfrak{p}' \in \mathrm{Max}(A) \mid \bigwedge_{z\in Z} v(f_z(\mathfrak{p}') - f_z(\mathfrak{p})) > v(c')\}$$

of $\mathfrak{p}$ in $\mathrm{Max}(A)$ is contained in $\varphi(\mathcal{V})$.

Indeed, let $\mathfrak{p}' \in \mathcal{U}$ and choose $\mathfrak{q}' \in \mathrm{Max}(B)$ with $\varphi(\mathfrak{q}') = \mathfrak{p}'$. Then, $v(f_z(\mathfrak{p}') - f_z(\mathfrak{p})) > v(c')$ for each $z \in Z$, so (5.7) holds. We set $\mathfrak{q}'' = \sigma^{-1}\mathfrak{q}'$ and notice that $\varphi(\mathfrak{q}'') = \mathfrak{p}'$. By (5.7), $v(z(\mathfrak{q}'') - z(\mathfrak{q})) > v(c) - \alpha$ for each $z \in Z$. In addition, by (5.3), $v(a_{i,Z,z}) \geq \alpha$ for $i = 1, \dots, r$ and for each $z \in Z$. It follows from (5.2) that

$$\begin{aligned} v(y_i(\mathfrak{q}'') - y_i(\mathfrak{q})) &= v\Big(\sum_{z\in Z} a_{i,Z,z}(z(\mathfrak{q}'') - z(\mathfrak{q}))\Big) \\ &\geq \min_{z\in Z}\big(v(a_{i,Z,z}) + v(z(\mathfrak{q}'') - z(\mathfrak{q}))\big) > v(c) \end{aligned}$$

for $i = 1, \dots, r$. Consequently, $\mathfrak{q}'' \in \mathcal{V}$, as claimed. □

Proposition 4.5. *Let $(\tilde{K}, v)$ be an algebraically closed valued field and $\varphi\colon W \to V$ a finite morphism of absolutely irreducible varieties defined over $\tilde{K}$ with V normal. Then, the v-continuous map $\varphi\colon W(\tilde{K}) \to V(\tilde{K})$ is v-open.*

Proof. The morphism φ decomposes into a purely inseparable finite morphism followed by a separable finite morphism. Since inseparable finite morphisms induce v-homeomorphisms on the corresponding sets of K-rational points, we may assume that φ is separable.

By definition, V has a cover consisting of affine Zariski-open subsets V_i whose inverse images W_i under φ are also affine, such that for each i, $\Gamma(V_i, \mathcal{O}_V)$ is an integrally closed domain, $\Gamma(W_i, \mathcal{O}_W)$ is an integral domain which is finitely generated as a module over $\Gamma(V_i, \mathcal{O}_V)$. It follows that $\Gamma(W_i, \mathcal{O}_W)$ is integral over $\Gamma(V_i, \mathcal{O}_V)$. Since every Zariski-open subset of a variety is also v-open, our proposition is a consequence of Lemma 4.4. □

Remark 4.6. Proposition 4.5 is related to [GPR95, Thm. 9.4(1)]. The latter result says that if (K, v) is an arbitrary Henselian field and $\varphi\colon W \to V$ is a smooth surjective morphism of absolutely irreducible varieties V and W defined over K, then the map $\varphi\colon W(K) \to V(K)$ is v-open.

5 A Density Property of Smooth Curves over PAC Fields

The aim of this short section is to prove a density result for curves over PAC fields.

We start with an arbitrary field K. As in Section 2 we consider for each $\mathbf{a} = (a_0{:}a_1) \in \mathbb{P}^1(K)$ the conic $H_\mathbf{a}$ defined in $\mathbb{P}^1_K \times \mathbb{P}^1_K$ by the equation $a_1 X_0 Y_0 = a_0 X_1 Y_1$. We consider a smooth projective absolutely irreducible curve D defined over K. Using the Segre embedding [GoW10, p. 112, Prop. 4.39], we may consider $H_\mathbf{a}$ also as a closed subscheme of $\mathbb{P}^3_K$. Let $\varphi\colon D \to \mathbb{P}^1_K$ be a non-constant separable morphism. We assume that

(6.1) φ is smooth over (0:1).

Let $\delta\colon D \times D \to \mathbb{P}^1_K \times \mathbb{P}^1_K$ be the product $\varphi \times \varphi$ and consider the inverse image $I_\mathbf{a} = \delta^{-1}(H_\mathbf{a})$.

Recall that we use a tilde to denote the constant extension from K to $\tilde{K}$ of algebro-geometrical objects.

Lemma 5.1. *Under Assumption (6.1), for almost all $\mathbf{a} \in \mathbb{P}^1(K)$, the scheme $I_\mathbf{a}$ is an absolutely irreducible smooth curve defined over K.*

Proof. Since D is absolutely irreducible and defined over K, the curve $\tilde{D}$ is $\tilde{K}$-irreducible, and the morphism $\tilde{\varphi}\colon \tilde{D} \to \mathbb{P}^1_{\tilde{K}}$ is surjective [Har77, p. 137, Prop. 6.8] and smooth over (0:1). Also, $\tilde{I}_{\mathbf{a}} = (\tilde{\delta})^{-1}(\tilde{H}_{\mathbf{a}})$. By Corollary 3.4, for almost all $\mathbf{a} \in \mathbb{P}^1(\tilde{K})$, $\tilde{I}_{\mathbf{a}}$ is irreducible. Moreover, by the latter corollary, $\tilde{I}_{\mathbf{a}}$ is also smooth. Hence, each point $\mathbf{p}$ of $\tilde{I}_{\mathbf{a}}$ is regular. Therefore, $\mathcal{O}_{\tilde{I}_{\mathbf{a}},\mathbf{p}}$ is an integral domain. It follows that $I_{\mathbf{a}}$ is an absolutely irreducible curve defined over K. □

Lemma 5.2. *Let $\psi\colon W \to V$ be a finite morphism of integral schemes over a field K such that V is normal. Consider the inclusion of the function field E of V into the function field F of W that ψ induces and assume that F/E is Galois. Suppose that the natural action of $G = \mathrm{Gal}(F/E)$ on the generic point of W extends to an action on W over V such that $\mathcal{O}_V$ is the fixed subsheaf of the induced action of G on $\mathcal{O}_W$. Then, for every $\mathbf{q}, \mathbf{q}' \in W$ with $\psi(\mathbf{q}) = \psi(\mathbf{q}')$ there exists $\sigma \in G$ such that $\sigma\mathbf{q} = \mathbf{q}'$.*

Proof. Let $\mathbf{p} \in V$ and $\mathbf{q}, \mathbf{q}' \in W$ such that $\psi(\mathbf{q}) = \mathbf{p}$ and $\psi(\mathbf{q}') = \mathbf{p}$. Let $V_0 = \mathrm{Spec}(A)$ be an affine Zariski-open neighborhood of $\mathbf{p}$ in V. Since V is integral and normal, A is an integrally closed domain with $\mathrm{Quot}(A) = E$. Since ψ is finite, $W_0 = \varphi^{-1}(V_0)$ is also affine, say $W_0 = \mathrm{Spec}(B)$, where B is an integral domain which is integral over A. By assumption, $F = \mathrm{Quot}(B)$ is a finite Galois extension of E. Also, G acts on B with A being the fixed ring of B under G. Finally, we may identify $\mathbf{q}$ and $\mathbf{q}'$ with prime ideals of B and $\mathbf{p}$ with the prime ideal of A lying under both $\mathbf{q}$ and $\mathbf{q}'$. By [Bou89, p. 331, Thm. 2(i)], there exists $\sigma \in G$ such that $\sigma\mathbf{q} = \mathbf{q}'$, as claimed. □

The proof of the following lemma uses a trick of Prestel [FrJ08, p. 204, proof of Prop. 11.5.3].

Lemma 5.3. *Let K be a PAC field and let v be a valuation of $\tilde{K}$. Let E be the function field of $\mathbb{P}^1_K$ and let $\hat{F}$ be a finite Galois extension of E which is regular over K. Suppose the morphism $\varphi\colon D \to \mathbb{P}^1_K$ introduced at the beginning of this section (in particular φ satisfies (6.1)) is the normalization of $\mathbb{P}^1_K$ in $\hat{F}$ [Liu06, p. 120, Def. 1.24].*

Let $\mathbf{p}$ be a point in $D(\tilde{K})$ such that $\tilde{\varphi}(\mathbf{p}) = (1{:}0)$ and let $\mathcal{V}$ be a v-open neighborhood of $\mathbf{p}$ in $D(\tilde{K})$. Then $\mathcal{V} \cap D(K) \neq \emptyset$.

Proof. Let $G = \mathrm{Gal}(\hat{F}/E)$. Since $\hat{F}$ is a regular extension of K, we may identify G with $\mathrm{Gal}(\hat{F}\tilde{K}/E\tilde{K})$. Note that $\hat{F}\tilde{K}/E\tilde{K}$ is the function field extension that corresponds to the morphism $\tilde{\varphi}\colon \tilde{D} \to \mathbb{P}^1_{\tilde{K}}$. Thus, the action of G on D extends to an action of G on $\tilde{D}$. Moreover, $\mathbb{P}^1_{\tilde{K}}$ is normal and $\mathcal{O}_{\mathbb{P}^1_{\tilde{K}}}$ is the fixed subsheaf under the action of G on $\mathcal{O}_{\tilde{D}}$. Also, by [Liu06, p. 121, Prop. 1.25], φ is finite, hence $\tilde{\varphi}$ is finite. It follows from Lemma 5.2 (with $\tilde{K}$ replacing K, $\tilde{\varphi}\colon \tilde{D} \to \mathbb{P}^1_{\tilde{K}}$ replacing $\psi\colon W \to V$, and $\hat{F}\tilde{K}/E\tilde{K}$ replacing F/E) that if $\mathbf{q}, \mathbf{q}' \in D(\tilde{K})$ and $\tilde{\varphi}(\mathbf{q}) = \tilde{\varphi}(\mathbf{q}')$, then there exists $\sigma \in G$ such that $\sigma\mathbf{q} = \mathbf{q}'$. Since σ fixes the elements of K, we have

(6.2) if $\mathbf{q} \in D(K)$, then $\mathbf{q}' = \sigma\mathbf{q} \in \sigma(D(K)) = D(K)$.

By Proposition 4.5, $\mathcal{U} = \tilde{\varphi}(\mathcal{V})$ is a v-open neighborhood of (1:0) in $\mathbb{P}^1(\tilde{K})$. Hence, there exists $a \in K^\times$ such that

(6.3) if $b \in \tilde{K}^\times$ and $v(b) \geq v(a)$, then $(1{:}b) \in \mathcal{U}$.

By definition, $I_{(1:a^2)} \subset D \times D$. Avoiding finitely many elements of $K^\times$, we may use Lemma 5.1 to choose a in K such that the curve $I_{(1:a^2)}$ is absolutely irreducible and is defined over K. Moreover, $H_{(1:a^2)} = \delta(I_{(1:a^2)})$ is defined by the equation $a^2X_0Y_0 = X_1Y_1$. Let U be the nonempty Zariski-open subset of $H_{(1:a^2)}$ defined by the inequalities $X_0 \neq 0$ and $Y_0 \neq 0$. Let $V = \delta^{-1}(U)$. Since K is PAC, there exists $(\mathbf{q}, \mathbf{r}) \in V(K)$. That is, $\mathbf{q}, \mathbf{r} \in D(K)$, $\varphi(\mathbf{q}) = (1{:}b)$, $\varphi(\mathbf{r}) = (1{:}c)$, and $bc = a^2$ for some $b, c \in K^\times$. Thus, $v(b) + v(c) = 2v(a)$. We may assume without loss that $v(b) \geq v(a)$. By (6.3), $\tilde{\varphi}(\mathbf{q}) = (1{:}b) \in \mathcal{U}$. Hence, there exists $\mathbf{q}' \in \mathcal{V}$ with $\tilde{\varphi}(\mathbf{q}') = \tilde{\varphi}(\mathbf{q})$. Since $\mathbf{q} \in D(K)$, it follows from (6.2) that $\mathbf{q}' \in D(K)$. Consequently, $\mathcal{V} \cap D(K) \neq \emptyset$. □

6 On the Density Property of PAC Fields

We prove Kollár's result saying that if V is an absolutely irreducible variety defined over a PAC field K and v is a valuation of $\tilde{K}$, then $V(K)$ is v-dense in $V(\tilde{K})$.

Remark 6.1. Let F/E be a Galois extension of degree n.

Claim: *If w, w' are valuations of F such that $O_w = O_{w'}$ and $w|_E = w'|_E$, then $w = w'$* Indeed, let $x \in F$. Since $(w(F^\times) : w(E^\times))|n$, there exists $a \in E$ such that $nw(x) = w(a)$. Hence, $w(x^n a^{-1}) = 0$, so $x^n a^{-1} \in O_w^\times$. Therefore, $x^n a^{-1} \in O_{w'}^\times$, hence, $w'(x^n a^{-1}) = 0$, so $nw'(x) = w'(a)$. Since $w(a) = w'(a)$, we get $nw(x) = nw'(x)$, consequently $w(x) = w'(x)$, as claimed.

In particular, if an element σ of $\mathrm{Gal}(F/E)$ belongs to the decomposition group D_w of w over E, then $O_{w\circ\sigma} = O_w$. In addition $w \circ \sigma|_E = w|_E$. Hence, by the claim, $w = w \circ \sigma$.

Now suppose that $w_1, \ldots, w_m$ are all of the extensions to F (up to equivalence) of a valuation v of E. Then $O_{w_1}, \ldots, O_{w_m}$ are all of the valuation rings of F whose intersections with E are O_v. Then for each $1 \leq i \leq n$ there exists $\sigma_i \in \mathrm{Gal}(F/E)$ such that $O_{w_1} = \sigma_i O_{w_i}$, that is $O_{w_1 \circ \sigma_i} = O_{w_i}$. By the preceding paragraph, $w_i = w_1 \circ \sigma_i$.

Lemma 6.2. *Let (E, v) be a valued field, F a finite separable extension of E, and $\hat{F}$ the Galois closure of F/E. Suppose v totally splits in F. Then v totally splits in $\hat{F}$.*

Proof. Let w be a valuation of $\hat{F}$ lying over v. It suffices to prove that the decomposition group $D_{w/v}$ of w over v is trivial. Consider $\sigma \in D_{w/v}$.

Claim: $\sigma \in \mathrm{Gal}(\hat{F}/F)$. Let $d = [F : E]$. By assumption F has d distinct valuations $v_1, \ldots, v_d$ extending v. For each $1 \leq i \leq d$ we extend v_i to a valuation

w_i of $\hat{F}$ such that $w_1 = w$. By Remark 6.1, there exists $\sigma_i \in \mathrm{Gal}(\hat{F}/E)$ such that $w_i = w \circ \sigma_i$ and $\sigma_1 = 1$. If some $1 \le j \le d$ satisfies $\sigma_i \mathrm{Gal}(\hat{F}/F) = \sigma_j \mathrm{Gal}(\hat{F}/F)$, then for each $x \in F$ we have $v_i(x) = w_i(x) = w(\sigma_i x) = w(\sigma_j x) = w_j(x) = v_j(x)$, so $v_i = v_j$, hence $i = j$. Thus, $\sigma_1 \mathrm{Gal}(\hat{F}/F), \ldots, \sigma_d \mathrm{Gal}(\hat{F}/F)$ are distinct cosets of $\mathrm{Gal}(\hat{F}/F)$ in $\mathrm{Gal}(\hat{F}/E)$. Since $(\mathrm{Gal}(\hat{F}/E) : \mathrm{Gal}(\hat{F}/F)) = d$, we have $\mathrm{Gal}(\hat{F}/E) = \dot{\bigcup}_{i=1}^d \sigma_i \mathrm{Gal}(\hat{F}/F)$.

It follows that $\sigma = \sigma_i \eta$ for some $1 \le i \le d$ and $\eta \in \mathrm{Gal}(\hat{F}/F)$. If $2 \le i \le d$, then $v_1 = w|_F = w \circ \sigma|_F = w \circ \sigma_i \circ \eta|_F = w_i|_F = v_i$, which is a contradiction. It follows that $i = 1$, so $\sigma \in \mathrm{Gal}(\hat{F}/F)$, as claimed.

Now, since v totally splits in F, it totally splits in each of the conjugates F' of F over E. By the claim, σ belongs to $\mathrm{Gal}(\hat{F}/F')$. Since the compositum of all of the fields F' is $\hat{F}$, we conclude that $\sigma = 1$, as asserted. □

When we speak about a function field of one variable F/K, we always assume that F/K is a regular extension [FrJ08, Section 3.1]. If $D = \sum_{i=1}^n a_i P_i$ is a divisor of F/K with distinct prime divisors $P_1, \ldots, P_n$ and integral coefficients $a_1, \ldots, a_n$, we write $v_P(D) = a_i$ for a prime divisor P of F/K, if $P = P_i$ for some i between 1 and n, and $v_P(D) = 0$ otherwise. The **divisor** and the **pole divisor** of an element $f \in F^\times$ are $\mathrm{div}(f) = \sum_P v_P(f)P$ and $\mathrm{div}_\infty(f) = -\sum_{v_P(f)<0} v_P(f)P$, where P ranges over all prime divisors of F/K and v_P is here the normalized valuation of F associated with P. Note that $\deg(\mathrm{div}_\infty(f)) = [F : K(f)]$. We say that $\mathrm{div}_\infty(f)$ **totally splits** in F if $\mathrm{div}_\infty(f) = P_1 + \cdots + P_m$, where $m = [F : K(f)]$ and $P_1, \ldots, P_m$ are distinct prime divisors of F/K. This holds if and only if the valuation v_∞ of $K(f)/K$ defined by $v_\infty(f) = -1$ totally splits in F.

Lemma 6.3. *Let F be a function field of one variable over a PAC field K. Then F/K has a separating transcendental element f such that the Galois closure $\hat{F}$ of $F/K(f)$ is a regular extension of K. Moreover, given a prime divisor P of $F\tilde{K}/\tilde{K}$, we may choose $f \in F$ such that $v_P(f) > 0$ and $\mathrm{div}_\infty(f)$ totally splits in F and in $\hat{F}$.*

Proof. The prime divisor P is already defined over a finite extension M of K. Let $\sigma_1, \ldots, \sigma_d$ be the distinct K-embeddings of M into $\tilde{K}$. If $p = \mathrm{char}(K) > 0$, let $q = p^j$ be the inseparable degree of M/K. If $\mathrm{char}(K) = 0$, put $q = 1$. Then $D = \sum_{i=1}^d q\sigma_i P$ is a positive divisor of F/K and $v_P(D) > 0$.

Since K is PAC, F/K has for each positive integer m distinct prime divisors $Q_1, \ldots, Q_m$ of degree 1 with $Q_i \ne P$, $i = 1, \ldots, m$. Taking m sufficiently large, there exists by Riemann-Roch an element $f \in F^\times$ such that

(7.1a) $\mathrm{div}(f) + Q_1 + \cdots + Q_m \ge D$,

and

(7.1b) $\mathrm{div}_\infty(f) = Q_1 + \cdots + Q_m$.

By (7.1a), $v_P(f) > 0$. By (7.1b), $[F : K(f)] = \deg(\mathrm{div}_\infty(f)) = m$, so by (7.1b) again, $\mathrm{div}_\infty(f)$ totally splits in F. In particular, $F/K(f)$ is a finite separable extension. Let $\hat{F}$ be the Galois closure of $F/K(f)$. Then, by Lemma 6.2, $\mathrm{div}_\infty(f)$ totally splits in $\hat{F}$. In particular, $\hat{F}$ has a K-valuation with residue field K. Thus, by [FrJ08, p. 42, Lemma 2.6.9], $\hat{F}$ is regular over K. □

Remark 6.4 (Comparison of proofs)**.** We say that a field K is **stable** if each finitely generated regular extension F of K has a separating transcendence base $\mathbf{t}$ such that the Galois closure $\hat{F}$ of $F/K(\mathbf{t})$ is regular over K. The stability property of PAC fields is proved in [FrJ76]. The essential case in the proof is that where F is a function field of one variable over K. In that case [FrJ76, Thm. 2.3] constructs for each large prime number l a separating transcendental element t for F/K such that the pole divisor of t over $\tilde{K}(t)$ decomposes as $P_1 + \cdots + P_{l-2} + 2P_{l-1}$, where $P_1, \ldots, P_{l-2}, P_{l-1}$ are distinct prime divisors of $F\tilde{K}/\tilde{K}$. This leads to the conclusion that the Galois closure $\hat{F}$ of $F/K(t)$ satisfies $\mathrm{Gal}(\hat{F}/K(t)) \cong S_l$, from which the regularity of $\hat{F}/K$ easily follows [FrJ76, Lemma 2.1].

On the other hand, the proof of Lemma 6.3, due to Kollár, generates a separating transcendental element f of F/K for each prime divisor P of F/K of degree 1 such that P is a zero of f and the pole divisor of f over $K(t)$ totally splits in F and is of arbitrary large degree, not necessarily prime. This implies that each of the pole divisors of f in $\hat{F}/K$ is of degree 1, so $\hat{F}/K$ is regular. However, that proof gives no clue for the Galois group $\mathrm{Gal}(\hat{F}/K(f))$.

The next lemma is a standard result of algebraic geometry (see [Lan58, p. 152, Cor.] or [Har77, p. 43, Prop. 6.8]).

Lemma 6.5. *Let $f\colon C \to C'$ be a rational map of absolutely irreducible curves defined over a field K with C' projective and C normal. Then, f is a morphism.*

Proof. We have to prove that f is defined at each point $\mathbf{q}$ of C. Replacing C, if necessary by an affine open neighborhood of $\mathbf{q}$, we may assume that C is affine. Let $\mathbf{x}$ be a generic point of C and $\mathbf{y} = (y_0{:}y_1{:}\cdots{:}y_n)$ a homogeneous generic point of C' such that $f(\mathbf{x}) = \mathbf{y}$. Assume without loss that $y_0, y_1, \ldots, y_n$ belong to the function field F of C' over K. Since C is a normal curve, $\mathcal{O}_{C,\mathbf{q}}$ is a discrete valuation ring. Denote the corresponding valuation of F/K by $v_{\mathbf{q}}$. Now let u be an element of F with $v_{\mathbf{q}}(u) = \min(v_{\mathbf{q}}(y_0), v_{\mathbf{q}}(y_1), \ldots, v_{\mathbf{q}}(y_n))$. Then each of the elements $u^{-1}y_i$, $i = 0, \ldots, n$, belongs to $\mathcal{O}_{C,\mathbf{q}}$ and at least one of them is a unit. Hence, f is defined at $\mathbf{q}$ and its value is $((u^{-1}y_0)(\mathbf{q}){:}(u^{-1}y_1)(\mathbf{q}){:}\cdots{:}(u^{-1}y_n)(\mathbf{q}))$. □

Theorem 6.6 (Density theorem)**.** *Let K be a PAC field, v a valuation of $\tilde{K}$, and V an absolutely irreducible variety defined over K. Then, $V(K)$ is v-dense in $V(\tilde{K})$.*

Proof. We break up the proof into several parts.

PART A: *We prove that $C(K)$ is v-dense in $C(\tilde{K})$ for each absolutely irreducible projective normal curve C which is defined over K.* Let $\mathbf{p} \in C(\tilde{K})$ and let $\mathcal{U}$ be a v-open neighborhood of $\mathbf{p}$ in $C(\tilde{K})$. Denote the function field of C over K by F. Then F is an algebraic function field of one variable over K which is regular over K [FrJ08, p. 175, Cor. 10.2.2(a)]. Let P be a prime divisor of $F\tilde{K}/\tilde{K}$ whose center at $C_{\tilde{K}}$ is $\mathbf{p}$. Lemma 6.3 gives an $f \in F$ such that $F/K(f)$ is a finite separable extension and the Galois closure $\hat{F}$ of $F/K(f)$ is a regular extension of K. It follows that we may identify $G = \mathrm{Gal}(\hat{F}\tilde{K}/\tilde{K}(f))$ with $\mathrm{Gal}(\hat{F}/K(f))$ via the restriction map. Moreover,

(7.2) $v_P(f) > 0$ and $\mathrm{div}_\infty(f)$ totally splits in $\hat{F}$.

We consider f also as a rational map from C into $\mathbb{P}^1_K$. By Lemma 6.5, f is a morphism. Since C is projective, f is proper [GoW10, p. 386, Cor. 13.41]. By (7.2), f is not constant, hence each of the fibers of f is finite (Otherwise there exists a point $\mathbf{a} \in \mathbb{P}^1_K$ such that $f^{-1}(\mathbf{a})$ is infinite. Since the fiber is closed and C is an irreducible curve, $f^{-1}(\mathbf{a}) = C$, hence $f(C) = \mathbf{a}$, in contrast to the former conclusion.) i.e. f is quasi-finite. It follows that $f\colon C \to \mathbb{P}^1_K$ is a finite morphism [GoW10, p. 358, Cor. 89]. The corresponding function field extension is $F/K(f)$.

Now let $\pi\colon D \to C$ be the projective normalization of C in $\hat{F}$ [Lan58, p. 143, Thm. 5], in particular, D is normal and π is finite [Liu06, p. 121, Prop. 1.25]. Then $\varphi = f \circ \pi$ is a finite morphism of D onto $\mathbb{P}^1_K$. It follows that $\varphi\colon D \to \mathbb{P}^1_K$ is the normalization of $\mathbb{P}^1_K$ in $\hat{F}$ [Liu06, p. 120, Def. 1.24].

We may interpret (7.2) as

(7.3) $f(\mathbf{p}) = (1{:}0)$ and $|\varphi^{-1}((0{:}1))| = [\hat{F} : K(f)]$.

In particular, φ is unramified over (0:1). Since the local ring $\mathcal{O}_{\mathbb{P}^1_K,(0:1)}$ is a discrete valuation ring, it is a Dedekind domain. Therefore, each of the local rings of D lying over $\mathcal{O}_{\mathbb{P}^1_K,(0:1)}$ is flat over $\mathcal{O}_{\mathbb{P}^1_K,(0:1)}$ [Liu06, p. 11, Corollary 2.14]. It follows that φ is étale over (0:1), hence smooth over (0:1).

Since $\pi\colon D \to C$ is finite, so is $\tilde{\pi}\colon \tilde{D} \to \tilde{C}$ [GoW10, p. 325, Prop. 12.11(2)]. Hence, the map $\tilde{\pi}\colon D(\tilde{K}) \to C(\tilde{K})$ is surjective [GoW10, p. 339, Prop. 12.43(2)] and v-continuous. Let $\mathbf{q}$ be a point in $D(\tilde{K})$ lying over $\mathbf{p}$. By (7.3), $\tilde{\varphi}(\mathbf{q}) = (1{:}0)$ and $\mathcal{V} = \tilde{\pi}^{-1}(\mathcal{U})$ is an open neighborhood of $\mathbf{q}$ in $D(\tilde{K})$. By Lemma 5.3, there exists $\mathbf{q}' \in \mathcal{V} \cap D(K)$. Then, $\mathbf{p}' = \pi(\mathbf{q}') \in \mathcal{U} \cap C(K)$. Thus, $C(K)$ is v-dense in $C(\tilde{K})$.

PART B: *We prove that $C(K)$ is v-dense in $C(\tilde{K})$ for each absolutely irreducible curve C which is defined over K.* Again, let $\mathbf{p} \in C(\tilde{K})$ and let $\mathcal{U}$ be a v-open neighborhood of $\mathbf{p}$ in $C(\tilde{K})$. Let C_{simp} be the Zariski-open subset of C consisting of all simple points. Then $C_{\mathrm{simp}}(\tilde{K})$ is v-open and v-dense in $C(\tilde{K})$ [GeJ75, Lemma 2.2], in particular, $\mathcal{U}_0 = C_{\mathrm{simp}}(\tilde{K}) \cap \mathcal{U}$ is nonempty and v-open. Replacing $\mathcal{U}$ by $\mathcal{U}_0$ and C by C_{simp}, we may assume that C is smooth. Similarly, replacing C by a nonempty Zariski-open affine subset, we may assume that $C \subseteq \mathbb{A}_K^n$ for some positive integer n.

Let C^* be the Zariski-closure of C in $\mathbb{P}_K^n$. In particular, we may view C as a Zariski-open subset of C^*. Let $\pi\colon D \to C^*$ be the projective normalization of C^* [Lan58, p. 143, Thm. 5]. Since C is normal, the restriction of π to $\pi^{-1}(C)$ is an isomorphism. Since $\pi^{-1}(C)$ is Zariski-open in D, $\tilde{\pi}^{-1}(\mathcal{U})$ is a nonempty v-open subset of $D(\tilde{K})$. By Part A, $\tilde{\pi}^{-1}(\mathcal{U}) \cap D(K) \neq \emptyset$. Hence, by [GeJ75, Lemma 2.2], $\tilde{\pi}^{-1}(\mathcal{U}) \cap \tilde{\pi}^{-1}(C(K)) \neq \emptyset$. It follows that $\mathcal{U} \cap C(K) \neq \emptyset$, as desired.

Part C: *The general case.* Again, let $\mathcal{U}$ be a nonempty v-open subset of V. By [GeJ75, Lemma 2.4], $V(K_s)$ is v-dense in $V(\tilde{K})$. Hence, we may assume that $\mathcal{U} \cap V(K_s)$ contains a point $\mathbf{p}$. Let C be an absolutely irreducible subcurve of V defined over K with $\mathbf{p} \in C(K_s)$ [JaR98, Lemma 10.1]. Then, $\mathcal{U} \cap C(\tilde{K})$ is a v-open neighborhood of $\mathbf{p}$ in $C(\tilde{K})$. By Part B, $\mathcal{U} \cap C(K) \neq \emptyset$. Consequently, $\mathcal{U} \cap V(K) \neq \emptyset$, as desired. □

7 Embedding Lemma

The essential step in the proof of Abraham Robinson's result about the model completeness of the theory of algebraically closed valued fields is the embedding lemma 7.3 that we prove below. Our presentation follows that of Alexander Prestel in [Pre86, pp. 236–241]. In this section and the following ones we allow valuations of fields to be trivial.

Lemma 7.1. *Let K be an algebraically closed field, E a field extension of K, and v a valuation of E. We denote the valuation ring of v by O_v and use a bar to denote reduction modulo v.*

(a) $\bar{K}$ is algebraically closed, $v(K^\times)$ is a divisible group, and division in $v(K^\times)$ by each $n \in \mathbb{N}$ is unique. Moreover, if K is an algebraic closure of a subfield K_0, then $v(K^\times)$ is the divisible closure of $v(K_0^\times)$.

(b) The ordered group $\Gamma = v(K^\times)$ is dense in itself. That is, for all $\alpha, \beta \in \Gamma$ with $\alpha < \beta$ there exists $\gamma \in \Gamma$ such that $\alpha < \gamma < \beta$. Moreover, if v is non-trivial, then for each $\alpha \in \Gamma$ there exist $\delta, \delta' \in \Gamma$ such that $\delta < \alpha < \delta'$.

(c) Let x be an element of E such that $\bar{x}$ is transcendental over $\bar{K}$. Then for all $a_0, \dots, a_n \in K$ we have $v(\sum_{i=0}^n a_i x^i) = \min(v(a_0), \dots, v(a_n))$.

(d) Let x be an element of $E^\times$ such that $v(x) \notin v(K^\times)$. Then the order of $v(x)$ in $v(E^\times)$ is infinite and $v(K(x)^\times) = v(K^\times) \oplus \mathbb{Z}v(x)$.

Proof.

Proof of (a): Consider a polynomial $\bar{f}(X) = X^n + \bar{a}_{n-1}X^{n-1} + \cdots + \bar{a}_0$, with $n \geq 1$ and $a_0, \ldots, a_{n-1} \in O_v \cap K$. Since K is algebraically closed, there exists $x \in K$ with $f(x) = 0$. If $v(x) < 0$, then $v(x^{-1}) > 0$ and $1 + a_{n-1}x^{-1} + \cdots + a_0x^{-n} = 0$. Taking residues on both sides, we get the contradiction $1 = 0$. Thus, $v(x) \geq 0$ and $\bar{f}(\bar{x}) = 0$, as desired.

Now let $a \in K^\times$ and let n a positive integer. Then $a^{1/n} \in K^\times$ and $v(a) = nv(a^{1/n})$. Hence, $v(K^\times)$ is divisible. Since $v(K^\times)$ is an ordered group, division by n is unique. Finally, $e = (v(K_0(a)^\times) : v(K_0^\times)) < \infty$. Hence, $ev(a) = v(a_0)$ for some $a_0 \in K_0$.

Proof of (b): Using (a), we may take $\gamma = \frac{\alpha+\beta}{2}$ to prove the first claim in (b). If v is non-trivial, there exists a positive $\varepsilon \in \Gamma$. Then $\alpha - \varepsilon < \alpha < \alpha + \varepsilon$, as desired.

Proof of (c): We choose a $0 \leq j \leq n$ with $v(a_j) = \min(v(a_0), \ldots, v(a_n))$ and let $b_i = a_i a_j^{-1}$, $i = 1, \ldots, n$. Then $\bar{b}_j = 1$ and $\bar{b}_i \in \bar{K}$ for $i = 0, \ldots, n$. Since $\bar{x}$ is transcendental over $\bar{K}$, we have $\sum_{i=0}^n \bar{b}_i\bar{x}^i \neq 0$, so $v(\sum_{i=0}^n b_ix^i) = 0$. Therefore, $v(\sum_{i=0}^n a_ix^i) = v(a_j) + v(\sum_{i=0}^n b_ix^i) = \min(v(a_0), \ldots, v(a_n))$, as claimed.

Proof of (d): First note that the order of $v(x)$ is not only infinite but even infinite modulo $v(K^\times)$. Indeed, if there exist $n \in \mathbb{N}$ and $a \in K^\times$ such that $nv(x) = v(a)$, then $v(x) = v(a^{1/n}) \in v(K^\times)$, in contrast to the assumption on x.

Next consider an element $\sum_{i=0}^n a_ix^i$ in $K[x]$. If $0 \leq i < j \leq n$, then $v(a_ix^i) \neq v(a_jx^j)$. Otherwise, $(j-i)v(x) = v(a_ia_j^{-1}) \in v(K)$, in contrast to the preceding paragraph. It follows that

$$v\Big(\sum_{i=0}^n a_ix^i\Big) = \min(v(a_0), v(a_1) + v(x), \ldots, v(a_n) + nv(x)) \in v(K^\times) + \mathbb{Z}v(x).$$

By the preceding paragraph, the sum on the right hand side is direct. □

An **embedding** of a valued field (E, v) into a valued field (F, w) is a pair (φ, φ'), where $\varphi\colon E \to F$ is an embedding of fields, $\varphi'\colon v(E^\times) \to w(F^\times)$ is an embedding of ordered groups, and $w(\varphi(e)) = \varphi'(v(e))$ for each $e \in E^\times$. In the sequel we sometimes abuse our language and write φ also for φ' and also for the pair (φ, φ'). If K is a common subfield of E and F, φ is the identity on K, and φ' is the identity on $v(K^\times)$, we say that φ is a K-**embedding**.

Lemma 7.2. *Let $\varphi_0\colon E_0 \to F_0$ be an isomorphism of fields. Let (E, v) and (F, w) be valued fields such that E is an algebraic extension of E_0 and F is a field extension of F_0 which is algebraically closed. Let $\varphi_0'\colon v(E_0^\times) \to w(F_0^\times)$ be an isomorphism of valued groups. Suppose $w(\varphi_0(e)) = \varphi_0'(v(e))$ for each $e \in E_0^\times$. Then it is possible to extend φ_0 to an embedding $\varphi\colon E \to F$ and to extend φ_0' to an embedding $\varphi'\colon v(E^\times) \to w(F^\times)$ of ordered groups such that $w(\varphi(e)) = \varphi'(v(e))$ for each $e \in E^\times$.*

Proof. We choose an algebraic closure $\tilde{E}_0$ of E_0 that contains E and an algebraic closure $\tilde{F}_0$ of F_0 in F. Then we extend φ_0 to an isomorphism $\tilde{\varphi}\colon \tilde{E}_0 \to \tilde{F}_0$. By Chevalley's theorem, v extends to a valuation $\tilde{v}$ of $\tilde{E}_0$. Let $\tilde{\gamma} \in v(\tilde{E}_0^\times)$ (resp. $\tilde{\delta} \in w(\tilde{F}_0^\times)$). Then, there exists $n \in \mathbb{N}$ and there exists a unique $\gamma_0 \in v(E_0^\times)$ (resp. $\delta_0 \in w(F_0^\times)$) such that $\gamma_0 = n\tilde{\gamma}$ (resp. $\delta_0 = n\tilde{\delta}$) (Lemma 7.1(a)). Hence, φ_0' uniquely extends to an isomorphism $\widetilde{\varphi'}\colon \tilde{v}(\tilde{E}_0^\times) \to w(\tilde{F}_0^\times)$. Then, $\tilde{w} = w|_{\tilde{F}_0}$ and $\widetilde{\varphi'} \circ \tilde{v} \circ \tilde{\varphi}^{-1}$ are valuations of $\tilde{F}_0$ that coincide on F_0. Hence, there exists $\sigma \in \mathrm{Aut}(\tilde{F}_0/F_0)$ such that $\tilde{w} \circ \sigma = \widetilde{\varphi'} \circ \tilde{v} \circ \tilde{\varphi}^{-1}$ [Efr06, p. 131, Thm. 14.3.2], so $\tilde{w} = \widetilde{\varphi'} \circ \tilde{v} \circ (\sigma \circ \tilde{\varphi})^{-1}$. Then, $\varphi = \sigma \circ \tilde{\varphi}|_E$ is an embedding of E into F that extends φ_0 and $\varphi' = \widetilde{\varphi'}|_{E^\times}$ is an embedding of $v(E^\times)$ into $w(F^\times)$ that extends φ_0' such that $w(\varphi(e)) = \varphi'(v(e))$ for each $e \in E^\times$. □

Recall that a structure $\mathcal{A}$ of a language $\mathcal{L}$ with a domain A is **$\aleph_1$-saturated** if it satisfies the following condition: Let $\varphi_1, \varphi_2, \varphi_3, \ldots$ be formulas of $\mathcal{L}$ in the free variables $X_1, X_2, X_3, \ldots$ with parameters in A. Suppose for each n there exist $a_1, a_2, a_3, \ldots \in A$ such that $\varphi_1(\mathbf{a}), \ldots, \varphi_n(\mathbf{a})$ hold in $\mathcal{A}$. Then there exist $x_1, x_2, x_3, \ldots \in A$ such that each $\varphi_n(\mathbf{x})$ holds in $\mathcal{A}$ [FrJ08, p. 143].

Lemma 7.3 (Embedding lemma)**.** *Let K be a countable algebraically closed field, (E, v) a valued field such that E is a function field of one variable over K, and (F, w) an $\aleph_1$-saturated algebraically closed non-trivial valued field such that $K \subseteq F$ and $v|_K = w|_K$. Then there exists a K-embedding $\varphi\colon (E, v) \to (F, w)$.*

Proof. Let $O = O_v \cap K = O_w \cap K$ and use a bar to denote reduction with respect to both v and w. In particular, $\bar{K}, \bar{E}, \bar{F}$ are the residue fields of K, E, F, respectively, $\bar{K} \subseteq \bar{E}, \bar{F}$, and both $\bar{K}$ and $\bar{F}$ are algebraically closed (Lemma 7.1(a)).

If $x \in E$ is transcendental over K, then by assumption, E is algebraic over $E_0 = K(x)$. Hence, in order to prove the lemma, it suffices by Lemma 7.2 to prove the following claim.

Claim: *There exist $x \in E$ and $y \in F$ transcendental over K, and there exists a K-isomorphism*

$$\varphi\colon (K(x), v(K(x)^\times)) \to (K(y), w(K(y)^\times))$$

such that $\varphi(x) = y$ and $w(\varphi(e)) = \varphi'(v(e))$ for each $e \in K(x)^\times$.

The proof of the Claim splits into three cases.

Case 1: $\bar{K} \neq \bar{E}$. We choose $x \in O_v$ such that $\bar{x} \notin \bar{K}$. Then $x \notin K$, so x is transcendental over K.

Since $\bar{K}$ is algebraically closed, $\bar{K}$ is infinite. Hence, for all $a_1, \ldots, a_n \in O$ there exists $y \in O$ such that $\bar{y} \neq \bar{a}_i$, so $w(y - a_i) = v(y - a_i) = 0$ for $i = 1, \ldots, n$. Since O is countable and (F, w) is $\aleph_1$-saturated, there exists $y \in O_w$ such that $w(y - a) = 0$ for each $a \in O$. This means that $\bar{y} \notin \bar{K}$. As in the preceding paragraph, y is transcendental over K. Therefore, there is a unique K-isomorphism $\varphi\colon K(x) \to$

$K(y)$ such that $\varphi(x) = y$. Moreover, since both $\bar{x}$ and $\bar{y}$ are transcendental over $\bar{K}$, Lemma 7.1(c) implies that $w(\sum_{i=0}^n a_i y^i) = v(\sum_{i=0}^n a_i x^i)$ for all $a_0, \ldots, a_n \in K$. Thus, $w(\varphi(e)) = v(e) = \varphi'(v(e))$, where $\varphi' = \mathrm{id}_{v(K(x)^\times)}$, for all $e \in K(x)^\times$, as desired.

Case 2: $v(E^\times) \neq v(K^\times)$. We choose $x \in E^\times$ such that

(8.1) $v(x) \notin v(K^\times)$.

We consider $a_1, \ldots, a_n \in K$ and assume without loss that

(7.2) $$v(a_1) \le \cdots \le v(a_m) < v(x) < v(a_{m+1}) \le \cdots \le v(a_n)$$

for some m between 0 and n. By convention, if $m = 0$, then relation (8.2) becomes $v(x) < v(a_1) \le \cdots \le v(a_n)$ and if $m = n$, then relation (8.2) simplifies to $v(a_1) \le \cdots \le v(a_n) < v(x)$. By assumption, $w(a_i) = v(a_i)$ for $i = 1, \ldots, n$. If $m = 0$, then Lemma 7.1(b) gives $y \in F$ such that $w(y) < w(a_1)$. If $m = n$, then Lemma 7.1(b) gives $y \in F$ such that $w(a_n) < w(y)$. Otherwise, Lemma 7.1(b) gives $y \in F$ such that $w(a_m) < w(y) < w(a_{m+1})$. Note that the first two cases use the assumption that w is non-trivial.

Since K is countable and (F, w) is $\aleph_1$-saturated, there exists $y \in F$ such that

(8.3) for all $a \in K$, $v(x) < v(a)$ implies that $w(y) < w(a)$, and $v(x) > v(a)$ implies that $w(y) > w(a)$.

In particular,

(8.4) $y \notin K$ and $w(y) \notin w(K^\times)$.

Since K is algebraically closed, both x and y are transcendental over K. Let $\varphi\colon K(x) \to K(y)$ be the unique K-isomorphism with $\varphi(x) = y$. By (8.1) (resp. (8.4)) and Lemma 7.1(d), the order of $v(x)$ (resp. $w(y)$) modulo $v(K^\times)$ is infinite and

(8.5) $v(K(x)^\times) = v(K^\times) \oplus \mathbb{Z}v(x)$ (resp. $w(K(y)^\times) = v(K^\times) \oplus \mathbb{Z}w(y)$).

Hence, there is an isomorphism $\varphi'\colon v(K(x)^\times) \to v(K(y)^\times)$ of ordered groups (by (8.3)) which is the identity map on $v(K^\times)$ such that $\varphi'(v(x)) = w(y)$. It follows from (8.3) that $\varphi'(v(x - a)) = w(y - a)$ for each $a \in K$. Hence,

$$\varphi'(v(a_0 \prod_{i=1}^m (x - a_i))) = w(a_0 \prod_{i=1}^m (y - a_i))$$

for all $a_0, a_1, \ldots, a_m \in K$ with $a_0 \neq 0$. Since K is algebraically closed, this implies that $w(\varphi(e)) = \varphi'(v(e))$ for each $e \in K(x)^\times$, as claimed.

Case 3: *$\bar{E} = \bar{K}$ and $v(E^\times) = v(K^\times)$*. We choose $x \in E$, transcendental over K and prove:

(8.6) For all $a_1, \dots, a_n \in K$ there exists $y \in K$ such that $w(y - a_i) = v(x - a_i)$, $i = 1, \dots, n$.

Indeed, by assumption there exists for each $1 \le i \le n$ an element $b_i \in K^\times$ such that $v(x - a_i) = v(b_i)$. We choose $1 \le j \le n$ such that $v(b_j) = \max(v(b_1), \dots, v(b_n))$. Then, $v((x - a_j)b_j^{-1}) = 0$, so by assumption, there exists $c \in K$ with $\bar{c} = \overline{(x - a_j)b_j^{-1}}$. Hence,

$$v\Big(\frac{x - (a_j + cb_j)}{b_j}\Big) = v\Big(\frac{x - a_j}{b_j} - c\Big) > 0.$$

Set $y = a_j + cb_j$. Then, $v(x - y) > v(b_j) \ge v(b_i) = v(x - a_i)$, hence

$$w(y - a_i) = v(y - a_i) = v((x - a_i) - (x - y)) = v(x - a_i), \qquad i = 1, \dots, n,$$

as claimed.

Since K is countable and (F, w) is $\aleph_1$-saturated, there exists $y \in F$ such that

(8.7) $w(y - a) = v(x - a)$ for all $a \in K$.

Since $x \notin K$, (8.7) implies that $y \notin K$. Since K is algebraically closed, y is transcendental over K. Let $\varphi\colon K(x) \to K(y)$ be the unique K-isomorphism with $\varphi(x) = y$. Again, since K is algebraically closed, each non-constant $u \in K[x]$ can be written as $u = c_0 \prod_{i=1}^m (x - c_i)$ with $c_0, c_1, \dots, c_m \in K$ and $c_0 \neq 0$, so $\varphi(u) = c_0 \prod_{i=1}^m (y - c_i)$. It follows from (8.7) that $w(\varphi(u)) = v(u)$. Hence, the latter relation holds for each $u \in K(x)^\times$, as desired. □

8 Algebraically Closed Valued Fields

The goal of this section is to prove Abraham Robinson's model completeness theorem for the theory of algebraically closed valued fields and also the stronger result about the elimination of quantifiers in that theory. For both goals we have to fix the first order language with which we want to speak about the algebraically closed fields. If we restrict ourselves only to the model completeness, it suffices to extend the language $\mathcal{L}(\text{ring})$ of rings [FrJ08, p. 135, Example 7.3.1] with a unary predicate symbol O whose interpretation in a valued field (K, v) is the valuation ring O_v.

We start with the language of rings $\mathcal{L}(\text{ring})$ with the constant symbols $0, 1$, the binary function symbol $+$ (for addition), the binary function symbol $\cdot$ (for multiplication), and the unary function symbol $-$ (for taking the negative)[2]. The

[2]Note that the symbol $-$ does not appear among the function symbols of $\mathcal{L}(\text{ring})$ in [FrJ08, p. 135, Example 7.3.1]. We have been forced here to include this symbol in $\mathcal{L}(\text{ring})$ in order to be able to prove Lemma 8.1(a).

axioms for fields in that language are:

$$(8.1)\quad \begin{aligned}
&(\forall X)(\forall Y)(\forall Z)[(X+Y)+Z = X+(Y+Z)];\\
&(\forall X)(\forall Y)[X+Y=Y+X];\\
&(\forall X)[X+0=X];\\
&(\forall X)[X+(-X)=0];\\
&(\forall X)(\forall Y)(\forall Z)[(XY)Z = X(YZ)];\\
&(\forall X)(\forall Y)[XY=YX];\\
&(\forall X)[1\cdot X = X];\\
&(\forall X)[X\neq 0 \rightarrow (\exists Y)[XY=1]];\\
&1\neq 0;\ \text{and}\\
&(\forall X)(\forall Y)(\forall Z)[X(Y+Z)=XY+XZ].
\end{aligned}$$

Note that a substructure R of a field K in the language $\mathcal{L}(\text{ring})$ is a subset of K that contains $0,1$ and is closed under addition, negation, and multiplication. Thus, R is an integral domain. If $K = \text{Quot}(R)$ is a valued field, then the main axiom for valued fields, "for all nonzero x we have $x\in O$ or $x^{-1}\in O$", does not make sense over R. This forces us to replace the monadic symbol O by the **division relation** associated with each valued field.

For each valued field (K,v) we define a binary relation $|_v$ on K by

$$(8.2)\qquad a|_v b \iff v(a)\le v(b).$$

It satisfies the following axioms (where we omit the index v):

(9.3a) $1|0$.

(9.3b) $(\forall X)[X|X]$.

(9.3c) $(\forall X)(\forall Y)(\forall Z)[X|Y \wedge Y|Z \rightarrow X|Z]$.

(9.3d) $(\forall X)(\forall Y)[X|Y \vee Y|X]$.

(9.3e) $(\forall X)(\forall Y)(\forall Z)[X|Y \rightarrow XZ|YZ]$.

(9.3f) $(\forall X)(\forall Y)(\forall Z)[X|Y \wedge X|Z \rightarrow X|(Y+Z)]$.

We call $|$ a **division relation** on K.

Conversely, if $|$ is a division relation on K, then $O = \{a\in K \mid 1|a\}$ is a valuation ring of K. Indeed, by (9.3a) and (9.3b), $0,1\in O$. By (9.3d), we have $1|-1$ or $-1|1$. By (9.3e), the latter case implies $1|-1$. Hence, in any case $-1\in O$. If $a,b\in O$, then $1|a$ and $1|b$, so $a+b\in O$, by (9.3f). By (9.3e), $a|ab$, so by (9.3c), $1|ab$, which implies $ab\in O$. Finally, if $a\in K^\times$, then $1|a$ or $a|1$ (by (9.3d)). In the latter case $1|a^{-1}$ (by (9.3e)), so in each case either $a\in O$ or $a^{-1}\in O$. Thus,

O defines a valuation of K (which may be trivial) whose division relation is $|$. It follows that the correspondence between valuations and division relations on K is bijective. We denote the division relation on K that corresponds to a valuation v of K by $|_v$. The advantage of the division relation is that it allows to treat valuation rings as first order structures.

The language of valued fields will therefore be the extension of $\mathcal{L}(\text{ring})$ by the division symbol $|$. We denote the extended language by $\mathcal{L}_{\text{val}}(\text{ring})$. Let T_{val} be the theory of $\mathcal{L}_{\text{val}}(\text{ring})$ that consists of the axioms (9.1) of fields and the axioms (9.3) for $|$.

Lemma 8.1. *Let (E, v) and (F, w) be valued fields. Let $\varphi_0\colon (R, |_{v,0}) \to (S, |_{w,0})$ be an isomorphism of substructures of $(E, |_v)$ and $(F, |_w)$, respectively. Let $K = \text{Quot}(R)$ and $L = \text{Quot}(S)$. Then*

(a) φ_0 extends to an isomorphism $\varphi\colon (K, v|_K) \to (L, w|_L)$ of valued fields.

(b) If E and F are algebraically closed, then φ extends to an isomorphism $\tilde{\varphi}\colon (\tilde{K}, v|_{\tilde{K}}) \to (\tilde{L}, w|_{\tilde{L}})$.

Proof. As usual, φ_0 extends to an isomorphism $\varphi\colon K \to L$ of the quotient fields. If $x, y \in K$ satisfy $x|_v y$, then there exists $b \in R$, $b \neq 0$, such that $bx, by \in R$, and then $bx|_v by$. Hence, $\varphi(bx)|_w \varphi(by)$, so $\varphi(b)\varphi(x)|_w \varphi(b)\varphi(y)$. Since $\varphi(b) \neq 0$, we have $\varphi(x)|_w \varphi(y)$. It follows that $\varphi\colon (K, v|_K) \to (L, w|_L)$ is an isomorphism of valued fields, which proves (a).

In order to prove (b), we use the valuations rather than the division relations. Let $\psi\colon \tilde{K} \to \tilde{L}$ be an isomorphism of fields that extends φ. Then, there exists $\sigma \in \text{Aut}(\tilde{L}/L)$ such that $\sigma \circ \psi\colon (\tilde{K}, v|_{\tilde{K}}) \to (\tilde{L}, w|_{\tilde{L}})$ extends $\varphi\colon (K, v|_K) \to (L, w|_L)$ [Efr06, p. 131, Thm. 14.3.21]. Thus, $\tilde{\varphi} = \sigma \circ \psi$ is the desired isomorphism. □

Whenever we speak about a "formula $\varphi(X_1, \ldots, X_n)$" we mean that the free variables of that formula belong to the set $\{X_1, \ldots, X_n\}$.

Definition 8.2. Let T be a theory in a first order language $\mathcal{L}$.

(a) We say that T is **model complete** if whenever a model $\mathcal{A}$ of T is a substructure of another model $\mathcal{B}$ of T, the model $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$.

(b) We say that T has the **amalgamation property** if whenever two models $\mathcal{B}, \mathcal{C}$ of T contain a common $\mathcal{L}$-substructure $\mathcal{A}$, there exists a model $\mathcal{D}$ of T and embeddings $f\colon \mathcal{B} \to \mathcal{D}$ and $g\colon \mathcal{C} \to \mathcal{D}$ that coincide on $\mathcal{A}$.

(c) We say that T admits **elimination of quantifiers** if for every formula $\varphi(X_1, \ldots, X_n)$ of the language $\mathcal{L}$ there exists a quantifier free formula $\psi(X_1, \ldots, X_n)$ such that for every model $\mathcal{A}$ of T with a domain A and for all $a_1, \ldots, a_n \in A$, the truth of $\varphi(\mathbf{a})$ in $\mathcal{A}$ is equivalent to the truth of $\psi(\mathbf{a})$ in $\mathcal{A}$.

We cite two theorems about the concepts just defined.

Proposition 8.3. *Let T be a theory in a first order language $\mathcal{L}$.*

(a) T admits elimination of quantifiers if and only if T is model complete and has the amalgamation property [Pre86, p. 193, Satz 3.22].

(b) T admits elimination of quantifiers if for every two models $\mathcal{B}, \mathcal{C}$ of T with domains B, C respectively, for every finitely generated common substructure $\mathcal{A}$ with domain A, and for every quantifier free formula $\eta(X)$ with parameters in A, the existence of $b \in B$ such that $\eta(b)$ holds in $\mathcal{B}$ implies the existence of $c \in C$ such that $\eta(c)$ holds in $\mathcal{C}$ [Pre86, p. 187, Satz 3.20].

A valued field (K, v) is non-trivial if and only if there exists $a \in K$ such that $v(a) > 0$, equivalently if

(9.4) there exists $x \in K$ such that $\neg(1|x)$.

Theorem 8.4 (Abraham Robinson)**.** *The theory of algebraically closed non-trivial valued fields in the language $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$ admits elimination of quantifiers, hence it is model complete and has the amalgamation property.*

Proof. By Proposition 8.3, it suffices to consider algebraically closed non-trivial valued fields (E, v) and (F, w), a common finitely generated substructure $(K, |)$ of $(E, |_v)$ and $(F, |_w)$, a quantifier free formula $\eta(X)$ with parameters in K such that there exists $x \in E$ with $E \models \eta(x)$, and to prove that there exists $y \in F$ such that $F \models \eta(y)$. In particular, K is a countable integral domain. By Lemma 8.1, we may replace K with the algebraic closure of $\mathrm{Quot}(K)$. We therefore assume that K is an algebraically closed field and use henceforth valuations rather than division relations.

Now recall that if $(F^*, w^*) = (F, w)^{\mathbb{N}}/\mathcal{D}$ is a nonprincipal ultrapower of (F, w), then (F^*, w^*) is $\aleph_1$-saturated [FrJ08, p. 143, Lemma 7.7.4] and is an elementary extension of (F, w) [FrJ08, p. 144, Prop. 7.7.5]. Replacing (F, w) by (F^*, w^*), if necessary, we may assume that (F, w) is $\aleph_1$-saturated.

Let η and x be as in the first paragraph of the proof. By the embedding lemma 7.3, there exists a K-embedding $\varphi\colon (K(x), v|_{K(x)}) \to (F, w)$ of valued fields. Set $y = \varphi(x)$. Since $\eta(X)$ is quantifier free, $\eta(x)$ holds in $(K(x), v|_{K(x)})$. Hence, $\eta(y)$ holds in $(K(y), w|_{K(y)})$ and therefore also in (F, w), as desired. □

9 Existential Closedness of PAC Fields

Using the density theorem 6.6, we prove that every PAC valued field (K, v) is existentially closed in each regular valued field extension (F, v) such that F is also PAC.

Theorem 9.1. *Let $(F, v)/(K, v)$ be an extension of non-trivial valued fields such that K is PAC and F/K is regular. Then (K, v) is existentially closed in (F, v) in the language $\mathcal{L}_{\rm val}(\text{ring})$.*

Proof. We break up the proof into two parts.

PART A: *Simplified form for existential formulas.* Let $\mathbf{X} = (X_1, \ldots, X_m)$ and $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be tuples of variables and let $f(\mathbf{X}, \mathbf{Y})$ and $g(\mathbf{X}, \mathbf{Y})$ be polynomials with coefficients in K. We may replace the formula $f(\mathbf{X}, \mathbf{Y})|g(\mathbf{X}, \mathbf{Y})$ by the equivalent formula $v(g(\mathbf{X}, \mathbf{Y})) \geq v(f(\mathbf{X}, \mathbf{Y}))$ and the formula $f(\mathbf{X}, \mathbf{Y}) \nmid g(\mathbf{X}, \mathbf{Y})$ by the equivalent formula $v(f(\mathbf{X}, \mathbf{Y})) > v(g(\mathbf{X}, \mathbf{Y}))$. Thus, every existential formula $\varphi(X_1, \ldots, X_m)$ in $\mathcal{L}_{\rm val}(\text{ring})$ with parameters in K is equivalent (in the theory of valued fields) to a formula of the form

$$
\begin{aligned}
(9.1)\qquad (\exists Y_1) \cdots (\exists Y_n) \bigvee_{i \in I} \bigwedge_{j \in J} \big[f_{ij}(\mathbf{X}, \mathbf{Y}) = 0 \wedge f'_{ij}(\mathbf{X}, \mathbf{Y}) \neq 0 \\
\wedge\, v(g_{ij}(\mathbf{X}, \mathbf{Y})) \geq v(g'_{ij}(\mathbf{X}, \mathbf{Y})) \\
\wedge\, v(h_{ij}(\mathbf{X}, \mathbf{Y})) > v(h'_{ij}(\mathbf{X}, \mathbf{Y}))\big]
\end{aligned}
$$

where I and J are finite sets, and $f_{ij}, f'_{ij}, g_{ij}, g'_{ij}, h_{ij}, h'_{ij}$ are polynomials with coefficients in K for all $(i, j) \in I \times J$. We have to prove that

(10.2) if there exists $\mathbf{x} \in F^m$ such that $\varphi(\mathbf{x})$ holds in (F, v), then there exists $\mathbf{a} \in K^m$ such that $\varphi(\mathbf{a})$ holds in (K, v).

First note that the disjunction symbol commutes with the existential quantifiers. Moreover, if (10.2) holds for one of the disjuncts of φ, it also holds for φ. Thus, it suffices to consider φ of the form

$$
\begin{aligned}
(9.3)\qquad (\exists Y_1) \cdots (\exists Y_n) \bigwedge_{j \in J} \big[f_j(\mathbf{X}, \mathbf{Y}) = 0 \wedge f'_j(\mathbf{X}, \mathbf{Y}) \neq 0 \\
\wedge\, v(g_j(\mathbf{X}, \mathbf{Y})) \geq v(g'_j(\mathbf{X}, \mathbf{Y})) \\
\wedge\, v(h_j(\mathbf{X}, \mathbf{Y})) > v(h'_j(\mathbf{X}, \mathbf{Y}))\big],
\end{aligned}
$$

where $f_j, f'_j, g_j, g'_j, h_j, h'_j \in K[\mathbf{X}, \mathbf{Y}]$ for all $j \in J$.

The formula $f'_j(\mathbf{X}, \mathbf{Y}) \neq 0$ is equivalent to $(\exists Z_{j1})[Z_{j1} f'_j(\mathbf{X}, \mathbf{Y}) - 1 = 0]$. The formula $v(g_j(\mathbf{X}, \mathbf{Y})) \geq v(g'_j(\mathbf{X}, \mathbf{Y}))$ is equivalent to

$$
\begin{aligned}
&[g_j(\mathbf{X}, \mathbf{Y}) = 0 \wedge g'_j(\mathbf{X}, \mathbf{Y}) = 0] \\
&\qquad \vee \big[g'_j(\mathbf{X}, \mathbf{Y}) \neq 0 \wedge (\exists Z_{j2})[g_j(\mathbf{X}, \mathbf{Y}) = Z_{j2} g'_j(\mathbf{X}, \mathbf{Y}) \wedge v(Z_{j2}) \geq 0]\big].
\end{aligned}
$$

Finally, the formula $v(h_j(\mathbf{X}, \mathbf{Y})) > v(h'_j(\mathbf{X}, \mathbf{Y}))$ is equivalent to

$$[h_j(\mathbf{X},\mathbf{Y})=0\wedge h'_j(\mathbf{X},\mathbf{Y})\neq 0]$$
$$\vee\big[h_j(\mathbf{X},\mathbf{Y})\neq 0\wedge(\exists Z_{j3})[h_j(\mathbf{X},\mathbf{Y})=Z_{j3}h'_j(\mathbf{X},\mathbf{Y})\wedge v(Z_{j3})>0]\big].$$

Since Z_{j1},Z_{j2},Z_{j3} do not occur among the coordinates of $\mathbf{Y}$, we may pull over the quantifiers $\exists Z_{j1}$, $\exists Z_{j2}$, and $\exists Z_{j3}$ to the left of (10.3). Then we rename each Z_{jk} as Y_r for some $r>n$ and finally enlarge n and repeat the first two simplification steps to conclude that φ has the form

$$(\exists Y_1)\cdots(\exists Y_n)\bigwedge_{j\in J}[f_j(\mathbf{X},\mathbf{Y})=0\wedge v(Y_1)\succ_1 0\wedge\cdots\wedge v(Y_n)\succ_n 0],$$

where for $i=1,\ldots,n$ the relation $\succ_i$ is either $\geq$, or $>$, or the trivial relation $0=0$.

Part B: *Existential closedness.* Let $\mathbf{x}\in F^m$ be such that $\varphi(\mathbf{x})$ holds in (F,v). Then there exists $\mathbf{y}\in F^n$ such that

$$\bigwedge_{j\in J}[f_j(\mathbf{x},\mathbf{y})=0\wedge v(y_1)\succ_1 0\wedge\cdots\wedge v(y_n)\succ_n 0].$$

Since F/K is a regular extension, so is $K(\mathbf{x},\mathbf{y})/K$. Thus, $(\mathbf{x},\mathbf{y})$ is a generic point of an affine absolutely irreducible variety W defined in $\mathbb{A}^{m+n}$ over K [FrJ08, p. 175, Cor. 10.2.2]. Now we extend v to a valuation v of $\tilde{F}$ and again denote the restriction of v to $\tilde{K}$ by v. By Theorem 8.4, the first order theory of algebraically closed non-trivial valued field is model complete. In particular, $(\tilde{K},v)$ is an elementary substructure of $(\tilde{F},v)$. Therefore, there exists $(\tilde{\mathbf{a}},\tilde{\mathbf{b}})\in W(\tilde{K})$ such that $v(\tilde{b}_i)\succ_i 0$ for each i. Since $W(K)$ is v-dense in $W(\tilde{K})$ (Theorem 6.6), we conclude that there exists $(\mathbf{a},\mathbf{b})\in W(K)$ such that $v(b_i)\succ_i 0$ for each i. Consequently, $\varphi(\mathbf{a})$ holds in (K,v). □

10 Model Companion

Model companions and model completions of first order theories generalize the relation that the theory of algebraically closed fields has relative to the theory of all fields. In this section we prove the existence of a model companion for the theory of non-trivial valued fields in a language that allows only extensions L/K of fields such that K is algebraically closed in L. In the next section we add more predicates to the language that force the field extensions we consider to be regular. This results in a model completion of the corresponding theory.

Definition 10.1. Let T and $\tilde{T}$ be theories in a first order language $\mathcal{L}$. We say that $\tilde{T}$ is a **model companion** of T if the following holds:

(11.1a) Each model of $\tilde{T}$ is a model of T.

(11.1b) Each model of T can be embedded into a model of $\tilde{T}$.

(11.1c) $\tilde{T}$ is model complete (Definition 8.2(a)).

We say that $\tilde{T}$ is a **model completion** of T if in addition

(11.1d) T has the amalgamation property (Definition 8.2(b)).

Remark 10.2. By (11.1a) and (11.1b) of Definition 10.1, (11.1d) is equivalent to the statement

(11.1.d') $\tilde{T}$ has the amalgamation property.

In this case $\tilde{T}$ admits, by Proposition 8.3(a), elimination of quantifiers.

Example 10.3. (a) The theory of algebraically closed fields is the model completion of the theory of fields in $\mathcal{L}(\text{ring})$. (Essentially [FrJ08, p. 168, Cor. 9.3.2]).

(b) The theory RCF of real closed fields is the model companion of the theory OF of ordered fields in the language $\mathcal{L}(\text{ring}, <)$, where $<$ is the ordering symbol [Pre86, Kor. 4.8]. By [VdD78, p. 40], RCF is even the model completion of OF.

(c) The theory ACF_{val} of algebraically closed non-trivial valued fields is the model completion of the theory F_{val} of valued fields in the language $\mathcal{L}_{\text{val}}(\text{ring})$. This follows from Chevalley's extension theorem of valuations and from Theorem 8.4 of Abraham Robinson [Pre86, p. 241, Kor. 4.18].

Example 10.4. We augment the language $\mathcal{L}(\text{ring})$ to a language $\mathcal{L}_R(\text{ring})$ by an n-ary relation symbol R_n for each positive integer n. Let T_R be the theory of fields together with the axioms

$$R_n(X_1, \ldots, X_n) \leftrightarrow (\exists Z)[Z^n + X_1 Z^{n-1} + \cdots + X_n = 0]. \tag{10.2}$$

For each field K and every n we interpret R_n in K in the unique way such that (11.2) holds. Then consider K also as a model of T_R. If an $\mathcal{L}(\text{ring})$-structure L is a field, then an $\mathcal{L}_R(\text{ring})$-substructure K of the $\mathcal{L}_R(\text{ring})$-structure L is an integral domain contained in L such that every equation $X^n + a_{n-1}X^{n-1} + \cdots + a_0 = 0$ with coefficients in K that has a root in L has a root in K. In particular, if K is a field, then K is algebraically closed in L.

By [FrJ08, p. 663, Thm. 27.2.3], T_R has a model companion $\tilde{T}_R$. A field K is a model of $\tilde{T}_R$ if and only if K is 1-**imperfect** (i.e. $\text{char}(K) = 0$ or $\text{char}(K) = p > 0$ and $[K : K^p] = p$), ω-free (i.e. every finite embedding problem over K is solvable), and PAC. However, T_R has no model completion, because T_R does not have the amalgamation property [FrJ08, p. 664, Example 27.2.4].

Let T_{val} be the theory of $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$ that consists of the usual axioms of fields (9.1) of Section 8 and the axioms (9.3) of Section 8 for $|$.

Adding the division symbol to the language $\mathcal{L}_R(\mathrm{ring})$, we get a first order language $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$ for valued fields such that a valued field $(F, |)$ is a substructure of another valued field $(F', |')$ if and only if F is an algebraically closed subfield of F' and the restriction of $|'$ to F is $|$. Let $T_{R,\mathrm{val}}$ be the theory in $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$ that consists of T_R and the axioms (9.3) of Section 8.

Remark 10.5. Every valued field (K, v) has an extension (K', v'), where K'/K is regular and v' is a non-trivial valuation of K'.

Indeed, if v is non-trivial, let $K' = K$ and $v' = v$. Otherwise, choose an indeterminate t, let $K' = K(t)$ and v' any of the valuations of K'/K (e.g. the one with $v'(t) = -1$).

Theorem 10.6. *The theory $T_{R,\mathrm{val}}$ of valued fields has a model companion $\tilde{T}_{R,\mathrm{val}}$ in the language $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$. A non-trivial valued field (K, v) is a model of $\tilde{T}_{R,\mathrm{val}}$ if and only if K is* 1*-imperfect, ω-free, and PAC.*

Proof. Let $\tilde{T}_{R,\mathrm{val}}$ be the theory $\tilde{T}_R$ of Example 10.4 together with the axioms (9.3) and (9.4) of Section 8 for non-trivial valued fields.

If (K, v) is a model of $\tilde{T}_{R,\mathrm{val}}$, then K is a model of $\tilde{T}_R$. By Example 10.4, K is 1-imperfect, ω-free, and PAC. Conversely, if a field K is 1-imperfect, ω-free, and PAC, then by Example 10.4, K is a model of $\tilde{T}_R$. Hence, if v is a non-trivial valuation of K, then (K, v) is a model of $\tilde{T}_{R,\mathrm{val}}$.

If (K, v) is a valued field, we extend it to another model (K', v') with K'/K regular and v' non-trivial (Remark 10.5). By Example 10.4 and by Definition 10.1(b), there exists a field extension L of K' which is 1-imperfect, ω-free and PAC such that K' is algebraically closed in L. Then, K is algebraically closed in L. By Chevalley, v' extends to a valuation w of L [Lan58, p. 8, Thm. 1]. In particular w is non-trivial. Hence, (L, w) is a model of $\tilde{T}_{R,\mathrm{val}}$ that extends (K, v) in the language $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$. Thus, $\tilde{T}_{R,\mathrm{val}}$ satisfies Condition (11.1a) and (11.1b) of Definition 10.1.

Claim: *If $(K, v) \subseteq (L, w)$ is an extension of models of $\tilde{T}_{R,\mathrm{val}}$, then the field L is a regular extension of K.* Indeed, by Example 10.4, K is algebraically closed in L. Moreover, by the second paragraph of the proof, K is 1-imperfect. Hence, by [FrJ08, p. 47, Lemma 2.7.5], L/K is a regular extension.

Next observe that if (K, v) and (L, w) are models of $\tilde{T}_{R,\mathrm{val}}$ and $(K, v) \subseteq (L, w)$, then K is PAC, and L/K is a regular extension (by the Claim). Moreover, both v and w are non-trivial valuations. Hence, by Theorem 9.1, (K, v) is existentially closed in (L, w) in the language $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$. Since each of the axioms R_n is equivalent to an existential formula of $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$ (Example 10.4), (K, v) is existentially closed in (L, w) in the language $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$. By Abraham Robinson, $\tilde{T}_{R,\mathrm{val}}$

is model complete [FrJ08, p. 659, Lemma 27.1.11], that is it satisfies Condition (11.1c) of Definition 10.1. Consequently, $\tilde{T}_{R,\mathrm{val}}$ is a model companion of $T_{R,\mathrm{val}}$. □

11 The Non-Amalgamation Property of $T_{R,\mathrm{val}}$

Example 27.2.4 of [FrJ08] constructs fields K and L of positive characteristic p that are algebraically closed in no common field extension. However, K and L are isomorphic, so that example does not prove that the theory T_R does not have the amalgamation property, as the paragraph before that example claims.

In this section we correct that example by constructing three fields E, K, L of characteristic p such that E is algebraically closed in both K and L but K and L are algebraically closed in no common field extension. This implies that none of the theories T_R and $T_{R,\mathrm{val}}$ has the amalgamation property.

Our construction uses the notions of differentials of fields. For each field L of characteristic p we consider the vector space $\mathrm{Der}(L,L)$ over $\mathbb{F}_p$ of all derivations of L into L and the map $d_L\colon L \to \mathrm{Hom}(\mathrm{Der}(L,L),L)$ defined by $(d_La)(D) = Da$ for each $a \in L$. It satisfies the relations

$$(11.1)\qquad p \cdot d_La = 0,\ d_L(a+b) = d_La + d_Lb,\ \text{and } d_L(ab) = ad_Lb + bd_La$$

for all $a, b \in L$. Each d_La with $a \in L$ is called a **differential** of L. Repeated application of (12.1) leads for $a_0, a_1, \ldots, a_n \in L$ to the following formula:

$$(11.2)\qquad d_L(a_0^p a_1^{i_1} \cdots a_n^{i_n}) = \sum_{j=1}^{n} a_0^p a_1^{i_1} \cdots a_{j-1}^{i_{j-1}} \cdot i_j a_j^{i_j-1} \cdot a_{j+1}^{i_{j+1}} \cdots a_n^{i_n} \cdot d_La_j.$$

Lemma 11.1. *Let L be a field extension of $\mathbb{F}_p$ and let $a \in L$. Then $d_La = 0$ if and only if $a \in L^p$.*

Proof. If $a = a_0^p$ for some $a_0 \in L^p$, then $d_La = pa_0^{p-1} \cdot d_La_0 = 0$. Conversely, assume that $a \notin L^p$. Then a can be completed to a p-basis of L [FrJ08, p. 45, Lemma 2.7.1]. By [Lan58, Thm. 1] or [Gey13, Prop. (d) of Section 1.18], the trivial derivative of L^p extends to a derivative $D \in \mathrm{Der}(L,L)$ such that $Da = 1$. Hence, $(d_La)(D) = Da \neq 0$, so $d_La \neq 0$. □

Lemma 11.2. *Let E be a field extension of $\mathbb{F}_p$ with $[E : E^p] = p^n$, where $n \geq 2$ is an integer, let $a_1, \ldots, a_n$ be a p-basis for E over E^p, and let x be an indeterminate. For $i = 2, \ldots, n$ let*

$$(11.3)\qquad y_i = a_{i-1}^{1/p} x + a_i^{1/p}$$

and let $L = E(x, y_2, \ldots, y_n)$. Then E is algebraically closed in L. Moreover, for $i = 2, \ldots, n$ we have

$$(11.4)\qquad [E(x, y_2, \ldots, y_i) : E(x)] = p^{i-1}.$$

Proof. We break the proof into three parts.

PART A: *Proof of (12.4).* By assumption, $[E(a_i^{1/p}) : E] = p$ for $i = 1, \dots, n$ and the fields $E(a_1^{1/p}), \dots, E(a_n^{1/p})$ are linearly disjoint over E. Hence, $E' = E(a_1^{1/p}, \dots, a_n^{1/p})$ satisfies $[E' : E] = p^n$, so $[E'(x) : E(x)] = p^n$. By (12.3), $y_i \in E'(x)$ for $i = 2, \dots, n$, so $L \subseteq E'(x)$. Moreover, by (12.3), $y_i^p \in E(x)$, so $[E(x, y_2, \dots, y_{i+1}) : E(x, y_2, \dots, y_i)] \le p$ for $i = 2, \dots, n-1$ and $[E(x, y_2) : E(x)] \le p$. Finally, by (12.3), $L(a_1^{1/p}) = E(x, y_2, \dots, y_n, a_1^{1/p}) = E'(x)$ and $[E'(x) : L] \le p$. It follows that (12.4) holds for $i = 2, \dots, n$ and also

$$[E'(x) : L] = p. \tag{11.5}$$

PART B: *For each $e \in E$ with $d_E e \neq 0$ we have $d_L e \neq 0$.* Otherwise, there exists $e \in E$ such that

$$d_E e \neq 0 \text{ but } d_L e = 0. \tag{11.6}$$

We denote the set of all n-tuples $\mathbf{j} = (j_1, \dots, j_n)$ with $0 \le j_1, \dots, j_n \le p-1$ by J. By our assumption on $a_1, \dots, a_n$, there exist $e_{\mathbf{j}} \in E$, $\mathbf{j} \in J$, such that

$$e = \sum_{\mathbf{j} \in J} e_{\mathbf{j}}^p a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}. \tag{11.7}$$

Taking differentials of both sides of (12.7) and using formula (12.2), we get $\varepsilon_1, \dots, \varepsilon_n \in E$ such that

$$d_E e = \sum_{i=1}^n \varepsilon_i d_E a_i \text{ and } 0 = d_L e = \sum_{i=1}^n \varepsilon_i d_L a_i. \tag{11.8}$$

Next we raise (12.3) to the pth power and get

$$a_n = y_n^p - a_{n-1} x^p, \quad a_{n-1} = y_{n-1}^p - a_{n-2} x^p, \quad \dots, \quad a_2 = y_2^p - a_1 x^p. \tag{11.9}$$

Applying d_L on the latter equalities, we get

$$d_L a_n = -x^p d_L a_{n-1}, \quad d_L a_{n-1} = -x^p d_L a_{n-2}, \quad \dots, \quad d_L a_2 = -x^p d_L a_1. \tag{11.10}$$

Now we successively substitute $d_L a_n, d_L a_{n-1}, \dots, d_L a_2$ from (12.10) in the right equality of (12.8) to get a relation

$$0 = \Big(\sum_{i=1}^n \pm \varepsilon_i x^{(i-1)p}\Big) d_L a_1. \tag{11.11}$$

If $d_L a_1 = 0$, then by Lemma 11.1, $a_1 \in L^p$. Hence, by (12.9), $a_2, \dots, a_n \in L^p$. It follows that $L = E'(x)$, in contrast to (12.5). It follows from this contradiction that $d_L a_1$ is a nonzero element of L. Hence, by (12.11), $\sum_{i=1}^n \pm\varepsilon_i x^{(i-1)p} = 0$. Therefore, $\varepsilon_1,\dots,\varepsilon_n = 0$, so by (12.8), $d_E e = 0$, in contrast to (12.6).

PART C: *E is algebraically closed in L.* Let u be an element of L which is algebraic over E. Then u belongs to $E'(x)$ and is algebraic over E'. Hence, $u \in E'$. Since $(E')^p \subseteq E$, we have $u_0 = u^p \in E \cap L^p$. Therefore, $d_L u_0 = 0$. By Part B, $d_E u_0 = 0$. Hence, by Lemma 11.1, $u_0 \in E^p$. Consequently, $u \in E$, as claimed. □

Proposition 11.3. *None of the theories T_R and $T_{R,\mathrm{val}}$ have the amalgamation property.*

Proof. We choose a field E of positive characteristic p such that $[E : E^p] = p^3$ and let a, b, c be p-basis for E over E^p. For example, we may take a, b, c as algebraically independent elements over $\mathbb{F}_p$ and set $E = \mathbb{F}_p(a, b, c)$ [FrJ08, p. 45, proof of Lemma 2.7.2]. Then, let x be a transcendental element over E, set $y = a^{1/p}x + b^{1/p}$ and $z = b^{1/p}x + c^{1/p}$, and let $K = E(x, y)$ and $L = E(x, y, z)$. By Lemma 11.2, E is algebraically closed in both K and L.

We assume there exist a field M and embeddings $\varphi\colon K \to M$ and $\psi\colon L \to M$ that coincide on E such that $K' = \varphi(K)$ and $L' = \psi(L)$ are algebraically closed in M. We may assume without loss that $K' = K$ and φ is the identity map. Then $\varphi(E) = E$, so E is algebraically closed in M.

Raising y and z to the pth power we get

$$y^p = ax^p + b \text{ and } z^p = bx^p + c. \tag{11.12}$$

Hence, $x' = \psi(x)$, $y' = \psi(y)$, and $z' = \psi(z)$ satisfy

$$(y')^p = a(x')^p + b \text{ and } (z')^p = b(x')^p + c. \tag{11.13}$$

If $x' \neq x$, then by (12.12) and (12.13), $\left(\frac{y'-y}{x'-x}\right)^p = a$. Hence, $a^{1/p} \in M \cap \tilde{E} = E$, which is a contradiction.

If $x' = x$, then $y' = y$ and $z' = z$. Hence, $L' = L$, so $L \subseteq M$. However, by Lemma 11.2, L is a proper algebraic extension of K, so K is not algebraically closed in M, which is again a contradiction.

We conclude that an M as above does not exist, so T_R does not have the amalgamation property.

Now we choose a valuation v_E of E and extend v_E, by Chevalley, to valuations v_K and v_L of K and L, respectively. Then (E, v_E) is a $T_{R,\mathrm{val}}$-submodel of (K, v_K) and of (L, v_L). But (K, v_K) and (L, v_L) can not be embedded into a common $T_{R,\mathrm{val}}$-model (M, v_M) over (E, v_E), because K will then be algebraically closed in M, in contrast to the construction of K above. Thus, $T_{R,\mathrm{val}}$ does not have the amalgamation property. □

In the next section we rectify the deficiency of $T_{R,\mathrm{val}}$ expressed in Proposition 11.3 by adding more relation symbols to the language $\mathcal{L}_{R,\mathrm{val}}$. The new relations will ensure that if a model of the new language is contained in another model, then the underlying field of the latter model will be a regular extension of the underlying field of the former model.

12 Elimination of Quantifiers

Let (K,v) and (L,w) be valued fields and consider them as structures for the language $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$ (Example 10.4). If (K,v) is a substructure for (L,w), then by that example, K is algebraically closed in L. However, L is not necessarily a regular extension of K. We rectify this deficiency of $\mathcal{L}_{R,\mathrm{val}}(\mathrm{ring})$ by adding more relations to the language and prove that the theory of the non-trivial valued fields in the new language has a model completion admitting elimination of quantifiers.

Example 12.1. As in [FrJ08, p. 664, Sec. 27.3] we augment the language $\mathcal{L}_R(\mathrm{ring})$ (Example 10.4) to a language $\mathcal{L}_{R,Q}(\mathrm{ring})$ by adding n-ary relation symbols $Q_{p,n}$, one for each prime number p and each positive integer n. Let $T_{R,Q}$ be the theory of $\mathcal{L}_{R,Q}(\mathrm{ring})$ consisting of T_R (Example 10.4) together with the axioms

$$(12.1)\quad Q_{p,n}(X_1,\ldots,X_n) \leftrightarrow p=0 \wedge (\exists U_{\mathbf{i}})_{\mathbf{i}\in I}\big(\sum_{\mathbf{i}\in I} U_{\mathbf{i}}^p X_1^{i_1}\cdots X_n^{i_n}=0 \wedge \bigvee_{\mathbf{i}\in I} U_{\mathbf{i}}\neq 0\big),$$

one for each pair (p,n), where I is the set of all n-tuples $(i_1,\ldots,i_n)$ of integers between 0 and $p-1$. Given a field K, we may uniquely regard K as a model of $T_{R,Q}$. Indeed, if $\mathrm{char}(K)=p>0$, we define $Q_{p,n}$ as the set of all n-tuples of elements of K satisfying the right hand side of (13.1), i.e. all n-tuples of p-dependent elements of K. In this case $Q_{p',n}$ is the empty relation for each prime number $p'\neq p$. Therefore, if $K\subseteq L$ is an extension of fields considered as models of $T_{R,Q}$, if $p=\mathrm{char}(K)>0$, and if $x_1,\ldots,x_n\in K$ are p-independent in K, then $\neg Q_{p,n}(x_1,\ldots,x_n)$ is true in K, hence in L, and therefore $x_1,\ldots,x_n$ are p-independent in L. It follows that L/K is a separable extension [FrJ08, p. 38, Lemma 2.6.1]. Considering L/K as an $\mathcal{L}_R(\mathrm{ring})$-extension, we find that K is algebraically closed in L (Example 10.4). Therefore, L/K is a regular extension [FrJ08, p. 39, Lemma 2.6.4].

Conversely, every regular extension L/K of fields is an extension of structures for the language $\mathcal{L}_{R,Q}(\mathrm{ring})$.

Definition 12.2. We say that a field K is ω**-imperfect** if either $\mathrm{char}(K)=0$ or $\mathrm{char}(K)=p>0$ and $[K^{1/p}:K]=\infty$. In the latter case, if L is a separable field extension of K, then L is linearly disjoint from $K^{1/p}$ over K, hence $[L^{1/p}:L]=\infty$, so L is also ω-imperfect.

Note that if $\mathrm{char}(K) = p > 0$, K is a model of $T_{R,Q}$, and $x_1, \ldots, x_n$ are elements of K such that $\neg Q_{p,n}(x_1, \ldots, x_n)$ holds in K, then, by (13.1), $[K(x_1^{1/p}, \ldots, x_n^{1/p}) : K] = p^n$. Hence, if each of the sentences

$$(12.2) \qquad (\exists X_1) \cdots (\exists X_n) \neg Q_{p,n}(X_1, \ldots, X_n)$$

holds in K, then K is ω-imperfect.

The following result is [FrJ08, p. 665, Thm. 27.3.1].

Proposition 12.3. *The theory $T_{R,Q}$ has a model completion $\tilde{T}_{R,Q}$ whose models are the ω-imperfect ω-free PAC fields.*

The axioms of $\tilde{T}_{R,Q}$ differ from the axioms of $\tilde{T}_R$ by the sentences (13.2) that replace the axioms for 1-imperfectness.

Adding the division symbol to the language $\mathcal{L}_{R,Q}(\mathrm{ring})$, we get a first order language $\mathcal{L}_{R,Q,\mathrm{val}}(\mathrm{ring})$ for valued fields. Then we augment the theory $T_{R,Q}$ to a theory $T_{R,Q,\mathrm{val}}$ in the language $\mathcal{L}_{R,Q,\mathrm{val}}(\mathrm{ring})$ by adding the axioms (9.3) of Section 8 to $T_{R,Q}$. We also augment $\tilde{T}_{R,Q}$ to a theory $\tilde{T}_{R,Q,\mathrm{val}}$ of the language $\mathcal{L}_{R,Q,\mathrm{val}}(\mathrm{ring})$ by adding the axioms (9.3) and (9.4) of Section 8 to $\tilde{T}_{R,Q}$. In particular, each of the models (K, v) of $\tilde{T}_{R,Q,\mathrm{val}}$ is a non-trivial valued field.

Lemma 12.4. *The theory $T_{R,Q,\mathrm{val}}$ has the amalgamation property.*

Proof. Let (L_1, v_1) and (L_2, v_2) be two models of $T_{R,Q,\mathrm{val}}$ that contain a common substructure for $T_{R,Q,\mathrm{val}}$. By Lemma 8.1(a), we may assume that this model is a common valued subfield (K, v). In particular, L_1 and L_2 are regular extensions of K. Replacing (L_2, v_2) by an isomorphic valued field extension (L_2', v_2') of (K, v), we may assume, in addition, that L_1 and L_2 are algebraically independent over K. Hence, L_1 and L_2 are linearly disjoint over K [FrJ08, p. 41, Lemma 2.6.7]. In particular, the compositum $L = L_1L_2$ is a regular extension of K [FrJ08, p. 41, Cor. 2.6.8(b)]. By [FrJ08, p. 35, Lemma 2.5.5], L has a valuation w that extends both v_1 and v_2. Thus, (L, w) is a model of $T_{R,Q,\mathrm{val}}$ that extends both (L_1, v_1) and (L_2, v_2), as desired. □

We are now in a position to prove an analog of both Theorem 10.6 and Proposition 12.3.

Theorem 12.5. *The theory $T_{R,Q,\mathrm{val}}$ has a model completion $\tilde{T}_{R,Q,\mathrm{val}}$ whose models are the non-trivial valued fields (F, w) such that F is an ω-imperfect, ω-free PAC field. Moreover, $\tilde{T}_{R,Q,\mathrm{val}}$ admits elimination of quantifiers.*

Proof. If (K, v) is a model of $\tilde{T}_{R,Q,\mathrm{val}}$, then K is a model of $\tilde{T}_{R,Q}$. By Proposition 12.3, K is ω-imperfect, ω-free, and PAC. Conversely, if a field K is ω-imperfect,

ω-free, and PAC, then by Proposition 12.3, K is a model of $\tilde{T}_{R,Q}$. Hence, if v is a non-trivial valuation of K, then (K,v) is a model of $\tilde{T}_{R,Q,\mathrm{val}}$.

If (K,v) is a model of $T_{R,Q,\mathrm{val}}$, we replace it by a regular valued field extension, if necessary, to assume that v is non-trivial (Remark 10.5). By Proposition 12.3, K has a regular field extension L which is ω-imperfect, ω-free and PAC. Note that the regularity of L/K is forced by the language $T_{R,Q,\mathrm{val}}$, as noted in Example 12.1. By Chevalley, v extends to a valuation w of L [Lan58, p. 8, Thm. 1], so (L,w) is a model of $\tilde{T}_{R,Q,\mathrm{val}}$ that extends (K,v) in the language $\mathcal{L}_{R,Q,\mathrm{val}}(\mathrm{ring})$. Thus, $\tilde{T}_{R,Q,\mathrm{val}}$ satisfies Conditions (11.1a) and (11.1b) of Definition 10.1.

Next observe that if (E,v) and (F,w) are models of $\tilde{T}_{R,Q,\mathrm{val}}$ and $(E,v) \subseteq (F,w)$, then E is PAC and F/E is a regular extension. Moreover, both v and w are non-trivial valuations. Hence, by Theorem 9.1, (E,v) is existentially closed in (F,w) in the language $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$. Since each of the axioms R_n and $Q_{p,n}$ is equivalent to an existential formula of $\mathcal{L}_{\mathrm{val}}(\mathrm{ring})$ (Example 10.4 and Example 12.1), (E,v) is existentially closed in (F,w) in the language $\mathcal{L}_{R,Q,\mathrm{val}}(\mathrm{ring})$. Thus, $\tilde{T}_{R,Q,\mathrm{val}}$ is model complete, that is it satisfies Condition (11.1c) of Definition 10.1. Consequently, $\tilde{T}_{R,Q,\mathrm{val}}$ is a model companion of $T_{R,Q,\mathrm{val}}$. By Lemma 12.4, $\tilde{T}_{R,Q,\mathrm{val}}$ is a model completion of $T_{R,Q,\mathrm{val}}$. By Remark 10.2, $\tilde{T}_{R,Q,\mathrm{val}}$ admits elimination of quantifiers. □

References

[AlK70] A. Altman and S. Kleiman, *Introduction to Grothendieck Duality Theory,* Lecture Notes in Mathematics **146**, Springer-Verlag, Berlin, 1970.

[AtM69] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra,* Addison–Wesley, Reading, 1969.

[Ax68] J. Ax, *The elementary theory of finite fields,* Annals of Mathematics **88** (1968), 239–271.

[Bou89] N. Bourbaki, *Commutative Algebra, Chapters 1–7,* Springer, Berlin, 1989.

[Efr06] I. Efrat, *Valuations, Orderings, and Milnor K-Theory,* Mathematical surveys and monographs **124**, American Mathematical Society, Providence, 2006.

[FrJ76] M. Fried and M. Jarden, *Stable extensions with the global density property,* Canadian Journal of Mathematics **28** (1976), 774–787.

[FrJ05] M. D. Fried and M. Jarden, *Field Arithmetic, Second Edition, revised and enlarged by Moshe Jarden,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 2005.

[FrJ08] M. D. Fried and M. Jarden, *Field Arithmetic, Third Edition, revised by Moshe Jarden,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 2008.

[Gey13] W.-D. Geyer, *Field Theory,* manuscript, Erlangen, 2013.

[GeJ75] W.-D. Geyer and M. Jarden, *Fields with the density property,* Journal of Algebra **35** (1975), 178–189.

[GoW10] U. Görtz and T. Wedhorn, *Algebraic Geometry I,* Vieweg + Teubner Verlag, Wiesbaden 2010.

[GPR95] B. Green, F. Pop, and P. Roquette, *On Rumely's local-global principle,* Jahresbericht der Deutschen Mathematiker-Vereinigung **97** (1995), 43–74.

[Gro64] A. Grothendieck, *Éléments de Géométrie Algébrique IV, première partie,* Publications Mathématiques, IHES **20** (1964), 101–355.

[Har77] R. Hartshorne, *Algebraic Geometry,* Graduate Texts in Mathematics **52**, Springer, New York, 1977.

[JaR98] M. Jarden and A. Razon, *Rumely's local global principle for algebraic PSC fields over rings,* Transactions of AMS **350** (1998), 55-85.

[Jar91] M. Jarden, *Intersection of local algebraic extensions of a Hilbertian field,* in "Generators and Relations in Groups and Geometries", (A. Barlotti et al., eds), NATO ASI Series C **333** 343–405, Kluwer, Dordrecht, 1991.

[Kol07] J. Kollár, *Algebraic varieties over PAC fields,* Israel Journal of Mathematics **162** (2007), 89–101.

[Lan58] S. Lang, *Introduction to Algebraic Geometry,* Interscience Publishers, New York, 1958.

[Lan93] S. Lang, *Algebra, Third Edition,* Eddison-Wesley, Reading, 1993.

[Liu06] Q. Liu, *Algebraic Geometry and Arithmetic Curves,* Oxford Graduate Texts in Mathematics **6**, Oxford University Press, 2006.

[Mum88] D. Mumford, *The Red Book of Varieties and Schemes,* Lecture Notes in Mathematics **1358**, Springer, Berlin, 1988.

[Pre86] A. Prestel, *Einführung in die Mathematische Logik und Modelltheorie,* Vieweg, Braunschweig 1986.

[Sha77] I. R. Shafarevich, *Basic Algebraic Geometry,* Grundlehren der mathematischen Wissenschaften **213**, Springer Berlin, 1977.

[VdD78] L.P.D. v. d. Dries, *Model theory of fields*, Thesis, Utrecht, 1978.

[Wei62] A. Weil, *Foundations of Algebraic Geometry*, American Mathematical Society, Providence, 1962.

Wulf-Dieter Geyer
Friedrich-Alexander-Universität Erlangen-Nürnberg
Department Mathematik
Cauerstraße 11
D-91058 Erlangen
GERMANY
geyer@mi.uni-erlangen.de

Moshe Jarden
School of Mathematics
Tel Aviv University
Ramat Aviv, Tel Aviv 69978
ISRAEL
jarden@post.tau.ac.il